On approximating DSGE models by series expansions

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Abstract

The existing literature on perturbation methods for dynamic stochastic general equilibrium models points out that the approximated solution can generate spurious unstable dynamics. Various ad-hoc techniques have been suggested in order to overcome this problem. In this paper we show that by applying series-expansion techniques, as suggested by the mathematical literature on perturbation methods for non-linear dynamical systems, generates solutions that are (globally) stable – as long as the original system is locally stable – at any order of approximation. A feature not shared by the alternative high-order perturbation approaches discussed in the economic literature. This paper contributes to the existing literature in two ways. First, it points to the mathematical foundations of some of the ad-hoc remedies used in the literature (i.e. "pruning") for second-order approximations. Second, it offers a clear guide on how to proceed for orders larger than two. We compare the alternative perturbation techniques and highlight their properties.

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1 Introduction

Following the seminal work of Jin and Judd (2002) and Judd (1998, 2002) a number of papers has contributed to the literature on higher order approximations by suggesting alternative techniques to solve and simulate dynamic stochastic general equilibrium (DSGE) models. Most of this literature derives approximated dynamical systems that are non-linear in the state variables (e.g. Schmitt-Grohé and Uribe, 2004, Kim et al., 2008, Anderson et al., 2006, Collard and Juillard, 2001 and Fernandez-Villaverde and Rubio-Ramirez, 2006).[1] As a consequence, for sufficiently large shocks the approximated solution can imply explosive dynamics, even if the original system is still (saddle-path) stable for shocks of the same magnitude.[2] The possibility that spurious unstable dynamics emerge from these solutions is a well-known fact, documented, among others, by Fernandez-Villaverde and Rubio-Ramirez (2006), Aruoba et al. (2006) and Kim et al. (2008). The literature is not unanimous about the way to treat this problem. For example, Aruoba et al. (2006) suggest to disregard re-

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[1]Gomme and Klein (2010) provide an alternative solution method for the coefficient matrices of a second order approximation to the policy function. In their paper the authors use the same representation of the approximating policy function as in Schmitt-Grohé and Uribe (2004). Andreasen (2010), in a recent paper, provides accompanying software that extend Schmitt-Grohé and Uribe (2004) to third order. Amisano and Tristani (2011) show how to take second order approximations à la Schmitt-Grohé and Uribe (2004) when innovations are heteroskedastic. Similarly, Benigno et al. (2010) show how to compute second order approximations for models with time varying risk.

[2]In this paper we refer to stability as local asymptotic stability (Azariadis, 1993, p. 22), and, in relation to DSGE models, as local saddle-path stability. In general, we cannot exclude that the true dynamical system has multiple (stochastic) equilibria such that the point of approximation is not *globally* saddle-path stable (e.g. Tu, 1994). In this case the radius of convergence of the original function around the point of approximation would be delimited by the separatrix curve marking the boundary between regions in the phase plane with different stability properties. Therefore, none of the local perturbation methods discussed here would allow us to make statements concerning the global properties of the true dynamical system.

alizations that generate unbounded solutions.[3] Kim et al. (2008), in an influential paper, have suggested to "prune" the second order approximated solution by replacing the quadratic terms with cross-products of the first order solution and, hence, by generating a solution that is recursively linear.[4] A different approach is followed by Lombardo and Sutherland (2007), who develop a second-order solution technique based on the recursive linearity of the state-space representation. In that paper the authors resort to an "order" argument to maintain that cross-products of variables in higher order expressions should be computed using lower order terms. Their paper does not make any explicit link to the mathematical foundations or their approach. A gap that we fill in the present paper.

In this paper we borrow from the applied-mathematics literature to show that approximating DSGE models using the method of series expansions naturally generates solutions that are i) recursively linear and ii) (globally) stable as long as the first order dynamics is stable. As explained by Berglund (2001, p. 3) "... at each order we only need to solve [a] linear equation, where the [non-linear] term depends only on previously computed quantities."[5] We show that, contrary to our approach, naive "pruning" does not ensure stability even in those regions of the phase space where the original model is stable.

Solving non-linear DSGE models by perturbation methods, including the series

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[3]These authors also point out that the problem could be avoided by bounding the support of the shocks, as recommended by Jin and Judd (2002).

[4]Schmitt-Grohé and Uribe (2004) show the state-space solution in non-linear form but don't discuss the potential issues related to its non-linear structure. Nevertheless, the accompanying computer code posted by the authors on the web, `simu_2nd.m`, applies the "pruning" procedure to the second-order approximation. I thank Martin Uribe for having pointed out this fact to me.

[5]It is important to notice that there is another sense in which the solutions proposed in the literature are recursively linear. While here we refer to recursive linearity in the state variables of the state-space representation of the solution, the literature has often pointed out the recursive linearity of the problem of determining the coefficient matrices of the state-space representation. On the latter problem our paper has nothing to add.

expansion method advocated here, amounts to finding a polynomial approximation to the unknown policy function. The coefficients of such polynomial are typically related to the value of the partial derivatives of the original function evaluated at the point of approximation. As we argue in this paper, there is no disagreement between the series expansion method and the alternative method à la Schmitt-Grohé and Uribe (2004) as concerns the value of these partial derivatives. What distinguishes these methods is the actual structure of the approximating polynomial. Contrary to Kim et al. (2008) we do not modify what we consider to be the correct approximation, given the approximation method, in order to achieve some desired properties. We show instead how to obtain a polynomial approximation that is consistent with the series expansion method and that has the desired property of asymptotic stability, with no need of ad-hoc corrections. In this sense, the purpose of our paper is identical to that of Sims (2000), Jin and Judd (2002), Schmitt-Grohé and Uribe (2004) or Anderson et al. (2006), among others, although each of these papers focuses on a different aspect of the problem.

We don't discuss in details the origins of the perturbation method exposed in this paper.[6] The approach followed here is thoroughly discussed by Holmes (1995), Bush (1992) and Hinch (1991), among others. Nevertheless, we point out that our approach finds its theoretical foundations in the asymptotic properties of series expansions as well as in the Implicit Function Theorem and in Taylor's theorem in particular (Holmes, 1995 and Chicone, 2006), as does the perturbation method without "pruning" proposed by the literature cited above. These foundations are essential in order to discriminate among alternative approximation techniques. In this sense our paper offers a way to rationalize the "pruning" technique discussed by Kim et al. (2008) for second-order approximations. More importantly, our paper suggests a sys-

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[6]For an interesting survey see Gutzwiller (1998).

tematic way to generate stable solutions at any order of approximation that display desirable asymptotic properties.[7]

In this light, our paper fills an important gap in the literature since many have documented the problems that can arise at higher orders of approximation from ad-hoc corrections of the perturbation approach à la Schmitt-Grohé and Uribe (2004).[8]

The rest of the paper is organized as follows. Section 2 provides an overview of the series-expansion method. Section 3 discusses the difference between the series expansion method and "pruning" and provides a proof that the method of series expansion generates globally stable dynamics as long as the first order dynamics is stable. Section 4 applies the technique to a simple non-linear DSGE model. Section 5 discusses how existing computer codes can be used together with the series expansion method to any order of approximation. Section 6 concludes.

2   The method of series expansions

In this paper we are interested in solving problems of the general form

$$\mathbb{E}_t F\left(z_{t+1}, z_t, \sigma \varepsilon_t\right) = 0 \tag{1}$$

where F is a system of non-linear stochastic difference equations deriving from first order conditions of agents' problems, resource constraints and market clearing conditions, $\mathbb{E}_t$ is the mathematical expectation operator, ε_t is a vector of exogenous stochastic forcing processes with given low of motion, z_t is a vector of endogenous variables and σ is a parameter (the perturbation parameter) such that if $\sigma = 0$ we

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[7]It is important to remember that there are two types of desirable asymptotic properties of series expansions. First, from Taylor's theorem, we know that for analytic functions, or within the radius of convergence, increasing the order of approximation will eventually recover the exact function. Second, for a given order of approximation, the approximation error is increasing in the size of the perturbation. The solution proposed here has both asymptotic properties.

[8]See for example den Haan and de Wind (2010) and Ruge-Murcia (2010).

know how to solve equation (1) as well as the derivatives of $F(\cdot)$.[9] For illustrative purposes, in this section, we describe the approximation method using a very simplified abstract version of the general case, and consider only a second-order expansion. This version would indeed not need approximations as the value of the variable at each point in time can be easily traced starting from given initial conditions. In the next section we will return to the more general case, showing an application to the neoclassical growth model.

In order to approximate the (scalar) model

$$y_t = f(\sigma, y_{t-1}) \tag{2}$$

we can proceed as follows (Holmes, 1995 and Berglund, 2001).[10]

First, we recognize that y_t, that solves equation (2), is a function of time and of the perturbation parameter σ. We are interested in an approximation of this function around $\sigma = 0$. Resorting to Taylor's theorem we choose to approximate y_t using the series expansion[11]

$$y_t = y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)}, \tag{3}$$

where $^{(x)}$ denotes the order x of approximation of the variable, i.e. $y_t^{(0)} = \mathcal{O}(\sigma^0)$ etc. Notice that $y_t^{(x)}$ is assumed not to depend on the perturbation parameter.[12] It

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[9]This representation can be thought of as the companion form of a more extended model with lagged variables. Also, terms like ε_{t+1} are taken care of by an appropriate definition of the terms in z_t.

[10]Obviously, in order to apply this method it is necessary that the solution admit series expansion in the perturbation parameter and that the function f be analytic. In particular the resulting Jacobian of the system of equations must be non-singular. When this condition is not satisfied singular-perturbation methods can be applied. Singular perturbation methods have been applied to portfolio problems by Judd and Guu (2001).

[11]To be consistent with the representation discussed in most of the literature (e.g. Holmes, 1995), at this stage we do not divide each term i of the expansion by the factorial of i ($i!$), as a Taylor expansion would require. The solution steps suggested by this literature correct for this omission. Alternatively, we could introduce these factorials, in which case no correction would be needed in the subsequent steps.

[12] Holmes (1995) and Judd (1998, p. 516) discuss the use of more general representations, i.e. of

is important to notice that this representation is identical to that discussed by Judd (1998, p. 456-457), who makes clear that the terms in $y_t^{(x)}$ are "derivatives of $[y_t(\sigma)]$ with respect to σ when $\sigma = 0$."

Next, we take an expansion of $f(\cdot)$ around $\sigma = 0$ (assume that $y_t^{(0)} = y_0$)

$$f(\sigma, y_t) \approx f(0, y_0) + f_\sigma \sigma + f_y(y_t - y_0) + \frac{1}{2}\left(f_{\sigma\sigma}\sigma^2 + f_{yy}(y_t - y_0)^2\right) + \mathcal{O}(\sigma^3), \quad (4)$$

where for simplicity we have assumed that $f_{\sigma y} = 0$.

Replace equations (3) and (4) in the original problem (2),

$$
\begin{aligned}
y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)} &= \\
f(0, y_0) + f_\sigma \sigma + f_y\left(\sigma y_{t-1}^{(1)} + \sigma^2 y_{t-1}^{(2)}\right) & \\
+ \frac{1}{2}\left(f_{\sigma\sigma}\sigma^2 + f_{yy}\left(\sigma y_{t-1}^{(1)} + \sigma^2 y_{t-1}^{(2)}\right)^2\right) + \mathcal{O}(\sigma^3). &
\end{aligned}
\quad (5)
$$

"By equating like powers" (Holmes, 1995, p. 27) we obtain that the zero order is

$$y_t^{(0)} = f(0, y_0),$$

that the first order is

$$y_t^{(1)} = f_y y_{t-1}^{(1)} + f_\sigma + \mathcal{O}(\sigma^2),$$

and that the second order is

$$y_t^{(2)} = f_y y_{t-1}^{(2)} + \frac{1}{2}\left(f_{\sigma\sigma} + f_{yy}\left(y_{t-1}^{(1)}\right)^2\right) + \mathcal{O}(\sigma^3).$$

Therefore

$$y_t - y_0 \approx \sigma\left(f_y y_{t-1}^{(1)} + f_\sigma\right) + \sigma^2\left(f_y y_{t-1}^{(2)} + \frac{1}{2}\left(f_{\sigma\sigma} + f_{yy}\left(y_{t-1}^{(1)}\right)^2\right)\right). \quad (6)$$

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the form $y_t = y_t^{(0)} + \sigma^\alpha y_t^{(1)} + \sigma^{2\alpha} y_t^{(2)} + \dots$. Quoting Holmes (1995, p. 19): "[This assumption] is nothing more than an educated guess. The motivation for making this assumption comes from the observation that in expanding functions one usually ends up using Taylor's theorem and [our guess] is simply a reflection of that type of expansions. The exponent α is to allow for a little flexibility." In this paper we focus only on the case of $\alpha = 1$, although this might not be the best choice in general.

Notice that there are terms of order higher than 2 in equation (5). These belong to the higher order terms in the residual, i.e. they belong in $\mathcal{O}\left(\sigma^{3}\right)$.[13]

It is important to notice that our approximation is the result of the application of the chain rule. In fact, by simply differentiating (2) with respect to σ, and bearing in mind the dependence of y_t on the perturbation parameter, we would have obtained the same result.[14]

The representation of the solution proposed by most of the recent economic literature on this topic (except for Lombardo and Sutherland, 2007) would amount to writing the second-order expansion as (assuming for simplicity that $y_0 = 0$)

$$y_t = f_y y_{t-1} + \sigma f_\sigma + \frac{1}{2}\left(\sigma^2 f_{\sigma\sigma} + f_{yy}\left(y_{t-1}\right)^2\right) + \mathcal{O}\left(\sigma^3\right), \qquad (7)$$

that is, neglecting the order of the approximation of the variables.

Kim et al. (2008), among others, recognize that this solution could generate spurious dynamics, and propose a remedy consisting of using the firs-order solution to compute the cross-product terms. Up to second order this procedure would generate virtually the same solution as the one that we propose in this paper. In particular these authors suggest to rewrite equation (7) as[15]

$$y_{p,t}^{(2)} = f_y y_{p,t-1}^{(2)} + \sigma f_\sigma + \frac{1}{2}\left(\sigma^2 f_{\sigma\sigma} + f_{yy}\left(y_{p,t-1}^{(1)}\right)^2\right) + \mathcal{O}\left(\sigma^3\right). \qquad (8)$$

where subscript "p" denotes the pruned solution. Solving backward the first order

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[13]By first replacing equation (3) and then taking expansions directly in terms of σ would make this statement unnecessary. The procedure followed in the text has the advantage of highlighting that in order to include further terms, we have to bring in terms from the residual, i.e. increase the order of approximation.

[14]Alternatively we could have first substituted equation (3) into equation (2) and then expanded with respect to σ for all the orders of approximation up to the desired one. In this case, equation (3) should include the factorial terms that we have omitted.

[15]Notice that the normalization with respect to σ is applied differently in the two representations. This has no consequence for the result as it is apparent from the following algebra.

equation we obtain[16]

$$y_{p,t}^{(1)} = f_y^T y_{p,t-T}^{(1)} + \sigma \sum_{i=0}^{T-1} f_y^i f_\sigma, \tag{9}$$

Doing the same for equation (8) yields

$$y_{p,t}^{(2)} = f_y^T y_{p,t-T}^{(2)} + \sigma \sum_{i=0}^{T-1} f_y^i \left(f_\sigma \right) + \sigma^2 \frac{1}{2} \sum_{i=0}^{T-1} f_y^i \left(f_{\sigma\sigma} + f_{yy} \left(\frac{y_{p,t-1}^{(1)}}{\sigma^2} \right)^2 \right). \tag{10}$$

Replacing the backward solution in equation (6) yields

$$y_t = \sigma f_y^T y_{t-T}^{(1)} + \sigma^2 f_y^T y_{t-T}^{(2)} + \sigma \sum_{i=0}^{T-1} f_y^i \left(f_\sigma \right) + \sigma^2 \frac{1}{2} \sum_{i=0}^{T-1} f_y^i \left(f_{\sigma\sigma} + f_{yy} \left(y_{t-1}^{(1)} \right)^2 \right). \tag{11}$$

Clearly, as long as $|f_y| < 1$, $\lim_{T\to\infty} y_{p,t}^{(2)} = y_t$. In general the discrepancy between the two solutions depends on the initial conditions. In typical exercises (e.g. impulse responses or unconditional moments) second order pruning and series expansions generate the same dynamic paths.[17]

This result also suggest that equation (7) does not generate the same solution as the other two representations. This is true for any (non-zero) size of the perturbation (so even with supports that ensure that the function remains within its radius of convergence, e.g. Aruoba et al., 2006). Obviously since these are all asymptotic approximations, the discrepancy between them vanishes as we reduce the size of the perturbation.

In light of this result our contribution offers theoretical motivations for the "pruned" representation of the solution up to second order of approximation. Nevertheless, further below we show that, in general, for higher order of approximation the equivalence does not hold.

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[16]See Ljungqvist and Sargent (2000, p. 11).

[17]This result is true also for system of equations. It simply rely on the fact that the coefficient (matrix) on the linear term in each equation (f_y) is the same at both orders of approximation.

The asymptotic properties of the series expansion method that we propose is grounded in Taylor's theorem. Appendix A further discusses the convergence properties of our method (SE) applied to a simple ad-hoc model (of the kind used by den Haan and de Wind, 2010) and compares the results with those obtained by using two alternative perturbation approaches, i.e. the one à la Schmitt-Grohé and Uribe (2004) without pruning (NP) and the one suggested by Aruoba et al. (2006). The latter approach consists of dropping the "explosive" draws from the simulation (DD). We show that both SE and DD approaches seem to converge relatively quickly to the true solution, while NP seems to reduce the number of "explosive" paths as the order of approximation increases. We argue that the SE approach provides a more systematic way to generate stable paths than the ad-hoc and relatively subjective DD approach.

3 Series expansion and "pruning" at higher orders of approximation

For second-order expansions, Kim et al. (2008) have proposed a "pruning" technique that ensures stability of the approximated dynamical system: i.e. they replace cross products of variables with products of variables solved at first order of approximation. Up to second order, "pruning" and the series-expansion method are equivalent. At higher order of approximation, "pruning" the solution by replacing cross-products of variables with products of lower-order approximations would still generate a recursively-linear system of difference equations. Nevertheless, stability of the system can no longer be ensured, even if the first order dynamics is stable. On the contrary, the series-expansion method preserves stability at any order of approximation. Furthermore, in general, the "pruned" solution will differ from that obtained

by applying the series expansion method as we show in this section.

Given a vector of non-predetermined (jump) variables y_t and a vector of predetermined (state) variables x_t, a solution to the model in equation (1) is generally postulated as (e.g. Schmitt-Grohé and Uribe, 2004):

$$y_t = g\left(x_t; \sigma\right), \tag{12}$$

and

$$x_{t+1} = h\left(x_t; \sigma\right) + \sigma \eta \varepsilon_{t+1}, \tag{13}$$

where $z_t' = [x_t'\, y_t']$.

For the sake of simplicity assume for a moment that y_t and x_t are scalars. Then, the n-th order approximation of $h\left(\cdot\right)$ according to the NP method yields

$$
\begin{aligned}
h\left(\cdot\right) \approx\ & h_0 + h_x x_t + h_\sigma \sigma + \sigma \eta \varepsilon_{t+1} \\
& \frac{1}{2} h_{xx} x_t^2 + \frac{1}{2} h_{\sigma\sigma} \sigma^2 + \frac{2}{6} h_{x\sigma} x_t \sigma + \\
& \frac{1}{6} h_{xxx} x_t^3 + \frac{3}{6} h_{\sigma\sigma x} x_t \sigma^2 + \frac{3}{6} h_{\sigma xx} \sigma x_t^2 + \frac{1}{6} h_{\sigma\sigma\sigma} \sigma^3 + \cdots + \mathcal{O}\left(\sigma^{n+1}\right)
\end{aligned}
$$

After collecting terms we obtain

$$
\begin{aligned}
h\left(\cdot\right) \approx\ & h_0 + \left[h_\sigma \sigma + \frac{1}{2} h_{\sigma\sigma} \sigma^2 + \frac{1}{6} h_{\sigma\sigma\sigma} \sigma^3\right] + \left[h_x + \frac{2}{6} h_{x\sigma} \sigma + \frac{3}{6} h_{\sigma\sigma x} \sigma^2\right] x_t + \sigma \eta \varepsilon_{t+1} + \\
& \left[\frac{3}{6} h_{\sigma xx} \sigma\right] x_t^2 + \frac{1}{6} h_{xxx} x_t^3 + \cdots + \mathcal{O}\left(\sigma^{n+1}\right)
\end{aligned}
$$

or, more compactly (and neglecting the approximation residual)

$$h\left(\cdot\right) - h_0 \approx D + A x_t + B x_t^2 + C x_t^3 + \sigma \eta \varepsilon_{t+1} + \dots. \tag{14}$$

This is the typical representation of the approximated solution discussed in the literature on perturbation methods (e.g. Anderson et al. (2006) for orders higher

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than two). "Pruning", as described by Kim et al. (2008) would amount to replacing the powers of x_t with cross-products of solutions of x_t to lower order, i.e.[18]

$$x_{t+1}^{(3)} = h\left(\cdot\right) - h_0 \approx D + Ax_t^{(3)} + Bx_t^{(1)}x_t^{(2)} + Cx_t^{(1)^3} + \sigma\eta\varepsilon_{t+1} + \ldots. \qquad (15)$$

The solution obtained applying the series-expansion method is:

$$h\left(\cdot\right) - h_0 \approx \sigma x_{t+1}^{(1)} + \sigma^2 x_{t+1}^{(2)} + \sigma^3 x_{t+1}^{(3)} + \ldots, \qquad (16)$$

where

$$x_{t+1}^{(1)} = h_x x_t^{(1)} + h_\sigma + \sigma\eta\varepsilon_{t+1}, \qquad (17)$$

$$x_{t+1}^{(2)} = h_x x_t^{(2)} + h_{x\sigma} x_t^{(1)} + \frac{1}{2} h_{xx}\left(x_t^{(1)}\right)^2 + \frac{1}{2} h_{\sigma\sigma}, \qquad (18)$$

$$\begin{aligned} x_{t+1}^{(3)} = {} & h_x x_t^{(3)} + \frac{3}{6} h_{\sigma\sigma x} x_t^{(1)} + h_{x\sigma} x_t^{(2)} + \\ & \frac{3}{6} h_{\sigma xx} x_t^{(1)} x_t^{(1)} + \frac{1}{6} h_{xxx}\left(x_t^{(1)}\right)^3 + \frac{1}{6} h_{\sigma\sigma\sigma}, \end{aligned} \qquad (19)$$

etc..

The key difference between the two solutions concerns their asymptotic stability properties. The series expansion method delivers asymptotically stable solutions as long as the first order dynamics is asymptotically stable. This result does not necessarily hold true for the "pruning" approach, as we now show.

First, represent the approximated solution, given by equations (15) and (17) to (19) as

$$x_t^{(j)} = \Phi_j x_{t-1}^{(j)} + B_{j,t}, \qquad (20)$$

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[18]The first and second order approximations are identical to those given in equations (17) and (18) below.

where $x^{(j)}$ is a vector of endogenous variables, Φ_j is a matrix of constant coefficients, $B_{j,t}$ is a vector-valued process independent of $x^{(j)}$ and j denotes the order of approximation. Then, let us state the following known fact in form of a lemma:

Lemma 1 *Provided that the eigenvalues of Φ_j lie within the unit circle, a particular solution $(x_t^{p,j})$ to this system can be found by solving it "backward", i.e. $x_t^{p,j} = \sum_{i=0}^{n-1} \Phi_j^i B_{j,t-i}$. The general solution to system (20) will be the sum of the solution to the homogeneous part and the particular solution, i.e. $x_t^{(j)} = \Phi_j^n x_{t-n}^{(j)} + x_t^{p,j}$.*

Proof The proof follows from the superposition principle and backward iteration (see Azariadis, 1993, p. 127). ▮

Using these results and following Azariadis (1993, p. 22) we can define asymptotic stability as follows:

Definition System (20) is asymptotically stable if $\lim_{t \to \infty} x_t^{(j)} = \lim_{t \to \infty} x_t^{p,j} < \infty$, where for a stochastic model $\lim_{t \to \infty} x_t^{p,j}$ is a probability distribution.[19]

Notice, furthermore, that the (kronecker) tensor product of two vectors v and w, with n_v and n_w elements respectively, gives the $n_v n_w$ cross products of all the elements in v with all the elements in w, i.e. $v \otimes w = [v_1 w_1, v_1 w_2, \ldots, v_{n_v} w_{n_w}]'$. This can be repeated any number of times for any number of vectors. Denote by n the number of vectors (possibly identical) involved in the tensor product. We call the tensor product involving multiple vectors as the n-th order tensor product. We can then state the following lemma:

Lemma 2 *The n-th order tensor product of asymptotically stable vectors of variables, divided by the n-th factorial (n!), is itself asymptotically stable $\forall n$.*

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[19]Notice that even if the underlying shocks are Gaussian, the limiting distribution of the endogenous variables, in general, will not be Gaussian at orders of approximation larger than one.

Proof The proof follows directly from the the fact that the limit of the product of two vector-valued functions is equal to the product of the limit of these functions (Apostol, 1967, vol. II, p. 248). The division by $n!$ ensures that the limit converges even for $n \to \infty$, since for any finite number a, $\lim_{n \to \infty} \dfrac{a^n}{n!} = 0$. ∎

Finally we are ready to state the following proposition:

Proposition 1 *Assume that the n-th order approximation of system* (1) *both by the method of series expansion and by the method of "pruning", generates n systems of difference equations – one per order of approximation up to n – that can be written in the form of equation* (20). *Then, the dynamics of the endogenous variables approximated to n-th order is globally asymptotically stable if the eigenvalues of the matrix Φ_j lie within the unit circle, $\forall j$.*

Proof On the basis of Lemma 1 and of Lemma 2, provided that the first order system is asymptotically stable, we know that the n-th order tensor product of the first-order solution vector divided by the n-th order factorial $(n!)$ has a well defined limit as $t \to \infty$, for any positive integer n. Since the forcing process of the second order solution is a linear transformation of second-order tensor products of first-order solutions, we have proved that the second order solution is asymptotically stable as long as the eigenvalues of the Φ_2 matrix lie within the unit circle. If this is the case, the n-th order tensor product of the second-order solution vector divided by the n-th order factorial $(n!)$ has a well defined limit as $t \to \infty$, for any positive integer n. Since the forcing process of the third order solution is a linear transformation of third-order tensor products of first- and second-order solutions, we have proved that the second order solution is asymptotically stable as long as the eigenvalues of the Φ_3 matrix lie within the unit circle. And so on . . . . ∎

By inspecting equations (17) to (19) we see that $\Phi_j = h_k \ \forall j$. So the whole approximated system is asymptotically stable as long as the first order system is

asymptotically stable, i.e. as long as $|h_k| < 1$. On the contrary, the stability property of equation (15) is governed by the eigenvalue A, which, in general, is different from h_k. In particular, there is no guarantee that $|A| < 1$, so that the system could be unstable even locally, despite the original system being locally saddle-path stable.[20] Furthermore, the "pruned" solution implies that the stability properties of the dynamical system are affected by the variance of the exogenous shocks: this is not the case for the SE approach.

Appendix C shows that the discrepancy between simple "pruning" and the series-expansion method is larger the higher the order of approximation: while at third order it is the matrix multiplying the linear terms that generates the discrepancy, at fourth order an additional source of discrepancy is generated by the matrix multiplying square terms, and so on.

4   An example: The neoclassical growth model

In this section we apply the method of series expansion to the Neo-Classical Growth Model. This model has also been used by Judd (1998), Lombardo and Sutherland (2007) and Schmitt-Grohé and Uribe (2004) to show how to apply their approximation techniques. While here we present only a second-order approximation, in Appendix C we extend the approximation to higher orders. Furthermore, there we show how to use existing software (i.e. Aruoba et al., 2006) to solve this model to any order of approximation. This section also illustrates that the solution can be found through direct factorization as opposed to postulating unknown solutions as in equations (12) and (13). A point similar to that made by Klein (2000) in relation to the solution of linearized rational expectation models and by Lombardo and

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[20] Ruge-Murcia (2010) has documented that simple "pruning" could be insufficient to ensure stability to third order.

Sutherland (2007) in relation to second order approximations.

Consider the following rational expectation model, consisting of an Euler consumption (c) equation, a capital (k) accumulation equation and an i.i.d. process for the (log) of the productivity shock (ε) with zero mean and variance normalized to one.[21] That is

$$c_t^{-\gamma} = \alpha\beta\mathbb{E}_t\left[e^{\sigma\varepsilon_{t+1}}k_{t+1}^{\alpha-1}c_{t+1}^{-\gamma}\right] \tag{21}$$

$$k_{t+1} = e^{\sigma\varepsilon_t}k_t^\alpha - c_t, \tag{22}$$

where $\sigma > 0$, $\alpha \in (0,1)$, $\gamma > 0$, $\beta \in (0,1)$ and $\mathbb{E}_t(\varepsilon_{t+1}) = 0$.

Notice that if $\sigma = 0$, the model is deterministic and has a closed form solution $k_0 = (\alpha\beta)^{\frac{1}{1-\alpha}}$ and $c_0 = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}$.

The first step of the series expansion method consists of assuming that the solution can be expressed in terms of a series expansion

$$c_t \approx c_0 + \sigma c_t^{(1)} + \sigma^2 c_t^{(2)} \tag{23}$$

and

$$k_t \approx k_0 + \sigma k_t^{(1)} + \sigma^2 k_t^{(2)}. \tag{24}$$

Take a second order expansion of equations (21) and (22) around $\sigma = 0$, dropping the expectation operator for notational convenience (implicitly associated to $t+1$ variables), i.e.

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[21]For the approximation to retain its asymptotic properties, the distribution of the shock must be such that the system remains within its radius of convergence. Here we leave this issue in the background referring the reader to Jin and Judd (2002) and Kim et al. (2008) for a discussion of this issue.

$$\mathcal{O}\left(\varepsilon^3\right) = \gamma c_0^{-\gamma-1}\hat{c}_t + \alpha\beta k_0^{\alpha-1}c_0^{-\gamma}\left[\sigma\varepsilon_{t+1} + k_0^{-1}\left(\alpha-1\right)\hat{k}_{t+1} - \gamma c_0^{-1}\hat{c}_{t+1}\right] \; + $$

$$-\frac{1}{2}\gamma\left(\gamma+1\right)c_0^{-\gamma-2}\hat{c}_t^2 + \frac{1}{2}\alpha\beta k_0^{\alpha-1}c_0^{-\gamma}\left[\sigma^2\varepsilon_{t+1}^2 + \gamma\left(\gamma+1\right)c_0^{-2}\hat{c}_{t+1}^2 + \left(\alpha-1\right)\left(\alpha-2\right)k_0^{-2}\hat{k}_{t+1}^2\right] \; + $$

$$+\alpha\beta k_0^{\alpha-1}c_0^{-\gamma}\left[k_0^{-1}\left(\alpha-1\right)\sigma\varepsilon_{t+1}\hat{k}_{t+1} - \gamma c_0^{-1}\sigma\varepsilon_{t+1}\hat{c}_{t+1} - \gamma\left(\alpha-1\right)k_0^{-1}c_0^{-1}\hat{c}_{t+1}\hat{k}_{t+1}\right]$$

and

$$\mathcal{O}\left(\varepsilon^3\right) = -\hat{k}_{t+1} + k_0^{\alpha}\sigma\varepsilon_t + \alpha k_0^{\alpha-1}\hat{k}_t - \hat{c}_t \; + $$

$$\sigma\alpha k_0^{\alpha-1}\varepsilon_t\hat{k}_t + \frac{1}{2}k_0^{\alpha}\sigma^2\varepsilon_t^2 + \frac{1}{2}\alpha\left(\alpha-1\right)k_0^{\alpha-2}\hat{k}_t^2.$$

Define $z_t = \left[\hat{c}_t\,\hat{k}_t\right]'$. Then can rewrite in matrix notation

$$\mathcal{O}\left(\varepsilon^3\right) = A_2 z_{t+1} + A_1 z_t + \sigma C_0 \varepsilon_t + $$

$$+ B_2 w_{t+1} + B_1 w_t + \sigma D_2 z_{t+1}\varepsilon_{t+1} + \sigma D_1 z_t \varepsilon_t + \sigma^2 C_1 \varepsilon_t^2 \;\; + \sigma^2 C_2 \varepsilon_{t+1}^2, \qquad (25)$$

where[22]

$$w_t = vech\left(z_t z_t'\right) = \left[\begin{array}{ccc} \hat{c}_t^2 & \left(\hat{c}_t\hat{k}_t\right) & \hat{k}_t^2 \end{array}\right]'$$

and where the matrices of coefficients are described in Appendix B.

We can also rewrite our assumption concerning the solution in matrix notation[23]

$$z_t = \sigma z_t^{(1)} + \sigma^2 z_t^{(2)}. \qquad (26)$$

Notice that

$$z_t z_t' = \sigma^2 z_t^{(1)} z_t^{(1)\prime} + \sigma^4 z_t^{(2)} z_t^{(2)\prime} + \sigma^3 z_t^{(1)} z_t^{(2)\prime} + \sigma^3 z_t^{(2)} z_t^{(1)\prime}. \qquad (27)$$

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[22]We are using the *vech* operator to eliminate repeated terms in kronecker products of identical vectors. For approximations of order larger than 2, appropriate elimination matrices can be applied to eliminate repeated terms in n-th tensor powers.

[23]Notice that we have reformulated the assumption concerning the solution: now we are expanding in deviations from the steady-state.

Since we are interested in solution up to order 2, we must drop the higher terms in σ.[24] Therefore

$$\sigma^2 \hat{w}_t = \sigma^2 vech \left(z_t^{(1)} z_t^{(1)\prime} \right).$$

Replacing equations (26) and (27) into (25) we obtain

$$A_2 \left(\sigma z_{t+1}^{(1)} + \sigma^2 z_{t+1}^{(2)} \right) + A_1 \left(\sigma z_t^{(1)} + \sigma^2 z_t^{(2)} \right) + \sigma C_0 \varepsilon_t +$$
$$+ B_2 \sigma^2 \hat{w}_{t+1} + B_1 \sigma^2 \hat{w}_t +$$
$$\sigma D_2 \left(\sigma z_t^{(1)} + \sigma^2 z_t^{(2)} \right) \varepsilon_t + \sigma D_1 \left(\sigma z_{t+1}^{(1)} + \sigma^2 z_{t+1}^{(2)} \right) \varepsilon_{t+1}$$
$$+ \sigma^2 C_1 \varepsilon_t^2 + \sigma^2 C_2 \varepsilon_{t+1}^2 \quad = 0.$$

By equating like powers we have

$$\sigma : \ A_2 z_{t+1}^{(1)} + A_1 z_t^{(1)} + C_0 \varepsilon_t = 0 \tag{28}$$

and

$$\sigma^2 : \ A_2 z_{t+1}^{(2)} + A_1 z_t^{(2)} + B_2 \hat{w}_{t+1} + B_1 \hat{w}_t \ +$$
$$+ D_2 z_{t+1}^{(1)} \varepsilon_{t+1} + D_1 z_t^{(1)} \varepsilon_t + C_2 \varepsilon_{t+1}^2 + C_1 \varepsilon_t^2 \ = \ 0. \tag{29}$$

Both these equations can be solved (recursively) using any solution technique for linear rational expectation models.[25] Alternatively we can use the approach suggested by Jin and Judd (2002) and followed by most of the related literature. In the next section and in Appendix C we follow the latter approach.

The final solution is obtained by solving for $z_t^{(1)}$ from (28) and for $z_t^{(2)}$ from (29) and replacing the solutions into (26).

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[24]As explained in the previous section, these higher order terms will match terms in the approximation residual.

[25]For example Christiano (2002), King and Watson (2002), Klein (2000), Uhlig (1999) etc..

5   Computer codes

As argued above, it is possible to apply the series expansion method by using a number of existing alternative computer codes, developed to solve DSGE models by perturbation methods. The previous section shows indeed that we need a software able to solve for the derivatives of the $g(\cdot)$ and $h(\cdot)$ functions. In particular, in Appendix C we use a Mathematica computer code, written by Aruoba et al. (2006), that generates the derivatives of these functions for the neoclassical growth model. We then use Matlab to construct and compare the NP and the SE solutions.[26]

6   Conclusion

We have shown that the series expansion method to solve non-linear equations discussed in the perturbation literature can be applied to DSGE models to obtain higher-order solutions that are recursively linear, in a very precise way. We have proved that this type recursive linearity has the advantage of avoiding spurious diverging dynamics. We have also proved that the recursive linearity generated by naive "pruning", does not ensure local asymptotic stability, even if the true model is locally asymptotically stable. In general, "pruning" and the series expansion method do not deliver the same approximation to the policy function.

Our paper shows that perturbation methods need not be "problematic" as some recent literature has claimed. Obviously, there are limitations to the accuracy of the approximation that low-order perturbations can achieve for particular models. And, in some cases, the radius of convergence of the series expansion could be so small

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[26]PerturbationAIM written by Anderson et al. (2006) could also be used to generate the derivatives of the $g(\cdot)$ and $h(\cdot)$ functions. Dynare (version 4.2) and Dynare++, at present, provide a solution in a form similar to equation (14), so that it is not directly possible to use it with the method of series expansion.

to make such an approach futile. Nevertheless, perturbation methods remain a very efficient way of analyzing a wide range of economic models (e.g. Kollmann et al., 2011a,b), and in some simple cases even analytically (e.g. Devereux and Sutherland, 2008). Furthermore, very often perturbation methods remain the only viable solution when a large number of state variables is involved.

By way of example, we have shown that, for an analytic function, the series-expansion method converges to the true dynamic path. We have shown that the alternative perturbation method that delivers non-linear dynamic equation shares this property but has the disadvantage of potentially producing exploding paths. In principle, exploding paths can be eliminated in simulation (as discussed in the literature) delivering bounded and converging approximations. Whether this approach or the one that we advocate is to be preferred will likely depend on the particular problem.

We have shown how to apply the series expansion method to the neoclassical growth model, solving it to fifth order of approximation using trivial extensions of existing computer codes.

Appendix

A   Convergence of the series expansion

In order to illustrate the convergence properties of the of the series expansion method, in this Appendix we use a very simple non-linear dynamic model, similar to the one used by den Haan and de Wind (2010). We don't compare our results with "pruning" since in our view the way to apply "pruning" to higher orders of approximation is to apply the series expansion method proposed here.

Consider the simple model

$$y_t = \gamma y_{t-1} + \mathbb{E}_t exp\left(\alpha y_{t-1}\left(\beta\sigma\varepsilon_{t+1} + 1\right)\right) + \sigma\varepsilon_t; \tag{30}$$

where $\gamma < 1, \alpha < 0$, $\beta \geq 0$, and where ε_t is white noise. This equation does not have a linear MA representation. Nevertheless it has a recursive structure so that we can easily simulate it and obtain the exact path of the variable y_t for given draws of the innovation. Moreover, we can apply perturbation methods and compare the simulation results with the exact solution.

Since ε_t is Gaussian, we know that

$$\mathbb{E}_t exp\left(\alpha y_{t-1}\left(\beta\sigma\varepsilon_{t+1} + 1\right)\right) = exp\left(\alpha y_{t-1} + \frac{1}{2}\left(\beta^2\sigma^2 y_{t-1}^2\right)\right).$$

Therefore, for $\beta \neq 0$ the stochastic-steady state differs from the non-stochastic one. We allow for the possibility of a stochastic steady state as this is typically the case in DSGE models and it is a source of distortion of approximations centered around the non-stochastic steady state (see den Haan and de Wind, 2010).

We consider three alternative perturbation procedures for different orders of approximation and up to order ten: i) the series expansion proposed here (SE); ii) the one used for example by Fernandez-Villaverde and Rubio-Ramirez (2006) among others (NP) and iii) the latter method adjusted by dropping draws that generate divergence over the next ten periods (DD).[27] We take 500 draws and retain only the last 251.

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[27]In essence in each period of the simulation we check if the series without further shocks, after ten periods displays values that are larger than the current one. If this is the case, we replace the current innovation with zero. To make the comparison sensible, we use the resulting series of actual draws also to compute the true value to which we compare this particular case. In practice we might have/want to take the distribution of the shocks as given. In this case the performance of DD would be worse since the true endogenous process would be subject to a different set of shocks. This is not necessarily the only way to generate stability by selecting draws. In essence we are selecting where we want to approximate the true function: a fully legitimate approach.

Table 1 considers the case in which the stochastic and non-stochastic steady state are identical ($\beta = 0$), as well as the case in which the two steady states differ ($\beta = 0.1$).[28] We set $\alpha = -1$ and $\gamma = 0.8$, while we vary the magnitude of the perturbation parameter ($\sigma = 1.2, 1, 0.9, 0.6$). The seed is reset to the same value at each order of approximation.

Consider the case of $\beta = 0$ in Table 1. In the first set of approximations, i.e. with a larger perturbation, the SE method shows clear convergence and the percentage mean error is relatively small at all orders of approximation. Convergence – not necessarily monotonic – is what theory predicts for analytic functions. The NP displays convergence, although for some (low) orders of magnitude it generates explosive dynamics. The DD method shows to be a decent remedy to this problem, as suggested by Aruoba et al. (2006). Although it does not explode (by design) it generates very poor approximations at low orders of approximation. Progressively reducing the size of the perturbation shows that all alternative approximations improve (as predicted by theory for asymptotic expansions). The NP method tends to explode at lower orders of approximation, although in an unpredictable way. This is to be expected since the approximation error is decreasing in the order of approximation. Since the true function does not explode, also the approximation, eventually, will not explode. Both the NP and DD method seem to be worse than the SE method for relatively low orders of approximation while they generate better approximations for high orders.

Allowing for a stochastic steady state makes the approximation around the non-stochastic steady state more distorted, as can be seen from the Table. den Haan and de Wind (2010) suggest to take as approximation point the stochastic steady

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[28]For sufficiently large values of beta and sufficiently large initial conditions, the true process diverges. We select beta and the standard deviation of the innovations so to be well within the stable range.

state. Nevertheless, Caldara et al. (2009) find no improvement in correcting the approximation in this way. The results reported in Table 1 for $\beta = 0.1$ are computed approximating around the (known) stochastic steady state. In general this point is unknown so that iterative procedures need to be used.[29] Approximating around the non-stochastic steady state, in our case produces much less accurate approximations.

In this simple backward-looking system it would be straightforward to extend the pruning procedure obtaining exactly our solution. It would amount to impose that the cross-product terms be constructed using first-order approximations, as explained earlier. den Haan and de Wind (2010, p. 21), in describing this procedure, make the coefficient on the linear term of the approximation depend on the order of approximation. According to the series expansion method, this coefficient is always the same at all orders of approximation. This discrepancy shows that not all interpretations of "pruning" at higher order of approximation coincide with the method proposed here.

B   Matrices of coefficients of second order-approximation

These are the matrices of coefficients of the second order approximation of the neo-classical growth model in levels.

$$A_2 = \begin{bmatrix} -\gamma\alpha\beta k_0^{\alpha-1}c_0^{-\gamma-1} & \alpha\beta\left(\alpha-1\right)k_0^{\alpha-2}c_0^{-\gamma} \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \gamma c_0^{-\gamma-1} & 0 \\ -1 & \alpha k_0^{\alpha-1} \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 \\ k_0^{\alpha} \end{bmatrix}, \quad C_1 = \frac{1}{2}\begin{bmatrix} 0 \\ k_0^{\alpha} \end{bmatrix}, \quad C_2 = \frac{1}{2}\begin{bmatrix} \alpha\beta k_0^{\alpha-1}c_0^{-\gamma} \\ 0 \end{bmatrix},$$

[29]See Evers (2010) and Juillard and Kamenik (2005).

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$$B_2 = \frac{1}{2} \begin{bmatrix} \gamma(\gamma+1)\alpha\beta k_0^{\alpha-1}c_0^{-\gamma-2} & -2\alpha\beta k_0^{\alpha-2}c_0^{-\gamma-1}\gamma(\alpha-1) & (\alpha-1)(\alpha-2)\alpha\beta k_0^{\alpha-3}c_0^{-\gamma} \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \frac{1}{2} \begin{bmatrix} -\gamma(\gamma+1)c_0^{-\gamma-2} & 0 & 0 \\ 0 & 0 & \alpha(\alpha-1)k_0^{\alpha-2} \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -\gamma\alpha\beta k_0^{\alpha-1}c_0^{-\gamma-1} & \alpha\beta k_0^{\alpha-2}c_0^{-\gamma}(\alpha-1) \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha k_0^{\alpha-1} \end{bmatrix}.$$

C Fifth order expansion of the neoclassical growth model using Aruoba et al. (2006)

Aruoba et al. (2006) (AFVRR) have written a Mathematica code to find the n-th partial derivatives of the $g(\cdot)$ and $h(\cdot)$ functions that can be used for any of the perturbation methods discussed in this paper. We use their file `perturbation_logs.nb`, and modify it to solve the neoclassical growth model described in our paper.[30]

The file `perturbation_logs.nb` defines scalar-valued $g(\cdot)$ and $h(\cdot)$ functions for each of the endogenous jump and state variables, whereas the exogenous shock is assumed to follow an AR(1) process in logs (with normally distributed innovation). In particular define

$$c_t = g(k_t, z_t, \sigma) \tag{31}$$

$$k_{t+1} = g(k_t, z_t, \sigma) \tag{32}$$

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[30]Their original file is set-up to solve a neoclassical growth model with endogenous labor. Therefore, our amendment is minimal. It is nevertheless straightforward to modify their code in order to solve any DSGE model.

and

$$z_{t+1} = \rho z_t + \sigma \eta \varepsilon_{t+1} \tag{33}$$

where $z_t = \log(A_t)$.

We follow Aruoba et al. (2006), Schmitt-Grohé and Uribe (2004) and Lombardo and Sutherland (2007), among others, and solve numerically our model in logs of consumption and capital. Furthermore, for the sake of comparison, we use the same parameterization used by Schmitt-Grohé and Uribe (2004) and Lombardo and Sutherland (2007): i.e. $\rho = 1$, $\delta = 1$, $\beta = 0.95$, $\alpha = 0.3$ and $\gamma = 2$.

Table 2 shows how the different partial derivatives are associated to the state variables evaluated at different orders of approximation according to the series expansion method. This Table complements the result shown in the text concerning the discrepancy between the series-expansion method and "pruning". One can see that according to the NP approach the term involving k_t^2 would be Bk_t^2, where $B \equiv [g_{kk} + g_{kk\sigma} + g_{kk\sigma\sigma}]$. According to the "pruning" approach, in order to generate a recursively linear system it would suffice to replace k_t^2 with the product of terms of lower order. The series expansion method that we advocate, on the contrary, suggests how the component of the coefficient B should precisely be assigned to the different terms.

After generating the partial derivatives of the $g(\cdot)$ and $h(\cdot)$ functions with per-turbation_logs.nb we save them using save_output_perturbation.nb. Finally we run simulate_perturbation.m and simulate_perturbation_AFVRR.m in Mat-lab. These codes retrieve the saved partial derivatives, constructs the appropriate vectors of (cross-products of) variables and, for given initial conditions and exoge-nous process, generates time series at the given order of approximation, using the

series-expansion method and the NP method respectively.[31]

Table 3 shows the response of capital and of consumption (in deviation from the stochastic steady state) to an unexpected increase in technology by 100%.[32] This example confirms the result discussed in Appendix A: as long as the NP solution is well behaved (i.e. does not diverge) both perturbation approaches deliver very similar results. Notice that in the first period both approaches generate the same response. This is the case only if we start from the non-stochastic steady state, as we do in this example.

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[31]The Mathematica code `perturbation_logs.nb` is available at http://economics.sas.upenn.edu/~jesusfv/companion.htm. The other codes mentioned here can be downloaded from https://sites.google.com/site/giovannilombardohomepage

[32]This is not exactly an impulse response function as we do not subtract the expected value of the variable.

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Table 1: Absolute percent mean error.

	$\beta = 0$				$\beta = 0.1$		
Order	SE	NP	DD		SE	NP	DD
		$\sigma = 1.2$				$\sigma = 1.2$	
2	11.0066	exploding	24.419		11.0725	exploding	28.1806
3	7.89805	16.7974	16.7974		7.9724	16.7273	16.7273
4	4.5704	exploding	19.1931		4.66354	exploding	18.1139
6	3.45446	exploding	4.64201		3.41486	exploding	4.60333
8	2.93758	3.91734	3.91734		2.90604	3.9317	3.9317
10	2.8259	0.58361	0.58361		2.78598	0.72189	0.72189
		$\sigma = 1$				$\sigma = 1$	
2	6.01543	exploding	26.0249		6.01681	exploding	25.4932
3	3.78238	8.36944	8.36944		3.7959	8.30864	8.30864
4	1.84586	exploding	6.93115		1.86158	exploding	6.87524
6	0.92493	1.96516	1.96516		0.92178	1.99822	1.99822
8	0.55499	0.26341	0.26341		0.56662	0.33433	0.33433
10	0.39594	0.02984	0.02984		0.4183	0.10741	0.10741
		$\sigma = 0.9$				$\sigma = 0.9$	
2	4.24978	exploding	14.5092		4.24527	exploding	14.3825
3	2.45879	5.51597	5.51597		2.46273	5.46996	5.46996
4	1.08499	7.0103	3.21948		1.0907	6.96278	3.20739
6	0.43473	0.64626	0.64626		0.44383	0.68091	0.68091
8	0.21539	0.0733	0.0733		0.23516	0.12112	0.12112
10	0.12601	0.00646	0.00646		0.15287	0.05923	0.05923
		$\sigma = 0.6$				$\sigma = 0.6$	
2	1.14357	3.74767	3.74767		1.1411	3.72673	3.72673
3	0.4691	1.05055	1.05055		0.46979	1.04046	1.04046
4	0.14359	0.31988	0.31988		0.14377	0.31195	0.31195
6	0.02413	0.01979	0.01979		0.03402	0.0296	0.0296
8	0.00574	0.00088	0.00088		0.02059	0.01743	0.01743
10	0.00147	3e-05	3e-05		0.01778	0.0173	0.0173

Table 2: Terms of higher order series-expansion

Coefficient\orders	(1)	(2)	(3)	(4)	...
g_k	$\mathrm{k}^{(1)}$	$\mathrm{k}^{(2)}$	$\mathrm{k}^{(3)}$	$\mathrm{k}^{(3)}$	...
g_z	$\mathrm{z}^{(1)}$	0	0	0	...
g_σ	1	0	0	0	...
g_{kk}	0	$\frac{\mathrm{k}^{(1)\,2}}{2}$	$\mathrm{k}^{(1)}\,\mathrm{k}^{(2)}$	$\frac{\mathrm{k}^{(2)\,2}}{2}+\mathrm{k}^{(1)}\,\mathrm{k}^{(3)}$	...
g_{kz}	0	$\mathrm{k}^{(1)}\,\mathrm{z}^{(1)}$	$\mathrm{k}^{(2)}\,\mathrm{z}^{(1)}$	$\mathrm{k}^{(3)}\,\mathrm{z}^{(1)}$	...
$g_{k\sigma}$	0	$\mathrm{k}^{(1)}$	$\mathrm{k}^{(2)}$	$\mathrm{k}^{(3)}$	...
g_{zz}	0	$\frac{\mathrm{z}^{(1)\,2}}{2}$	0	0	...
$g_{z\sigma}$	0	$\mathrm{z}^{(1)}$	0	0	...
$g_{\sigma\sigma}$	0	$\frac{1}{2}$	0	0	...
g_{kkk}	0	0	$\frac{\mathrm{k}^{(1)\,3}}{6}$	$\frac{\mathrm{k}^{(1)\,2}\,\mathrm{k}^{(2)}}{2}$	...
g_{llz}	0	0	$\frac{\mathrm{k}^{(1)\,2}\,\mathrm{z}^{(1)}}{2}$	$\mathrm{k}^{(1)}\,\mathrm{k}^{(2)}\,\mathrm{z}^{(1)}$	...
$g_{kk\sigma}$	0	0	$\frac{\mathrm{k}^{(1)\,2}}{2}$	$\mathrm{k}^{(1)}\,\mathrm{k}^{(2)}$	...
g_{kzz}	0	0	$\frac{\mathrm{k}^{(1)}\,\mathrm{z}^{(1)\,2}}{2}$	$\frac{\mathrm{k}^{(2)}\,\mathrm{z}^{(1)\,2}}{2}$	...
$g_{kz\sigma}$	0	0	$\mathrm{k}^{(1)}\,\mathrm{z}^{(1)}$	$\mathrm{k}^{(2)}\,\mathrm{z}^{(1)}$	...
$g_{k\sigma\sigma}$	0	0	$\frac{\mathrm{k}^{(1)}}{2}$	$\frac{\mathrm{k}^{(2)}}{2}$	...
g_{zzz}	0	0	$\frac{\mathrm{z}^{(1)\,3}}{6}$	0	...
$g_{zz\sigma}$	0	0	$\frac{\mathrm{z}^{(1)\,2}}{2}$	0	...
$g_{z\sigma\sigma}$	0	0	$\frac{\mathrm{z}^{(1)}}{2}$	0	...
$g_{\sigma\sigma\sigma}$	0	0	$\frac{1}{6}$	0	...

(Table continues)

Table 2: Terms of higher order series-expansion (continued)

Coefficient\orders	(1)	(2)	(3)	(4)	...
g_{kkkk}	0	0	0	$\frac{k^{(1)4}}{24}$	...
g_{kkkz}	0	0	0	$\frac{k^{(1)3}z^{(1)}}{6}$	...
$g_{kkk\sigma}$	0	0	0	$\frac{k^{(1)3}}{6}$	...
g_{kkzz}	0	0	0	$\frac{k^{(1)2}z^{(1)2}}{4}$	...
$g_{kkz\sigma}$	0	0	0	$\frac{k^{(1)2}z^{(1)}}{2}$	...
$g_{kk\sigma\sigma}$	0	0	0	$\frac{k^{(1)2}}{4}$	...
g_{kzzz}	0	0	0	$\frac{k^{(1)}z^{(1)3}}{6}$	...
$g_{kzz\sigma}$	0	0	0	$\frac{k^{(1)}z^{(1)2}}{2}$	...
$g_{kz\sigma\sigma}$	0	0	0	$\frac{k^{(1)}z^{(1)}}{2}$	...
$g_{k\sigma\sigma\sigma}$	0	0	0	$\frac{k^{(1)}}{6}$	...
g_{zzzz}	0	0	0	$\frac{z^{(1)4}}{24}$	...
$g_{zzz\sigma}$	0	0	0	$\frac{z^{(1)3}}{6}$	...
$g_{zz\sigma\sigma}$	0	0	0	$\frac{z^{(1)2}}{4}$	...
$g_{z\sigma\sigma\sigma}$	0	0	0	$\frac{z^{(1)}}{6}$	...
$g_{\sigma\sigma\sigma\sigma}$	0	0	0	$\frac{1}{24}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3: Response to 100% technology shock: Ratio of Aruoba et al. (2006) to Series Expansion

Order\period	1	2	3	4
Capital				
(1)	1	1	1	1
(2)	1	0.99766	0.9959	0.99564
(3)	1	0.99775	0.9912	0.98365
(4)	1	1.0016	1.0034	1.0042
(5)	1	1.0004	1.0009	1.0015
Consumption				
(1)	1	1	1	1
(2)	1	0.99488	0.98675	0.9757
(3)	1	0.99596	0.97217	0.90261
(4)	1	1.0033	1.0127	1.0427
(5)	1	1.0007	1.0033	1.0143