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Solving Dynamic General Equilibrium Models Using Log Linear Approximation

Log-linearization strategy

- Example #1: A Simple RBC Model.
  - Define a Model ‘Solution’
  - Motivate the Need to Somehow Approximate Model Solutions
  - Describe Basic Idea Behind Log Linear Approximations
  - Some Strange Examples to be Prepared For

‘Blanchard-Kahn conditions not satisfied’

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- Example #3: Stochastic RBC Model with Hours Worked (Matrix Generalization of Previous Results)
- Example #4: Example #3 with 'Exotic' Information Sets.

Example #1: Nonstochastic RBC Model

$$\text{Maximize}_{\{c_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

subject to:

$$C_t + K_{t+1} - (1-\delta)K_t \; = \; K_t^{\alpha}, \; K_0 \text{ given}$$

First order condition:

$$C_t^{-\sigma} - \beta C_{t+1}^{-\sigma}\left[\alpha K_{t+1}^{\alpha-1} + (1-\delta)\right],$$

or, after substituting out resource constraint:

$$v(K_t, K_{t+1}, K_{t+2}) \; = \; 0, \; t = 0, 1, ..., \text{ with } K_0 \text{ given.}$$

- 'Solution': a function, $K_{t+1} = g(K_t)$, such that

$$v(K_t, g(K_t), g[g(K_t)]) \;=\; 0, \text{ for all } K_t.$$

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This is an Infinite Number of Equations
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in an Infinite Number of Unknowns
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- With Only a Few Rare Exceptions this is Very Hard to Solve Exactly
  - Easy cases:
    * If $\sigma = 1, \delta = 1 \Rightarrow g(K_t) = \alpha\beta K_t^{\alpha}$.
    * If v is linear in K_t, K_{t+1}, K_{t+1}.
  - Standard Approach: Approximate v by a Log Linear Function.

Approximation Method Based on Linearization

- Three Steps
  - Compute the Steady State
  - Do a Log Linear Expansion About Steady State
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- Step 1: Compute Steady State -
  - Steady State Value of K, K^* -

$$
\begin{aligned}
& C^{-\sigma} - \beta C^{-\sigma}\left[\alpha K^{\alpha-1} + (1-\delta)\right] = 0, \\
\Rightarrow\ & \alpha K^{\alpha-1} + (1-\delta) = \frac{1}{\beta} \\
\Rightarrow\ & K^* = \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

  - K^* satisfies:

$$
v(K^*, K^*, K^*) = 0.
$$

- Step 2:
  - Replace v by First Order Taylor Series Expansion About Steady State:

$$v_1(K_t - K^*) + v_2(K_{t+1} - K^*) + v_3(K_{t+2} - K^*) \;=\; 0,$$

  - Here,

$$v_1 \;=\; \frac{dv_u(K_t, K_{t+1}, K_{t+2})}{dK_t}, \text{ at } K_t = K_{t+1} = K_{t+2} = K^*.$$

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  - Conventionally, do *Log-Linear Approximation*:

$$(v_1K)\,\hat{K}_t + (v_2K)\,\hat{K}_{t+1} + (v_3K)\,\hat{K}_{t+2} \ = \ 0,$$

$$\hat{K}_t \equiv \frac{K_t - K^*}{K^*}.$$

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  - Write this as:

$$\alpha_2 \hat{K}_t + \alpha_1 \hat{K}_{t+1} + \alpha_0 \hat{K}_{t+2} \ = \ 0,$$
$$\alpha_2 = v_1 K, \ \alpha_1 = v_2 K, \ \alpha_0 = v_3 K$$

- Step 3: Solve
  - Posit the Following Policy Rule:
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$$\alpha_2\hat{K}_t + \alpha_1 A\hat{K}_t + \alpha_0 A^2\hat{K}_t = 0,$$

    or

$$\alpha_2 + \alpha_1 A + \alpha_0 A^2 = 0.$$

  - A is the Eigenvalue of Polynomial

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  - A is the Eigenvalue of Polynomial
- In General: Two Eigenvalues.
  - Can Show: In RBC Example, One Eigenvalue is Explosive. The Other Not.
  - There Exist Theorems (see Stokey-Lucas, chap. 6) That Say You Should Ignore the Explosive A.

Some Strange Examples to be Prepared For

- Other Examples Are Possible:
  - Both Eigenvalues Explosive
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- Other Examples Are Possible:
  - Both Eigenvalues Explosive
  - Both Eigenvalues Non-Explosive
  - What Do These Things Mean?

Some Strange Examples to be Prepared For ...

- Example With Two Explosive Eigenvalues
- Preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^\gamma}{\gamma},\ \gamma\ <\ 1.$$

- Technology:
  - Production of Consumption Goods

$$C_t\ =\ k_t^\alpha n_t^{1-\alpha}$$

  - Production of Capital Goods

$$k_{t+1}\ =\ 1 - n_t.$$

Some Strange Examples to be Prepared For ...

- Planning Problem:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{\left[k_t^{\alpha}\left(1-k_{t+1}\right)^{1-\alpha}\right]^{\gamma}}{\gamma}$$

- Euler Equation:

$$\begin{aligned} v(k_t, k_{t+1}, k_{t+2}) &= -(1-\alpha)k_t^{\alpha\gamma}(1-k_{t+1})^{[(1-\alpha)\gamma-1]} + \beta\alpha k_{t+1}^{(\alpha\gamma-1)}\left(1-k_{t+2}\right)^{(1-\alpha)\gamma} \\ &= 0, \end{aligned}$$

$t = 0, 1, ...$

- Steady State:

$$k = \frac{\alpha\beta}{1-\alpha+\alpha\beta}.$$

Some Strange Examples to be Prepared For ...

- Log-linearize Euler Equation:

$$\alpha_0 \hat{k}_{t+2} + \alpha_1 \hat{k}_{t+1} + \alpha_2 \hat{k}_t \; = \; 0$$

- With $\beta = 0.58$, $\gamma = 0.99$, $\alpha = 0.6$, *Both* Roots of Euler Equation are explosive:

$$-1.6734, \; -1.0303$$

- Other Properties:
  - Steady State:

$$0.4652$$

  - Two-Period Cycle:

$$0.8882, \; 0.0870$$

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  - If Log Linearized Euler Equation Around Particular Steady State Has Only Explosive Roots
    - All Possible Equilibria Involve Leaving that Steady State
    - Log Linear Approximation Not Useful, Since it Ceases to be Valid Outside a Neighborhood of Steady State

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  - 'Blanchard-Kahn conditions not satisfied, too many explosive roots'

Some Strange Examples to be Prepared For ...

- Another Possibility:
  - Both roots stable
  - Many paths converge into steady state: multiple equilibria
  - How can this happen?
    - strategic complementarities between economic agents.
    - inability of agents to coordinate.
    - combination can lead to multiple equilibria, 'coordination failures'.
  - What is source of strategic complementarities?
    - nature of technology and preferences
    - nature of relationship between agents and the government.

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	work hard	take it easy
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  - Everyone ‘take it easy’ equilibrium is a coordination failure: if everyone could get together, they’d all choose to work hard.
  - Example closer to home: every firm in the economy has a ‘pet investment project’ which only seems profitable if the economy is booming
    * Equilibrium #1: each firm conjectures all other firms will invest, this implies a booming economy, so it makes sense for each firm to invest.
    * Equilibrium #2: each firm conjectures all other firms will not invest, so economy will stagnate and it makes sense for each firm not to invest.

Some Strange Examples to be Prepared For ...

– Example even closer to home:
  * firm production function -

$$y_t = A_t K_t^{\alpha} h_t^{1-\alpha},$$

$$A_t = Y_t^{\gamma},\ Y_t \text{ ~ economy-wide average output}$$

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$Y_t = y_t$ 'economy-wide average output is average of individual firms' production'

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* equilibrium condition -

$Y_t = y_t$ ‘economy-wide average output is average of individual firms’ production’

* household preferences -

$$\sum_{t=0}^{\infty} \beta^t u(C_t, h_t)$$

* γ large enough leads to two stable eigenvalues, multiple equilibria.

Some Strange Examples to be Prepared For ...

- Lack of commitment in government policy can create strategic complementarities that lead to multiple equilibria.

  – Simple economy: many atomistic households solve

$$\max u(c,h) = c - \frac{1}{2}l^2$$

$$c \leq (1-\tau)\,wh,$$

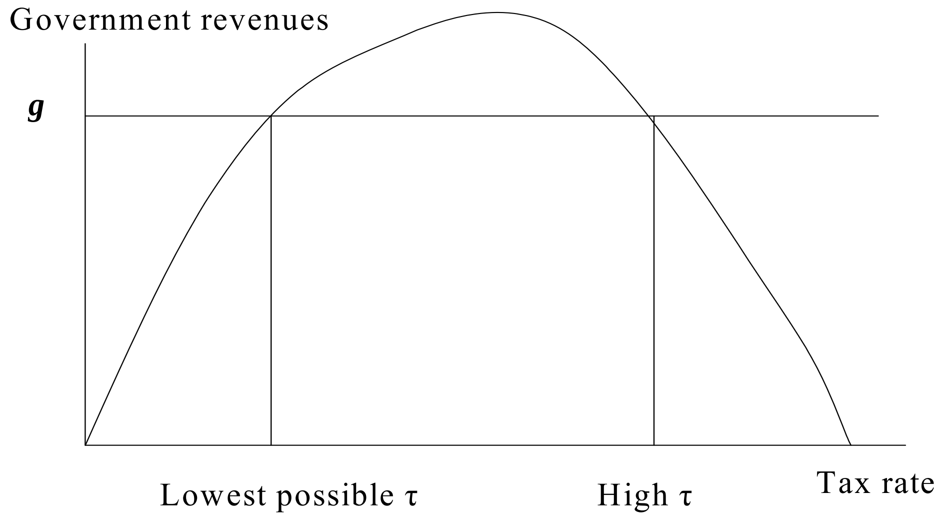
  w is technologically determined marginal product of labor.

  – Government chooses τ to satisfy its budget constraint

$$g \leq \tau wl$$

Some Strange Examples to be Prepared For ...

– Laffer curve



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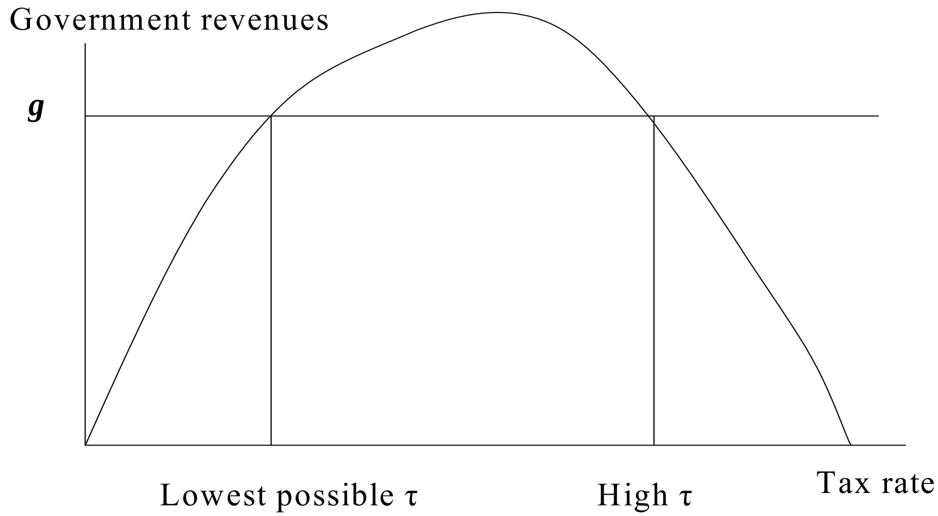

– Two scenarios depending on 'order of moves'

- commitment: (i) government sets τ (ii) private economy acts
  - lowest possible τ only possible outcome
- no commitment: (i) private economy determines h (ii) government chooses τ
  - at least two possible equilibria - lowest possible τ or high τ

Some Strange Examples to be Prepared For ...

- There is a coordination failure in the Lafffer curve example.....
  - If everyone could get together, they would all agree to work hard, so that the government sets low taxes ex post.
  - But, by assumption, people cannot get together and coordinate. (Also, because individuals have zero impact on government finances, it makes no sense for an *individual* person to work harder in the hope that this will allow the government to set low taxes.)
- There are strategic complementarities in the previous example
  - If I think everyone else will not work hard then, because this will require the government to raise taxes, I have an incentive to also not work hard.
- For an environment like this that leads to too many stable eigenvalues, see Schmitt-Grohe and Uribe paper on balanced budget, *JPE*.

Example #2: RBC Model With Uncertainty

- Model

$$\text{Maximize } E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

subject to

$$C_t + K_{t+1} - (1-\delta)K_t \;=\; K_t^{\alpha}\varepsilon_t,$$

where ε_t is a stochastic process with $E\varepsilon_t = \varepsilon$, say. Let

$$\hat{\varepsilon}_t \;=\; \frac{\varepsilon_t - \varepsilon}{\varepsilon},$$

and suppose

$$\hat{\varepsilon}_t \;=\; \rho\hat{\varepsilon}_{t-1} + e_t,\; e_t \tilde{} N(0, \sigma_e^2).$$

- First Order Condition:

$$E_t\left\{C_t^{-\sigma} - \beta C_{t+1}^{-\sigma}\left[\alpha K_{t+1}^{\alpha-1}\varepsilon_{t+1} + 1 - \delta\right]\right\} = 0.$$

- First Order Condition:

$$E_t v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0,$$

where

$$\begin{aligned} & v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\ = \; & \left(K_t^{\alpha}\varepsilon_t + (1-\delta)K_t - K_{t+1}\right)^{-\sigma} \\ & -\beta\left(K_{t+1}^{\alpha}\varepsilon_{t+1} + (1-\delta)K_{t+1} - K_{t+2}\right)^{-\sigma} \\ & \times\left[\alpha K_{t+1}^{\alpha-1}\varepsilon_{t+1} + 1 - \delta\right]. \end{aligned}$$

- Solution: a $g(K_t, \varepsilon_t)$, Such That

$$E_t v\left(g(g(K_t, \varepsilon_t), \varepsilon_{t+1}), g(K_t, \varepsilon_t), K_t, \varepsilon_{t+1}, \varepsilon_t\right) \;=\; 0,$$

For All K_t, ε_t.

- Hard to Find g, Except in Special Cases
  - One Special Case: v is Log Linear.

- Log Linearization Strategy:
  - Step 1: Compute Steady State of K_t when ε_t is Replaced by $E\varepsilon_t$
  - Step2: Replace v By its Taylor Series Expansion About Steady State.
  - Step 3: Solve Resulting Log Linearized System.
- Logic: If Actual Stochastic System Remains in a Neighborhood of Steady State, Log Linear Approximation Good

- Caveat: Strategy not accurate in all conceivable situations.
  - Example: suppose that where I live -

$$\varepsilon \equiv \text{temperature} = \begin{cases} \text{140 Fahrenheit, 50 percent of time} \\ \text{0 degrees Fahrenheit the other half} \end{cases}.$$

  - On average, temperature quire nice: $E\varepsilon = 70$ (like parts of California)

  - Let $K =$ capital invested in heating and airconditioning

    * EK *very*, *very* large!
    * Economist who predicts investment based on replacing ε by $E\varepsilon$ would predict $K = 0$ (as in many parts of California)

  - In standard model this is not a big problem, because shocks are not so big....steady state value of K (i.e., the value that results eventually when ε is replaced by $E\varepsilon$) is approximately $E\varepsilon$ (i.e., the average value of K when ε is stochastic).

Example #2: RBC Model With Uncertainty ...

- Step 1: Steady State:

$$K^* = \left[\frac{\alpha\varepsilon}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}.$$

- Step 2: Log Linearize -

$$\begin{aligned}
& v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
\simeq\ & v_1\left(K_{t+2} - K^*\right) + v_2\left(K_{t+1} - K^*\right) + v_3\left(K_t - K^*\right) \\
& + v_3\left(\varepsilon_{t+1} - \varepsilon\right) + v_4\left(\varepsilon_t - \varepsilon\right) \\
=\ & v_1 K^*\left(\frac{K_{t+2} - K^*}{K^*}\right) + v_2 K^*\left(\frac{K_{t+1} - K^*}{K^*}\right) + v_3 K^*\left(\frac{K_t - K^*}{K^*}\right) \\
& + v_3\varepsilon\left(\frac{\varepsilon_{t+1} - \varepsilon}{\varepsilon}\right) + v_4\varepsilon\left(\frac{\varepsilon_t - \varepsilon}{\varepsilon}\right) \\
=\ & \alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t.
\end{aligned}$$

- Step 3: Solve Log Linearized System
  - Posit:

$$\hat{K}_{t+1} = A\hat{K}_t + B\hat{\varepsilon}_t.$$

  - Pin Down A and B By Condition that log-linearized Euler Equation Must Be Satisfied.
    - Note:

$$\begin{aligned}\hat{K}_{t+2} &= A\hat{K}_{t+1} + B\hat{\varepsilon}_{t+1} \\ &= A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}.\end{aligned}$$

    - Substitute Posited Policy Rule into Log Linearized Euler Equation:

$$E_t\left[\alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t\right] = 0,$$

so must have:

$$\begin{aligned}&E_t\{\alpha_0\left[A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}\right] \\ &+\alpha_1\left[A\hat{K}_t + B\hat{\varepsilon}_t\right] + \alpha_2\hat{K}_t + \beta_0\rho\hat{\varepsilon}_t + \beta_0 e_{t+1} + \beta_1\hat{\varepsilon}_t\} = 0\end{aligned}$$

* Then,

$$
\begin{aligned}
E_t\left[\alpha_0\hat{K}_{t+2}+\alpha_1\hat{K}_{t+1}+\alpha_2\hat{K}_t+\beta_0\hat{\varepsilon}_{t+1}+\beta_1\hat{\varepsilon}_t\right] \\
= E_t\{\alpha_0\left[A^2\hat{K}_t+AB\hat{\varepsilon}_t+B\rho\hat{\varepsilon}_t+Be_{t+1}\right] \\
+\alpha_1\left[A\hat{K}_t+B\hat{\varepsilon}_t\right]+\alpha_2\hat{K}_t+\beta_0\rho\hat{\varepsilon}_t+\beta_0 e_{t+1}+\beta_1\hat{\varepsilon}_t\} \\
= \alpha(A)\hat{K}_t+F\hat{\varepsilon}_t \\
= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(A) &= \alpha_0 A^2+\alpha_1 A+\alpha_2, \\
F &= \alpha_0 AB+\alpha_0 B\rho+\alpha_1 B+\beta_0\rho+\beta_1
\end{aligned}
$$

* Find A and B that Satisfy:

$$
\alpha(A) = 0, \; F = 0.
$$

Example #3 RBC Model With Hours Worked and Uncertainty

- Maximize

$$E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

subject to

$$C_t + K_{t+1} - (1-\delta)K_t \; = \; f(K_t, N_t, \varepsilon_t)$$

and

$$E\varepsilon_t = \varepsilon,$$

$$\hat{\varepsilon}_t \; = \; \rho\hat{\varepsilon}_{t-1} + e_t, \; e_t \tilde{} N(0, \sigma_e^2)$$

$$\hat{\varepsilon}_t \; = \; \frac{\varepsilon_t - \varepsilon}{\varepsilon}.$$

- First Order Conditions:

$$E_t v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0$$

and

$$v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = 0.$$

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$$v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = 0.$$

where

$$\begin{aligned} & v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\ = \; & U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & -\beta U_c\left(f(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, N_{t+1}\right) \\ & \times\left[f_K(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + 1 - \delta\right] \end{aligned}$$

- First Order Conditions:

$$E_t v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0$$

and

$$v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = 0.$$

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and,

$$\begin{aligned} & v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\ = \; & U_N\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & +U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & \times f_N(K_t, N_t, \varepsilon_t). \end{aligned}$$

- Steady state K^* and N^* such that Equilibrium Conditions Hold with $\varepsilon_t \equiv \varepsilon$.

- Log-Linearize the Equilibrium Conditions:

$$
\begin{aligned}
& v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
= \; & v_{K,1} K^* \hat{K}_{t+2} + v_{K,2} N^* \hat{N}_{t+1} + v_{K,3} K^* \hat{K}_{t+1} + v_{K,4} N^* \hat{N}_t + v_{K,5} K^* \hat{K}_t \\
& + v_{K,6} \varepsilon \hat{\varepsilon}_{t+1} + v_{K,7} \varepsilon \hat{\varepsilon}_t
\end{aligned}
$$

$v_{K,j}$ ~ Derivative of v_K with respect to j^{th} argument, evaluated in steady state.

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$$
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= \; & v_{K,1}K^*\hat{K}_{t+2} + v_{K,2}N^*\hat{N}_{t+1} + v_{K,3}K^*\hat{K}_{t+1} + v_{K,4}N^*\hat{N}_t + v_{K,5}K^*\hat{K}_t \\
& + v_{K,6}\varepsilon\hat{\varepsilon}_{t+1} + v_{K,7}\varepsilon\hat{\varepsilon}_t
\end{aligned}
$$

$v_{K,j}$ ~ Derivative of v_K with respect to j^{th} argument, evaluated in steady state.

$$
\begin{aligned}
& v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\
= \; & v_{N,1}K^*\hat{K}_{t+1} + v_{N,2}N^*\hat{N}_t + v_{N,3}K^*\hat{K}_t + v_{N,4}\varepsilon\hat{\varepsilon}_{t+1}
\end{aligned}
$$

$v_{N,j}$ ~ Derivative of v_N with respect to j^{th} argument, evaluated in steady state.

- Representation Log-linearized Equilibrium Conditions
  - Let

$$z_t = \begin{pmatrix} \hat{K}_{t+1} \\ \hat{N}_t \end{pmatrix},\ s_t = \hat{\varepsilon}_t,\ \epsilon_t = e_t.$$

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  - Then, the linearized Euler equation is:

$$\begin{aligned} E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] &= 0, \\ s_t &= P s_{t-1} + \epsilon_t,\ \epsilon_t \sim N(0, \sigma_e^2),\ P = \rho. \end{aligned}$$

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  - Here,

$$\begin{aligned} \alpha_0 &= \begin{bmatrix} v_{K,1}K^* & v_{K,2}N^* \\ 0 & 0 \end{bmatrix}, \; \alpha_1 = \begin{bmatrix} v_{K,3}K^* & v_{K,4}N^* \\ v_{N,1}K^* & v_{N,2}N^* \end{bmatrix}, \\ \alpha_2 &= \begin{bmatrix} v_{K,5}K^* & 0 \\ v_{N,3}K^* & 0 \end{bmatrix}, \end{aligned}$$

$$\beta_0 = \begin{pmatrix} v_{K,6}\varepsilon \\ 0 \end{pmatrix}, \; \beta_1 = \begin{pmatrix} v_{K,7}\varepsilon \\ v_{N,4}\varepsilon \end{pmatrix}.$$

- Previous is a Canonical Representation That Essentially All Log Linearized Models Can be Fit Into (See Christiano (2002).)

- Again, Look for Solution

$$z_t = Az_{t-1} + Bs_t,$$

where A and B are pinned down by log-linearized Equilibrium Conditions.

- Now, A is *Matrix* Eigenvalue of *Matrix* Polynomial:

$$\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0.$$

- Also, B Satisfies Same System of Log Linear Equations as Before:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0.$$

- Go for the 2 Free Elements of B Using 2 Equations Given by

$$F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Finding the Matrix Eigenvalue of the Polynomial Equation,

$$\alpha(A) = 0,$$

  and Determining if A is Unique is a Solved Problem.

- See Anderson, Gary S. and George Moore, 1985, 'A Linear Algebraic Procedure for Solving Linear Perfect Foresight Models,' *Economic Letters*, 17, 247-52 or Articles in Computational Economics, October, 2002. See also, the program, DYNARE.

- Solving for B
  - Given A, Solve for B Using Following (Log Linear) System of Equations:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

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$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1)\, vec(A_2),$$

- Solving for B
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$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1)\, vec(A_2),$$

to Convert $F = 0$ Into

$$vec(F') = d + q\delta = 0,\ \delta = vec(B').$$

  - Find B By First Solving:

$$\delta = -q^{-1}d.$$

Example #4: Example #3 With 'Exotic' Information Set

- Suppose the Date t Investment Decision is Made Before the Current Realization of the Technology Shock, While the Hours Decision is Made Afterward.
- Now, Canonical Form Must Be Written Differently:

$$\mathcal{E}_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0,$$

where

$$\mathcal{E}_t X_t = \begin{bmatrix} E\left[X_{1t}|\hat{\varepsilon}_{t-1}\right] \\ E\left[X_{2t}|\hat{\varepsilon}_t\right] \end{bmatrix}.$$

- Convenient to Change s_t:

$$s_t = \begin{pmatrix} \hat{\varepsilon}_t \\ \hat{\varepsilon}_{t-1} \end{pmatrix},\ P = \begin{bmatrix} \rho & 0 \\ 1 & 0 \end{bmatrix},\ \epsilon_t = \begin{pmatrix} e_t \\ 0 \end{pmatrix}.$$

- Adjust β_i's:

$$\beta_0 = \begin{pmatrix} v_{K,6}\varepsilon & 0 \\ 0 & 0 \end{pmatrix},\ \beta_1 = \begin{pmatrix} v_{K,7}\varepsilon & 0 \\ v_{N,4}\varepsilon & 0 \end{pmatrix},$$

Example #4: Example #3 With 'Exotic' Information Set ...

- Posit Following Solution:

$$z_t = Az_{t-1} + Bs_t.$$

- Substitute Into Canonical Form:

$$\begin{aligned} &\mathcal{E}_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] \\ = \ &\alpha(A)z_{t-1} + \mathcal{E}_t F s_t + \mathcal{E}_t \beta_0 \epsilon_{t+1} = \alpha(A)z_{t-1} + \mathcal{E}_t F s_t = 0, \end{aligned}$$

- Then,

$$\begin{aligned} \mathcal{E}_t F s_t \ &= \ \mathcal{E}_t \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} s_t = \mathcal{E}_t \begin{bmatrix} F_{11}\hat{\varepsilon}_t + F_{12}\hat{\varepsilon}_{t-1} \\ F_{21}\hat{\varepsilon}_t + F_{22}\hat{\varepsilon}_{t-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & F_{12} + \rho F_{11} \\ F_{21} & F_{22} \end{bmatrix} s_t = \tilde{F} s_t. \end{aligned}$$

- Equations to be solved:

$$\alpha(A) = 0, \ \tilde{F} = 0.$$

- $\tilde{F}$ Only Has *Three* Equations How Can We Solve for the Four Elements of B?
- Answer: Only *Three* Unknowns in B Because B Must Also Obey Information Structure:

$$B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Summary so Far

- Solving Models By Log Linear Approximation Involves Three Steps:
  a. Compute Steady State
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- Step 3 Requires Finding A and B in:

$$z_t = Az_{t-1} + Bs_t,$$

  to Satisfy Log-Linearized Equilibrium Conditions:

$$\mathcal{E}_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right]$$

$$s_t = P s_{t-1} + \epsilon_t,\ \epsilon_t \sim \text{ iid}$$

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  a. Compute Steady State

  b. Log-Linearize Equilibrium Conditions

  c. Solve Log Linearized Equations.

- Step 3 Requires Finding A and B in:

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$$\mathcal{E}_t \left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right]$$
$$s_t = Ps_{t-1} + \epsilon_t,\ \epsilon_t \sim \text{ iid}$$

- We are Led to Choose A and B so that:

$$\begin{aligned} \alpha(A) &= 0, \\ \text{(standard information set) } F &= 0, \\ \text{(exotic information set) } \tilde{F} &= 0 \end{aligned}$$

  and Eigenvalues of A are Less Than Unity In Absolute Value.