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Solving and Estimating Dynamic General Equilibrium Models Using Log Linear Approximation

Overview

- A log-linearization strategy for solving DSGE models.

- Estimation

  – Putting the model into state space-observer form
  – Computing the impulse response functions implied by a DSGE model
  – Maximum likelihood
  – Bayesian inference

- A dividend from analysis:
  – formal connection between VARs and DSGE models

Log-linearization strategy

1. Example #1: A Simple RBC Model.
   - Define a Model 'Solution'
   - Motivate the Need to Somehow Approximate Model Solutions
   - Describe Basic Idea Behind Log Linear Approximations
   - Some Strange Examples to be Prepared For

     'Blanchard-Kahn conditions not satisfied'
2. Example #2: Putting the Stochastic RBC Model into General Canonical Form
3. Example #3: Stochastic RBC Model with Hours Worked (Matrix Generalization of Previous Results)
4. Example #4: Example #3 with 'Exotic' Information Sets.
5. Summary so Far.
6. Example #5: Sticky price model with no capital - log linearizing about flexible price equilibrium.

Example #1: Nonstochastic RBC Model

$$\text{Maximize}_{\{c_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

subject to:

$$C_t + K_{t+1} - (1-\delta)K_t \ = \ K_t^{\alpha}, \ K_0 \text{ given}$$

First order condition:

$$C_t^{-\sigma} - \beta C_{t+1}^{-\sigma} \left[\alpha K_{t+1}^{\alpha-1} + (1-\delta) \right],$$

or, after substituting out resource constraint:

$$v(K_t, K_{t+1}, K_{t+2}) \ = \ 0, \ t = 0, 1, ..., \text{ with } K_0 \text{ given.}$$

Example #1: Nonstochastic RBC Model ...

- ‘Solution’: a function, $K_{t+1} = g(K_t)$, such that

$$v(K_t, g(K_t), g[g(K_t)]) \;=\; 0, \text{ for all } K_t.$$

- Problem:

This is an Infinite Number of Equations
(one for each possible K_t)
in an Infinite Number of Unknowns
(a value for g for each possible K_t)

- With Only a Few Rare Exceptions this is Very Hard to Solve Exactly
  - Easy cases:
    * If $\sigma = 1, \delta = 1 \Rightarrow g(K_t) = \alpha\beta K_t^{\alpha}$.
    * If v is linear in K_t, K_{t+1}, K_{t+1}.
  - Standard Approach: Approximate v by a Log Linear Function.

Approximation Method Based on Linearization

- Three Steps
  - Compute the Steady State
  - Do a Log Linear Expansion About Steady State
  - Solve the Resulting Log Linearized System
- Step 1: Compute Steady State -
  - Steady State Value of K, K^* -

$$
\begin{aligned}
& C^{-\sigma} - \beta C^{-\sigma}\left[\alpha K^{\alpha-1} + (1-\delta)\right] = 0, \\
\Rightarrow\ & \alpha K^{\alpha-1} + (1-\delta) = \frac{1}{\beta} \\
\Rightarrow\ & K^* = \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

  - K^* satisfies:

$$v(K^*, K^*, K^*) = 0.$$

- Step 2:
  - Replace v by First Order Taylor Series Expansion About Steady State:

$$v_1(K_t - K^*) + v_2(K_{t+1} - K^*) + v_3(K_{t+2} - K^*) \ = \ 0,$$

  - Here,

$$v_1 \ = \ \frac{dv_u(K_t, K_{t+1}, K_{t+2})}{dK_t}, \text{ at } K_t = K_{t+1} = K_{t+2} = K^*.$$

  - Conventionally, do *Log-Linear Approximation*:

$$(v_1 K)\,\hat{K}_t + (v_2 K)\,\hat{K}_{t+1} + (v_3 K)\,\hat{K}_{t+2} \ = \ 0,$$

$$\hat{K}_t \equiv \frac{K_t - K^*}{K^*}.$$

  - Write this as:

$$\alpha_2 \hat{K}_t + \alpha_1 \hat{K}_{t+1} + \alpha_0 \hat{K}_{t+2} \ = \ 0,$$

$$\alpha_2 = v_1 K, \ \alpha_1 = v_2 K, \ \alpha_0 = v_3 K$$

- Step 3: Solve
  - Posit the Following Policy Rule:
    $$\hat{K}_{t+1} = A\hat{K}_t,$$
    Where A is to be Determined.
  - Compute A :
    $$\alpha_2 \hat{K}_t + \alpha_1 A \hat{K}_t + \alpha_0 A^2 \hat{K}_t = 0,$$
    or
    $$\alpha_2 + \alpha_1 A + \alpha_0 A^2 = 0.$$
  - A is the Eigenvalue of Polynomial
- In General: Two Eigenvalues.
  - Can Show: In RBC Example, One Eigenvalue is Explosive. The Other Not.
  - There Exist Theorems (see Stokey-Lucas, chap. 6) That Say You Should Ignore the Explosive A.

Some Strange Examples to be Prepared For

- Other Examples Are Possible:
  - Both Eigenvalues Explosive
  - Both Eigenvalues Non-Explosive
  - What Do These Things Mean?

Some Strange Examples to be Prepared For ...

- Example With Two Explosive Eigenvalues
- Preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^{\gamma}}{\gamma}, \; \gamma < 1.$$

- Technology:
  - Production of Consumption Goods

$$C_t = k_t^{\alpha} n_t^{1-\alpha}$$

  - Production of Capital Goods

$$k_{t+1} = 1 - n_t.$$

Some Strange Examples to be Prepared For ...

• Planning Problem:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{\left[k_t^\alpha \left(1-k_{t+1}\right)^{1-\alpha}\right]^\gamma}{\gamma}$$

• Euler Equation:

$$\begin{aligned} v(k_t, k_{t+1}, k_{t+2}) &= -(1-\alpha)k_t^{\alpha\gamma}(1-k_{t+1})^{[(1-\alpha)\gamma-1]} + \beta\alpha k_{t+1}^{(\alpha\gamma-1)}\left(1-k_{t+2}\right)^{(1-\alpha)\gamma} \\ &= 0, \end{aligned}$$

$t = 0, 1, ...$

• Steady State:

$$k = \frac{\alpha\beta}{1-\alpha+\alpha\beta}.$$

Some Strange Examples to be Prepared For ...

- Log-linearize Euler Equation:

$$\alpha_0 \hat{k}_{t+2} + \alpha_1 \hat{k}_{t+1} + \alpha_2 \hat{k}_t \;=\; 0$$

- With $\beta = 0.58$, $\gamma = 0.99$, $\alpha = 0.6$, *Both* Roots of Euler Equation are explosive:

$$-1.6734, \;\; -1.0303$$

- Other Properties:
  - Steady State:

$$0.4652$$

  - Two-Period Cycle:

$$0.8882, \; 0.0870$$

Some Strange Examples to be Prepared For ...

- Meaning of Stokey-Lucas Example
  - Illustrates the Possibility of All Explosive Roots
  - Economics:
    - If Somehow You Start At Single Steady State, Stay There
    - If You are Away from Single Steady State, Go Somewhere Else
  - If Log Linearized Euler Equation Around Particular Steady State Has Only Explosive Roots
    - All Possible Equilibria Involve Leaving that Steady State
    - Log Linear Approximation Not Useful, Since it Ceases to be Valid Outside a Neighborhood of Steady State
  - Could Log Linearize About Two-Period Cycle (That's Another Story...)
  - The Example Suggests That *Maybe* All Explosive Root Case is Unlikely

  - 'Blanchard-Kahn conditions not satisfied, too many explosive roots'

Some Strange Examples to be Prepared For ...

- Another Possibility:
  - Both Roots Stable
  - Many Paths Converge Into Steady State: Multiple Equilibria
  - Can Happen For Many Reasons
    - Strategic Complementarities Among Different Agents In Private Economy
    - Certain Types of Government Policy
  - This is a More Likely Possibility
  - Avoid Being Surprised by It By Always Thinking Through Economics of Model.

Some Strange Examples to be Prepared For ...

- Strategic Complementarities Between Agent A and Agent B
  - Payoff to agent A is higher if agent B is working harder
  - In following setup, strategic complementarities give rise to two equilibria:

Me	Everyone else	
	work hard	take it easy
work hard	3	0
take it easy	1	1

  - Example closer to home: every firm in the economy has a 'pet investment project' which only seems profitable if the economy is booming
    - ∗ Equilibrium #1: each firm conjectures all other firms will invest, this implies a booming economy, so it makes sense for each firm to invest.
    - ∗ Equilibrium #2: each firm conjectures all other firms will not invest, so economy will stagnate and it makes sense for each firm not to invest.

Some Strange Examples to be Prepared For ...

– Example even closer to home:

  * firm production function -

$$y_t = A_t K_t^{\alpha} h_t^{1-\alpha},$$

$$A_t = Y_t^{\gamma},\ Y_t \text{ ~ economy-wide average output}$$

  * resource constraint -

$$C_t + K_{t+1} - (1-\delta) K_t = Y_t$$

  * equilibrium condition -

$Y_t = y_t$ ‘economy-wide average output is average of individual firms’ production’

  * household preferences -

$$\sum_{t=0}^{\infty} \beta^t u(C_t, h_t)$$

  * γ large enough leads to two stable eigenvalues, multiple equilibria.

Some Strange Examples to be Prepared For ...

- Lack of Commitment in Government Policy Can Lead to Multiple Equilibria

  – Simple economy: households solve

$$\max u(c, h) = c - \frac{1}{2} l^2$$

$$c \leq (1 - \tau) wh,$$

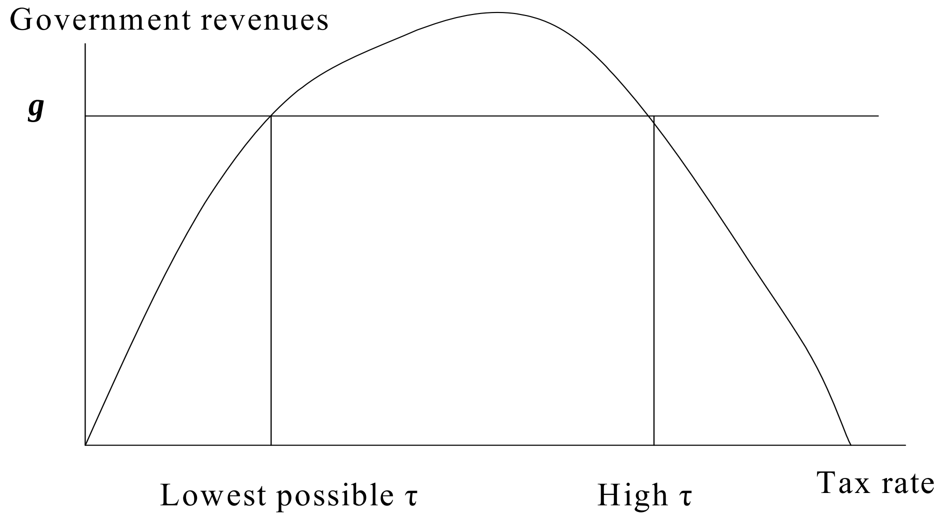
  w is technologically determined marginal product of labor.

  – Government chooses τ to satisfy its budget constraint

$$g \leq \tau wl$$

Some Strange Examples to be Prepared For ...

– Laffer curve



– Two scenarios depending on 'order of moves'
  - commitment: (i) government sets τ (ii) private economy acts
    - lowest possible τ only possible outcome
  - no commitment: (i) private economy determines h (ii) government chooses τ
    - at least two possible equilibria - lowest possible τ or high τ

- For an environment like this that leads to too many stable eigenvalues, see Schmitt-Grohe and Uribe paper on balanced budget, *JPE*.

Example #2: RBC Model With Uncertainty

- Model

$$\text{Maximize } E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

subject to

$$C_t + K_{t+1} - (1-\delta)K_t \; = \; K_t^{\alpha}\varepsilon_t,$$

where ε_t is a stochastic process with $E\varepsilon_t = \varepsilon$, say. Let

$$\hat{\varepsilon}_t \; = \; \frac{\varepsilon_t - \varepsilon}{\varepsilon},$$

and suppose

$$\hat{\varepsilon}_t \; = \; \rho\hat{\varepsilon}_{t-1} + e_t, \; e_t \tilde{} N(0, \sigma_e^2).$$

- First Order Condition:

$$E_t \left\{ C_t^{-\sigma} - \beta C_{t+1}^{-\sigma} \left[\alpha K_{t+1}^{\alpha-1}\varepsilon_{t+1} + 1 - \delta \right] \right\} = 0.$$

- First Order Condition:

$$E_t v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0,$$

where

$$\begin{aligned} & v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\ = \; & (K_t^{\alpha}\varepsilon_t + (1-\delta)K_t - K_{t+1})^{-\sigma} \\ & -\beta\left(K_{t+1}^{\alpha}\varepsilon_{t+1} + (1-\delta)K_{t+1} - K_{t+2}\right)^{-\sigma} \\ & \times \left[\alpha K_{t+1}^{\alpha-1}\varepsilon_{t+1} + 1 - \delta\right]. \end{aligned}$$

- Solution: a $g(K_t, \varepsilon_t)$, Such That

$$E_t v\left(g(g(K_t, \varepsilon_t), \varepsilon_{t+1}), g(K_t, \varepsilon_t), K_t, \varepsilon_{t+1}, \varepsilon_t\right) \; = \; 0,$$

For All K_t, ε_t.

- Hard to Find g, Except in Special Cases
  - One Special Case: v is Log Linear.

- Log Linearization Strategy:
  - Step 1: Compute Steady State of K_t when ε_t is Replaced by $E\varepsilon_t$
  - Step2: Replace v By its Taylor Series Expansion About Steady State.
  - Step 3: Solve Resulting Log Linearized System.
- Logic: If Actual Stochastic System Remains in a Neighborhood of Steady State, Log Linear Approximation Good

- Caveat: Strategy not accurate in all conceivable situations.
  - Example: suppose that where I live -

$$\varepsilon \equiv \text{temperature} = \begin{cases} \text{140 Fahrenheit, 50 percent of time} \\ \text{0 degrees Fahrenheit the other half} \end{cases}.$$

  - On average, temperature quire nice: $E\varepsilon = 70$ (like parts of California)

  - Let $K =$ capital invested in heating and airconditioning

    * EK *very*, *very* large!
    * Economist who predicts investment based on replacing ε by $E\varepsilon$ would predict $K = 0$ (as in many parts of California)

  - In standard model this is not a big problem, because shocks are not so big....steady state value of K (i.e., the value that results eventually when ε is replaced by $E\varepsilon$) is approximately $E\varepsilon$ (i.e., the average value of K when ε is stochastic).

Example #2: RBC Model With Uncertainty ...

- Step 1: Steady State:

$$K^* = \left[\frac{\alpha\varepsilon}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}.$$

- Step 2: Log Linearize -

$$
\begin{aligned}
& v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
\simeq\ & v_1\left(K_{t+2} - K^*\right) + v_2\left(K_{t+1} - K^*\right) + v_3\left(K_t - K^*\right) \\
& + v_3\left(\varepsilon_{t+1} - \varepsilon\right) + v_4\left(\varepsilon_t - \varepsilon\right) \\
=\ & v_1 K^* \left(\frac{K_{t+2} - K^*}{K^*}\right) + v_2 K^* \left(\frac{K_{t+1} - K^*}{K^*}\right) + v_3 K^* \left(\frac{K_t - K^*}{K^*}\right) \\
& + v_3\varepsilon\left(\frac{\varepsilon_{t+1} - \varepsilon}{\varepsilon}\right) + v_4\varepsilon\left(\frac{\varepsilon_t - \varepsilon}{\varepsilon}\right) \\
=\ & \alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t.
\end{aligned}
$$

- Step 3: Solve Log Linearized System
  - Posit:

$$\hat{K}_{t+1} = A\hat{K}_t + B\hat{\varepsilon}_t.$$

  - Pin Down A and B By Condition that log-linearized Euler Equation Must Be Satisfied.
    - Note:

$$\begin{aligned}\hat{K}_{t+2} &= A\hat{K}_{t+1} + B\hat{\varepsilon}_{t+1} \\ &= A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}.\end{aligned}$$

    - Substitute Posited Policy Rule into Log Linearized Euler Equation:

$$E_t\left[\alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t\right] = 0,$$

so must have:

$$\begin{aligned}E_t\{\alpha_0\left[A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}\right] \\ +\alpha_1\left[A\hat{K}_t + B\hat{\varepsilon}_t\right] + \alpha_2\hat{K}_t + \beta_0\rho\hat{\varepsilon}_t + \beta_0 e_{t+1} + \beta_1\hat{\varepsilon}_t\} &= 0\end{aligned}$$

* Then,

$$
\begin{aligned}
&E_t\left[\alpha_0\hat{K}_{t+2}+\alpha_1\hat{K}_{t+1}+\alpha_2\hat{K}_t+\beta_0\hat{\varepsilon}_{t+1}+\beta_1\hat{\varepsilon}_t\right]\\
&=E_t\{\alpha_0\left[A^2\hat{K}_t+AB\hat{\varepsilon}_t+B\rho\hat{\varepsilon}_t+Be_{t+1}\right]\\
&+\alpha_1\left[A\hat{K}_t+B\hat{\varepsilon}_t\right]+\alpha_2\hat{K}_t+\beta_0\rho\hat{\varepsilon}_t+\beta_0e_{t+1}+\beta_1\hat{\varepsilon}_t\}\\
&=\alpha(A)\hat{K}_t+F\hat{\varepsilon}_t\\
&=0
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(A) &= \alpha_0A^2+\alpha_1A+\alpha_2,\\
F &= \alpha_0AB+\alpha_0B\rho+\alpha_1B+\beta_0\rho+\beta_1
\end{aligned}
$$

* Find A and B that Satisfy:

$$\alpha(A)\ =\ 0,\ F=0.$$

Example #3 RBC Model With Hours Worked and Uncertainty

- Maximize

$$E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

subject to

$$C_t + K_{t+1} - (1-\delta)K_t \;=\; f(K_t, N_t, \varepsilon_t)$$

and

$$E\varepsilon_t = \varepsilon,$$

$$\hat{\varepsilon}_t \;=\; \rho\hat{\varepsilon}_{t-1} + e_t,\; e_t\tilde{\ } N(0, \sigma_e^2)$$

$$\hat{\varepsilon}_t \;=\; \frac{\varepsilon_t - \varepsilon}{\varepsilon}.$$

- First Order Conditions:

$$E_t v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0$$

and

$$v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = 0.$$

where

$$\begin{aligned} & v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\ = \; & U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & -\beta U_c\left(f(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, N_{t+1}\right) \\ & \times \left[f_K(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + 1 - \delta\right] \end{aligned}$$

and,

$$\begin{aligned} & v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\ = \; & U_N\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & +U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & \times f_N(K_t, N_t, \varepsilon_t). \end{aligned}$$

- Steady state K^* and N^* such that Equilibrium Conditions Hold with $\varepsilon_t \equiv \varepsilon$.

- Log-Linearize the Equilibrium Conditions:

$$
\begin{aligned}
& v_K(K_{t+2}, N_{t+1,}K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
= \; & v_{K,1}K^*\hat{K}_{t+2} + v_{K,2}N^*\hat{N}_{t+1} + v_{K,3}K^*\hat{K}_{t+1} + v_{K,4}N^*\hat{N}_t + v_{K,5}K^*\hat{K}_t \\
& + v_{K,6}\varepsilon\hat{\varepsilon}_{t+1} + v_{K,7}\varepsilon\hat{\varepsilon}_t
\end{aligned}
$$

$v_{K,j}$ ~ Derivative of v_K with respect to j^{th} argument, evaluated in steady state.

$$
\begin{aligned}
& v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\
= \; & v_{N,1}K^*\hat{K}_{t+1} + v_{N,2}N^*\hat{N}_t + v_{N,3}K^*\hat{K}_t + v_{N,4}\varepsilon\hat{\varepsilon}_{t+1}
\end{aligned}
$$

$v_{N,j}$ ~ Derivative of v_N with respect to j^{th} argument, evaluated in steady state.

- Representation Log-linearized Equilibrium Conditions
  - Let
$$z_t = \begin{pmatrix} \hat{K}_{t+1} \\ \hat{N}_t \end{pmatrix},\ s_t = \hat{\varepsilon}_t,\ \epsilon_t = e_t.$$
  - Then, the linearized Euler equation is:
$$E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0,$$
$$s_t = P s_{t-1} + \epsilon_t,\ \epsilon_t \sim N(0, \sigma_e^2),\ P = \rho.$$
  - Here,
$$\alpha_0 = \begin{bmatrix} v_{K,1}K^* & v_{K,2}N^* \\ 0 & 0 \end{bmatrix},\ \alpha_1 = \begin{bmatrix} v_{K,3}K^* & v_{K,4}N^* \\ v_{N,1}K^* & v_{N,2}N^* \end{bmatrix},$$
$$\alpha_2 = \begin{bmatrix} v_{K,5}K^* & 0 \\ v_{N,3}K^* & 0 \end{bmatrix},$$
$$\beta_0 = \begin{pmatrix} v_{K,6}\varepsilon \\ 0 \end{pmatrix},\ \beta_1 = \begin{pmatrix} v_{K,7}\varepsilon \\ v_{N,4}\varepsilon \end{pmatrix}.$$
- Previous is a Canonical Representation That Essentially All Log Linearized Models Can be Fit Into (See Christiano (2002).)

- Again, Look for Solution

$$z_t = Az_{t-1} + Bs_t,$$

where A and B are pinned down by log-linearized Equilibrium Conditions.

- Now, A is *Matrix* Eigenvalue of *Matrix* Polynomial:

$$\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0.$$

- Also, B Satisfies Same System of Log Linear Equations as Before:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0.$$

- Go for the 2 Free Elements of B Using 2 Equations Given by

$$F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Finding the Matrix Eigenvalue of the Polynomial Equation,

$$\alpha(A) = 0,$$

  and Determining if A is Unique is a Solved Problem.

- See Anderson, Gary S. and George Moore, 1985, 'A Linear Algebraic Procedure for Solving Linear Perfect Foresight Models,' *Economic Letters*, 17, 247-52 or Articles in Computational Economics, October, 2002. See also, the program, DYNARE.

- Solving for B
  - Given A, Solve for B Using Following (Log Linear) System of Equations:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

  - To See this, Use

$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1)\, vec(A_2),$$

  to Convert $F = 0$ Into

$$vec(F') = d + q\delta = 0,\ \delta = vec(B').$$

  - Find B By First Solving:

$$\delta = -q^{-1} d.$$

Example #4: Example #3 With ‘Exotic’ Information Set

- Suppose the Date t Investment Decision is Made Before the Current Realization of the Technology Shock, While the Hours Decision is Made Afterward.
- Now, Canonical Form Must Be Written Differently:

$$\mathcal{E}_t \left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t \right] \; = \; 0,$$

where

$$\mathcal{E}_t X_t = \left[\begin{array}{c} E\left[X_{1t} | \hat{\varepsilon}_{t-1} \right] \\ E\left[X_{2t} | \hat{\varepsilon}_t \right] \end{array} \right].$$

- Convenient to Change s_t:

$$s_t \; = \; \left(\begin{array}{c} \hat{\varepsilon}_t \\ \hat{\varepsilon}_{t-1} \end{array} \right), \; P = \left[\begin{array}{cc} \rho & 0 \\ 1 & 0 \end{array} \right], \; \epsilon_t = \left(\begin{array}{c} e_t \\ 0 \end{array} \right).$$

- Adjust β_i’s:

$$\beta_0 = \left(\begin{array}{cc} v_{K,6}\varepsilon & 0 \\ 0 & 0 \end{array} \right), \; \beta_1 = \left(\begin{array}{cc} v_{K,7}\varepsilon & 0 \\ v_{N,4}\varepsilon & 0 \end{array} \right),$$

- Posit Following Solution:

$$z_t = Az_{t-1} + Bs_t.$$

- Substitute Into Canonical Form:

$$\begin{aligned} &\mathcal{E}_t \left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] \\ = \ &\alpha(A) z_{t-1} + \mathcal{E}_t F s_t + \mathcal{E}_t \beta_0 \epsilon_{t+1} = \alpha(A) z_{t-1} + \mathcal{E}_t F s_t = 0, \end{aligned}$$

- Then,

$$\begin{aligned} \mathcal{E}_t F s_t \ &= \ \mathcal{E}_t \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} s_t = \mathcal{E}_t \begin{bmatrix} F_{11}\hat{\varepsilon}_t + F_{12}\hat{\varepsilon}_{t-1} \\ F_{21}\hat{\varepsilon}_t + F_{22}\hat{\varepsilon}_{t-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & F_{12} + \rho F_{11} \\ F_{21} & F_{22} \end{bmatrix} s_t = \tilde{F} s_t. \end{aligned}$$

- Equations to be solved:

$$\alpha(A) = 0, \ \tilde{F} = 0.$$

- $\tilde{F}$ Only Has *Three* Equations How Can We Solve for the Four Elements of B?
- Answer: Only *Three* Unknowns in B Because B Must Also Obey Information Structure:

$$B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Summary so Far

- Solving Models By Log Linear Approximation Involves Three Steps:
  a. Compute Steady State

  b. Log-Linearize Equilibrium Conditions

  c. Solve Log Linearized Equations.

- Step 3 Requires Finding A and B in:

$$z_t = Az_{t-1} + Bs_t,$$

to Satisfy Log-Linearized Equilibrium Conditions:

$$\mathcal{E}_t \left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right]$$
$$s_t = Ps_{t-1} + \epsilon_t,\ \epsilon_t \sim \text{ iid}$$

- We are Led to Choose A and B so that:

$$\begin{aligned} \alpha(A) &= 0, \\ \text{(standard information set) } F &= 0, \\ \text{(exotic information set) } \tilde{F} &= 0 \end{aligned}$$

and Eigenvalues of A are Less Than Unity In Absolute Value.

Example #5: A Sticky Price Model (Clarida-Gali-Gertler)

- Technology grows forever: equilibrium of model has no constant steady state.

- Deviations of the equilibrium from a particular benchmark does have a steady state.

  - Benchmark: best equilibrium achievable by monetary and fiscal policy

    - ‘Ramsey equilibrium’, ‘natural equilibrium’

  - Natural equilibrium is trivial to compute because of absence of capital.

  - Natural equilibrium supported by zero inflation monetary policy.

- Model is approximately log-linear around natural equilibrium allocations.

- Model:
  - Households choose consumption and labor.
  - Monopolistic firms produce and sell output using labor, subject to sticky prices
  - Monetary authority obeys a Taylor rule.

Household

- Preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}.$$

- Household efficiency conditions:

$$C_t^{-1} = \beta E_t C_{t+1}^{-1} R_t / \bar{\pi}_{t+1}, \ MRS_t = \exp(\tau_t) N_t^{\varphi} C_t = \frac{W_t}{P_t}.$$

- Take logs of intertemporal Euler equation:

$$\begin{aligned} -c_t &= \log \beta + r_t + \log \left[E_t C_{t+1}^{-1} / \bar{\pi}_{t+1} \right] \\ &= \log \beta + r_t + \log \left[E_t \exp(-c_{t+1} - \pi_{t+1}) \right] \\ &\simeq \log \beta + r_t + \log \left[\exp(-E_t c_{t+1} - E_t \pi_{t+1}) \right] \\ &= \log \beta + r_t - E_t c_{t+1} - E_t \pi_{t+1} \end{aligned}$$

- Note the approximation that was used here!

Household ...

- Household efficiency conditions (repeated):

$$C_t^{-1} = \beta E_t C_{t+1}^{-1} R_t / \bar{\pi}_{t+1},\ MRS_t = \exp\left(\tau_t\right) N_t^{\varphi} C_t = \frac{W_t}{P_t}.$$

- Log-linear approximation of intertemporal Euler equation:

$$c_t = -\left[r_t - E_t \pi_{t+1} - rr\right] + E_t c_{t+1}$$

$$rr \equiv -\log\beta,\ c_t \equiv \log C_t,\ \pi_t \equiv \log\bar{\pi}_t,\ r_t \equiv \log R_t$$

- Log of intratemporal Euler equation:

$$w_t - p_t = c_t + \varphi n_t + \tau_t \ (= \log MRS_t)$$

Firms

- Final Good Firms:
  - Buy $Y_t(i)$, $i \in [0,1]$ at prices $P_t(i)$ and sell Y_t at price P_t
  - Technology:

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \ \varepsilon \geq 1. \tag{1}$$

  - Demand for intermediate good (fonc for optimization of $Y_t(i)$):

$$Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} \tag{2}$$

  - Eqs (1) and (2) imply:

$$P_t = \left(\int_0^1 P_t(i)^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}} \tag{3}$$

Firms ...

- Intermediate Good Firms

  – Technology:

$$
\begin{aligned}
Y_t(i) &= A_t N_t(i), \quad a_t = \log A_t, \\
\Delta a_t &= \rho \Delta a_{t-1} + \varepsilon_t.
\end{aligned}
$$

  – Marginal cost of production for i^{th} firm (with subsidy, ν_t) :

$$
(1 - \nu_t) \frac{W_t}{A_t P_t}
$$

  – Calvo price-setting frictions:

    * A fraction, θ, of intermediate good firms cannot change price:

$$
P_i(t) = P_i(t-1)
$$

    * A fraction, $1 - \theta$, set price optimally:

$$
P_t(i) = \tilde{P}_t
$$

Best ('Natural') Equilibrium

- Cross-industry efficiency:

$$N_t\,(i) = N_t \text{ all } i$$

so that

$$Y_t = A_t N_t,\ y_t = a_t + n_t \tag{4}$$

- Labor efficiency (in logs):

$$\overbrace{c_t + \varphi n_t + \tau_t}^{\log\ MRS_t} = \overbrace{a_t}^{\log\ MP_{L,t}} \tag{5}$$

- Combine (4) and (5):

$$a = y_t + \varphi\,(y_t - a_t) + \tau_t$$

so that natural level of output is:

$$y_t^* = a_t - \frac{1}{1+\varphi}\tau_t$$

- Natural level of employment:

$$n_t^* = y_t^* - a_t = -\frac{1}{1+\varphi}\tau_t$$

Best ('Natural') Equilibrium ...

- Monetary and fiscal policy that supports optimal ('natural') allocations:
  - Monetary policy supports cross-industry efficiency by making $\tilde{P}$ constant, so that

$$P_t(i) = P_t = \tilde{P}$$

$$\pi_t = \frac{P_t}{P_{t-1}} = 1, \text{ all } t.$$

  - Fiscal policy supports labor efficiency by setting ν_t to eliminate distortions from monopoly power

- To determine 'natural' real interest rate, rr_t^*, substitute 'natural' output, y_t^* into household Euler equation:

$$\overbrace{\gamma + a_t - \frac{1}{1+\varphi}\tau_t}^{y_t^*} = -\left[rr_t^* - rr\right] + E_t \overbrace{\left(\gamma + a_{t+1} - \frac{1}{1+\varphi}\tau_{t+1}\right)}^{y_{t+1}^*}$$

or,

$$rr_t^* = rr + \rho \Delta a_t + \frac{1}{1+\varphi}(1-\lambda)\tau_t.$$

Best ('Natural') Equilibrium ...

- Natural rate:

$$rr_t^* = rr + \rho \Delta a_t + \frac{1}{1+\varphi} (1-\lambda) \tau_t.$$

  - Δa_t jumps
    - a_t will keep rising in future (if $\rho > 0$)
    - rise in c_t^* smaller than rise in c_{t+1}^*
    - people would like to use financial markets to smooth away from this
    - discourage this by having a high interest rate.

  - τ_t jumps
    - τ_t will be less high in the future (unless $\lambda > 1$)
    - c_t^* falls more than c_t^*
    - people want to smooth away
    - discourage this by having a high interest rate.

Equilibrium with Taylor Rule Monetary Policy

- Target interest rate, $\hat{r}_t$:

$$\hat{r}_t = \phi_\pi \pi_t + \phi_x x_t,\ x_t \equiv y_t - y_t^*.$$

- Actual interest rate, r_t :

$$\begin{aligned} r_t &= \alpha r_{t-1} + (1-\alpha)\,\hat{r}_t + u_t \\ u_t &= \delta u_{t-1} + \eta_t. \end{aligned}$$

- Policy rule:

$$r_t = \alpha r_{t-1} + u_t + (1-\alpha)\phi_\pi \pi_t + (1-\alpha)\phi_x x_t$$

Equilibrium with Taylor Rule Monetary Policy ...

- Intertemporal equation:

$$\text{Equilibrium} \; : \; y_t = -\left[r_t - E_t\pi_{t+1} - rr\right] + E_t y_{t+1}$$

$$\text{Benchmark} \; : \; y_t^* = -\left[rr_t^* - rr\right] + E_t y_{t+1}^*$$

- Subtract, to obtain ‘New Keynesian IS equation’:

$$x_t = -\left[r_t - E_t\pi_{t+1} - rr_t^*\right] + E_t x_{t+1}$$

- With staggered pricing, in presence of shocks $N_t(i)$ varies across i, so that:

$$y_t = \phi_t + n_t + a_t,\ \phi_t = \begin{cases} = 0 & \text{if } P_t(i) = P_t(j) \text{ for all } i, j \\ \leq 0 & \text{otherwise} \end{cases}.$$

- Along a nonstochastic steady state growth path, $\phi_t = 0$. In a small neighborhood of steady state (see Tak Jun, JME):

$$\phi_t \approx 0.$$

- Calvo reduced form inflation equation:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t,\ \kappa = \frac{(1-\theta)(1-\beta\theta)(1+\varphi)}{\theta}.$$

- We now have three equations (‘IS curve, Phillips curve and policy rule’) in three unknowns: π_t, r_t, x_t.

Equations of Taylor rule Equilibrium

$$\beta E_t \pi_{t+1} + \kappa x_t - \pi_t = 0 \text{ (Calvo pricing equation)}$$

$$-\left[r_t - E_t \pi_{t+1} - rr_t^*\right] + E_t x_{t+1} - x_t = 0 \text{ (intertemporal equation)}$$

$$\alpha r_{t-1} + u_t + (1-\alpha)\phi_\pi \pi_t + (1-\alpha)\phi_x x_t - r_t = 0 \text{ (policy rule)}$$

$$rr_t^* - \rho \Delta a_t - \frac{1}{1+\varphi}(1-\lambda)\tau_t = 0 \text{ (definition of natural rate)}$$

- Note:
  - Preference and technology shocks enter system through rr_t^*
  - Optimal equilibrium can be supported by setting nominal rate to natural rate:
  $$r_t = rr_t^*.$$
  - Practical issue: how to measure rr_t^*???

Solving the Sticky Price Model

- Exogenous shocks:

$$\begin{aligned} s_t &= \begin{pmatrix} \Delta a_t \\ u_t \\ \tau_t \end{pmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{pmatrix} \Delta a_{t-1} \\ u_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \varepsilon_t^{\tau} \end{pmatrix} \\ s_t &= P s_{t-1} + \epsilon_t \end{aligned}$$

- Equilibrium conditions:

$$\begin{bmatrix} \beta & 0 & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ r_{t+1} \\ rr^*_{t+1} \end{pmatrix} + \begin{bmatrix} -1 & \kappa & 0 & 0 \\ 0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} \\ (1-\alpha)\phi_\pi & (1-\alpha)\phi_x & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \pi_t \\ x_t \\ r_t \\ rr^*_t \end{pmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t-1} \\ x_{t-1} \\ r_{t-1} \\ rr^*_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} s_{t+1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma\psi\rho & 0 & -\frac{1}{\sigma+\varphi}(1-\lambda) \end{pmatrix} s_t$$

$$E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0$$

Solving the Sticky Price Model ...

- Collecting:

$$E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0$$

$$s_t - P s_{t-1} - \epsilon_t = 0.$$

- Solution:

$$z_t = A z_{t-1} + B s_t$$

- As before, want A such that

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0,$$

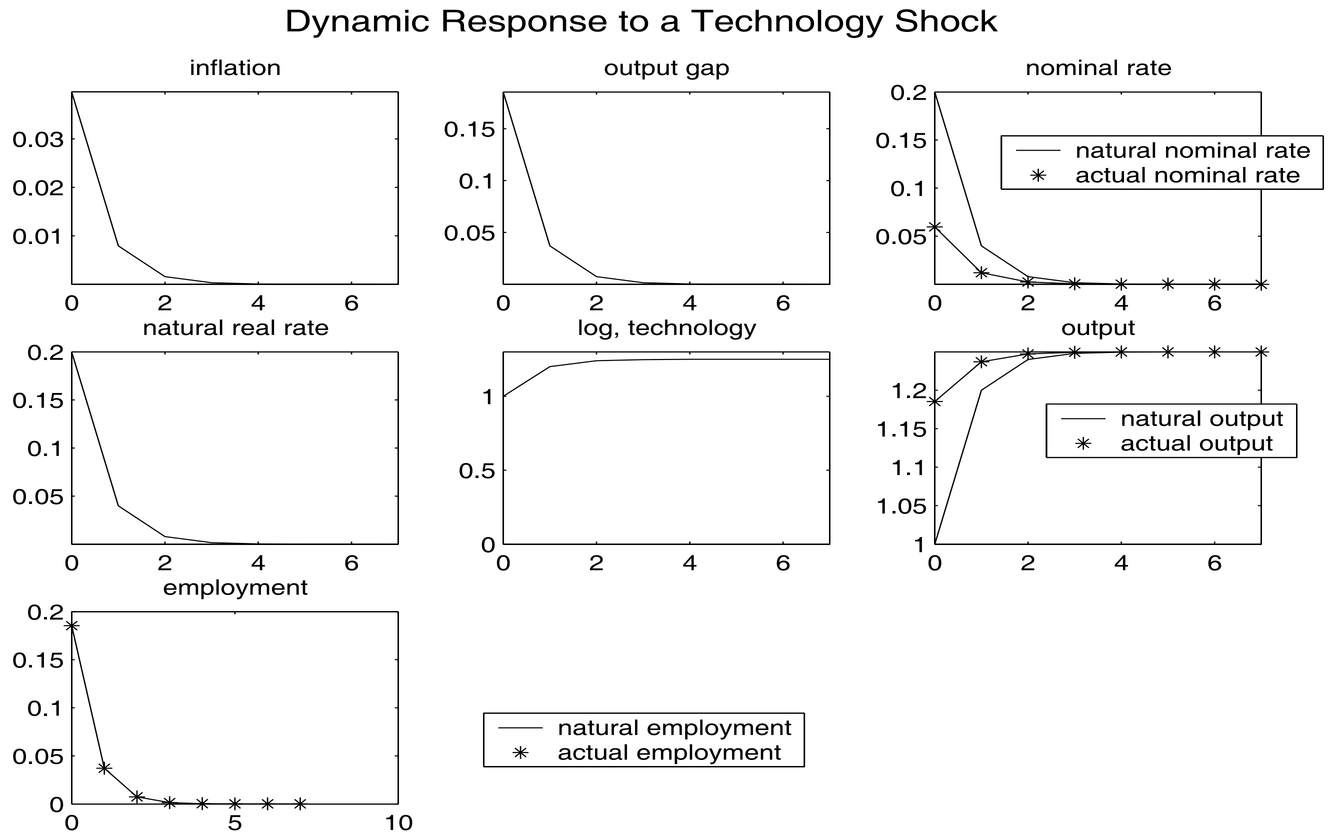
- Want B such that:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

- Note: if $\alpha = 0$, $A = 0$.

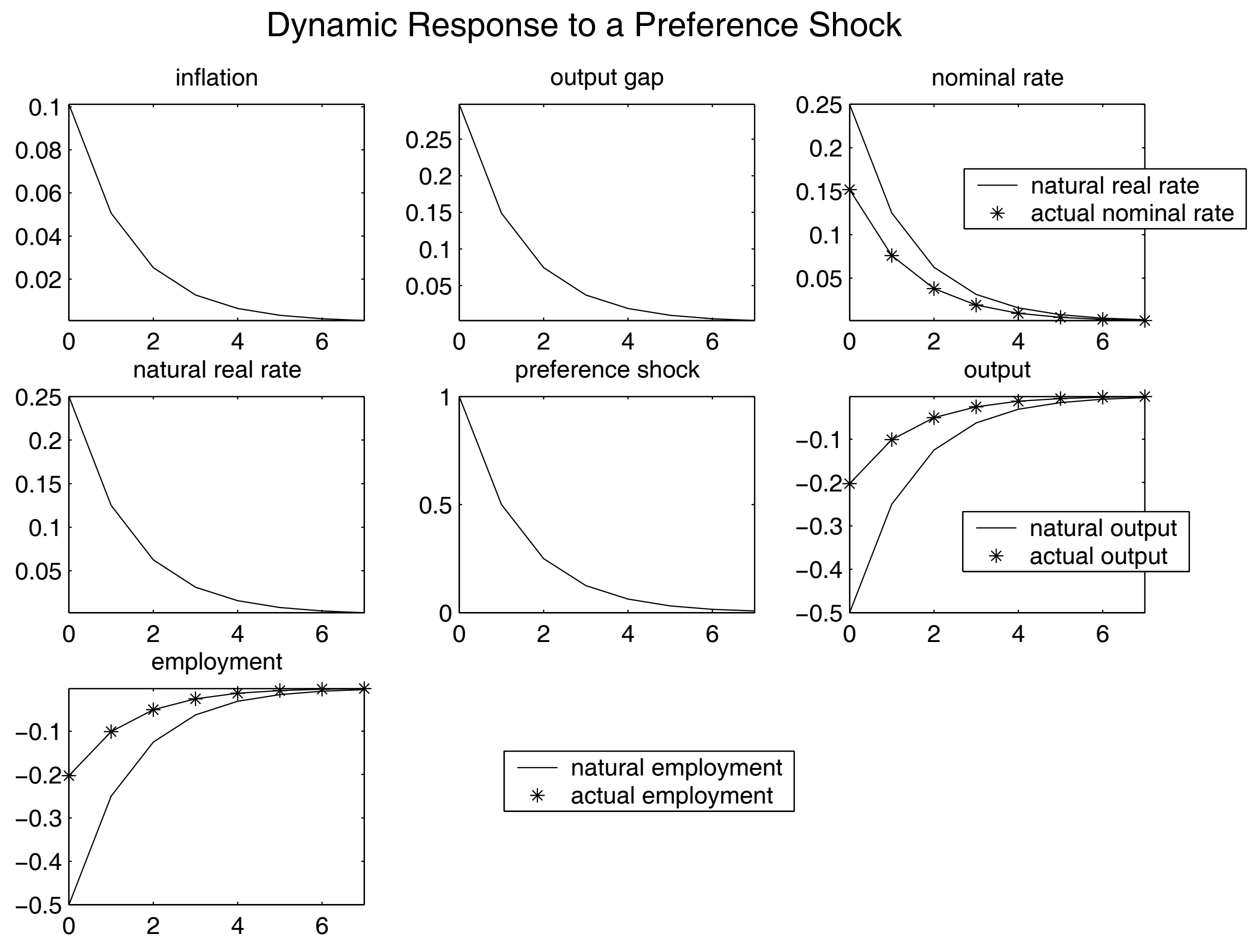
Examples with Sticky Price Model

$$\phi_x = 0,\ \phi_\pi = 1.5,\ \beta = 0.99,\ \varphi = 1,\ \rho = 0.2,\ \theta = 0.75,\ \alpha = 0,\ \delta = 0.2,\ \lambda = 0.5.$$



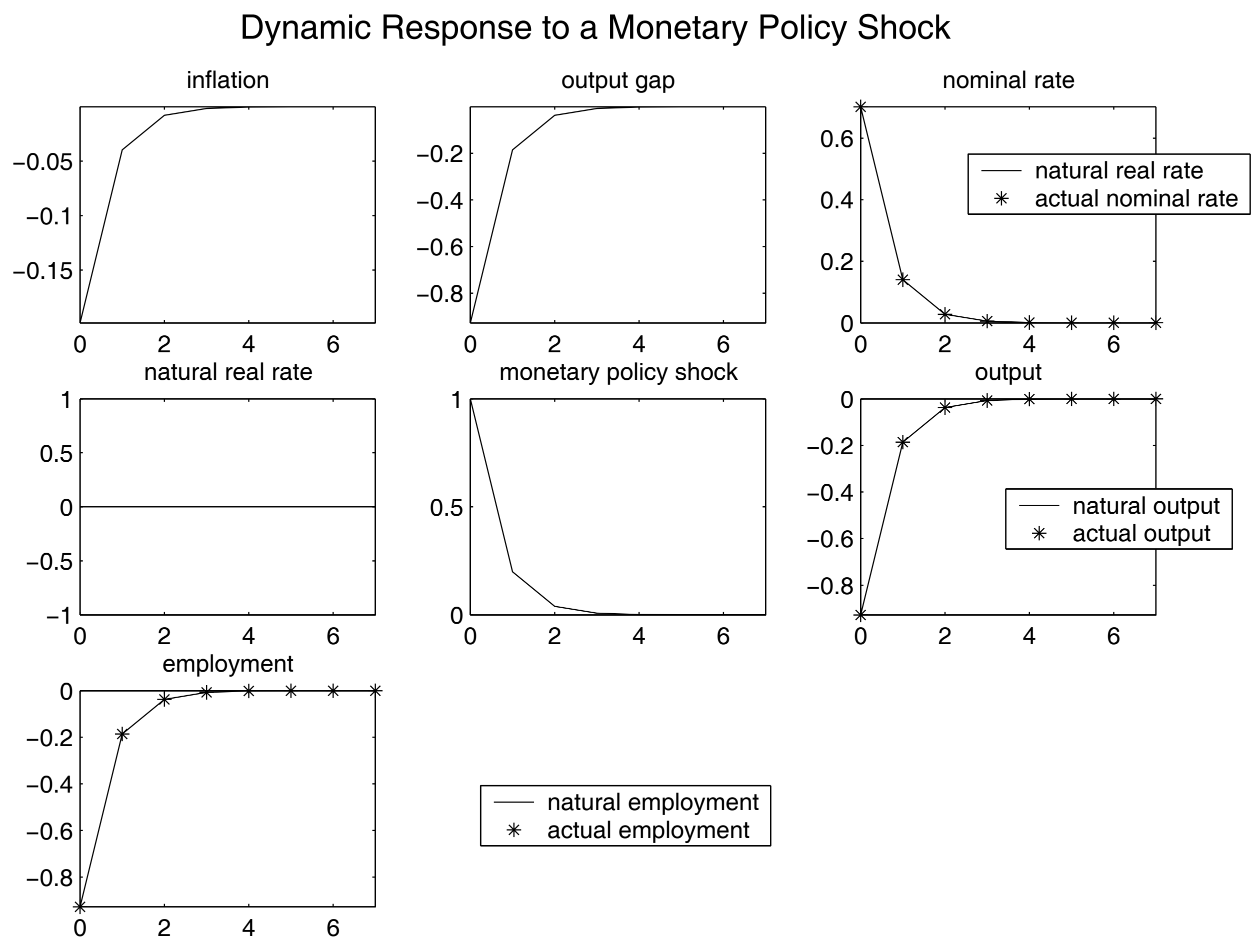
- Interest rate not increased enough, employment and inflation rise.

Examples with Sticky Price Model ...



- Under policy rule, interest rate not increased enough.
  - This encourages consumption above what is needed for the zero-inflation equilibrium.
  - The extra demand drives up output gap, inflation

Examples with Sticky Price Model ...



- Monetary policy shock drives up the interest rate
  - High interest rate discourages current consumption
  - Output, output gap and employment fall
  - Fall in costs causes inflation to drop.

Estimation of Model Parameters

- Limited information methods
  - Matching model impulse response functions with VAR model response functions
  - Rotemberg and Woodford, Christiano, Eichenbaum and Evans, others.
- Full information methods
  - Maximum likelihood
  - Bayesian maximum likelihood
- First: discuss state space-observer system.

State Space-Observer System

- Compact summary of the model, and of the data used in the analysis.
- Typically, data are available in log form. So, the following is useful:

  – If x is steady state of x_t :

$$\begin{aligned} \hat{x}_t &\equiv \frac{x_t - x}{x}, \\ &\Longrightarrow \frac{x_t}{x} = 1 + \hat{x}_t \\ &\Longrightarrow \log\left(\frac{x_t}{x}\right) = \log\left(1 + \hat{x}_t\right) \approx \hat{x}_t \end{aligned}$$

- Suppose we have a model solution in hand:

$$\begin{aligned} z_t &= Az_{t-1} + Bs_t \\ s_t &= Ps_{t-1} + \epsilon_t, \ E\epsilon_t\epsilon_t' = D. \end{aligned}$$

- Consider example #3, in which

$$z_t = \begin{pmatrix} \hat{K}_{t+1} \\ \hat{N}_t \end{pmatrix},\ s_t = \hat{\varepsilon}_t,\ \epsilon_t = e_t.$$

  Data used in analysis may include variables in z_t and/or other variables.

- Suppose variables of interest include employment and GDP.
  - GDP, y_t :

$$y_t = \varepsilon_t K_t^{\alpha} N_t^{1-\alpha},$$

    so that

$$\begin{aligned} \hat{y}_t &= \hat{\varepsilon}_t + \alpha \hat{K}_t + (1-\alpha)\hat{N}_t \\ &= \begin{pmatrix} 0 & 1-\alpha \end{pmatrix} z_t + \begin{pmatrix} \alpha & 0 \end{pmatrix} z_{t-1} + s_t \end{aligned}$$

  - Then,

$$Y_t^{data} = \begin{pmatrix} \log y_t \\ \log N_t \end{pmatrix} = \begin{pmatrix} \log y \\ \log N \end{pmatrix} + \begin{pmatrix} \hat{y}_t \\ \hat{N}_t \end{pmatrix}$$

- Model prediction for data:

$$
\begin{aligned}
Y_t^{data} &= \begin{pmatrix} \log y \\ \log N \end{pmatrix} + \begin{pmatrix} \hat{y}_t \\ \hat{N}_t \end{pmatrix} \\
&= \begin{pmatrix} \log y \\ \log N \end{pmatrix} + \begin{bmatrix} 0 & 1-\alpha \\ 0 & 1 \end{bmatrix} z_t + \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} z_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} s_t \\
&= a + H\xi_t
\end{aligned}
$$

$$
\xi_t = \begin{pmatrix} z_t \\ z_{t-1} \\ \hat{\varepsilon}_t \end{pmatrix}, \; a = \begin{bmatrix} \log y \\ \log N \end{bmatrix}, \; H = \begin{bmatrix} 0 & 1-\alpha & \alpha & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}
$$

- The *Observer Equation* may include measurement error, w_t :

$$
Y_t^{data} = a + H\xi_t + w_t, \; Ew_t w_t' = R.
$$

- Semantics: ξ_t is the *state* of the system (do not confuse with the economic state (K_t, ε_t)!).

- The state equation
  - Law of motion of the state, ξ_t

$$\xi_t = F\xi_{t-1} + u_t,\ Eu_t u_t' = Q$$

$$\begin{pmatrix} z_{t+1} \\ z_t \\ s_{t+1} \end{pmatrix} = \begin{bmatrix} A & 0 & BP \\ I & 0 & 0 \\ 0 & 0 & P \end{bmatrix} \begin{pmatrix} z_t \\ z_{t-1} \\ s_t \end{pmatrix} + \begin{pmatrix} B \\ 0 \\ I \end{pmatrix} \epsilon_{t+1},$$

$$u_t = \begin{pmatrix} B \\ 0 \\ I \end{pmatrix} \epsilon_t,\ Q = \begin{bmatrix} BDB' & 0 & BD \\ 0 & 0 & 0 \\ DB' & & D \end{bmatrix},\ F = \begin{bmatrix} A & 0 & BP \\ I & 0 & 0 \\ 0 & 0 & P \end{bmatrix}.$$

- Summary: State-Space, Observer System -

$$\xi_t \;=\; F\xi_{t-1} + u_t,\; Eu_t u_t' = Q,$$

$$Y_t^{data} \;=\; a + H\xi_t + w_t,\; Ew_t w_t' = R.$$

- Can be constructed from model parameters

$$\theta = (\beta, \delta, ...)$$

so

$$F = F(\theta),\; Q = Q(\theta),\; a = a(\theta),\; H = H(\theta),\; R = R(\theta).$$

- State space observer system very useful
  - Estimation of θ and forecasting ξ_t and Y_t^{data}
  - Can take into account situations in which data represent a mixture of quarterly, monthly, daily observations.
  - Software readily available on web and elsewhere.
  - Useful for solving the following forecasting problems:
    * Filtering:
    $$P\left[\xi_t | Y_{t-1}^{data}, Y_{t-2}^{data}, ..., Y_1^{data}\right],\ t = 1, 2, ..., T.$$
    * Smoothing:
    $$P\left[\xi_t | Y_T^{data}, ..., Y_1^{data}\right],\ t = 1, 2, ..., T.$$
    * Example: 'real rate of interest' and 'output gap' can be recovered from ξ_t using example #5.

Matching Impulse Response Functions

- Set $\epsilon_t = 1$ for $t = 1$, $\epsilon_t = 0$ otherwise

- Impulse response function: log deviation of data with shock from where data would have been in the absence of a shock -

$$
\begin{aligned}
u_t &= \begin{pmatrix} B \\ 0 \\ I \end{pmatrix} \epsilon_t, \\
\xi_t &= F\xi_{t-1} + u_t, \ \xi_0 = 0,
\end{aligned}
$$

$$
\text{impulse response function} \implies \tilde{Y}_t^{data} = H\xi_t, \ \text{for } t = 1, 2, ...
$$

- Choose model parameters, θ, to match $\tilde{Y}_t^{data}$ with corresponding estimate from VAR.

Maximum Likelihood Estimation

- State space-observer system:

$$\xi_{t+1} = F\xi_t + u_{t+1},\ Eu_t u_t' = Q,$$

$$Y_t^{data} = a_0 + H\xi_t + w_t,\ Ew_t w_t' = R$$

- Parameters of system: (F, Q, a_0, H, R). These are functions of model parameters, θ.

- Formulas for computing likelihood

$$P\left(Y^{data}|\theta\right) = P\left(Y_1^{data}, ..., Y_T^{data}|\theta\right).$$

are standard (see Hamilton's textbook).

Bayesian Maximum Likelihood

- Bayesians describe the mapping from prior beliefs about θ, summarized in $p(\theta)$, to new posterior beliefs in the light of observing the data, Y^{data}.

- General property of probabilities:

$$p\left(Y^{data},\theta\right)=\begin{cases} p\left(Y^{data}|\theta\right)\times p\left(\theta\right) \\ p\left(\theta|Y^{data}\right)\times p\left(Y^{data}\right) \end{cases},$$

  which implies Bayes' rule:

$$p\left(\theta|Y^{data}\right)=\frac{p\left(Y^{data}|\theta\right)p\left(\theta\right)}{p\left(Y^{data}\right)},$$

  mapping from prior to posterior induced by Y^{data}.

- Properties of the posterior distribution, $p\left(\theta|Y^{data}\right)$.
  - The value of θ that maximizes $p\left(\theta|Y^{data}\right)$ ('mode' of posterior distribution).
  - Graphs that compare the marginal posterior distribution of individual elements of θ with the corresponding prior.
  - Probability intervals about the mode of θ ('Bayesian confidence intervals')
  - Other properties of $p\left(\theta|Y^{data}\right)$ helpful for assessing model 'fit'.

- Computation of mode sometimes referred to as 'Basyesian maximum likelihood':

$$\theta^{\text{mod}\,e} = \arg\max_{\theta} \left\{ \log \left[p\left(Y^{data} | \theta \right) \right] + \sum_{i=1}^{N} \log \left[p_i \left(\theta_i \right) \right] \right\}$$

  maximum likelihood with a penalty function.

- Shape of posterior distribution, $p\left(\theta | Y^{data}\right)$, obtained by Metropolis-Hastings algorithm.
  - Algorithm computes

$$\theta(1), ..., \theta(N),$$

    which, as $N \to \infty$, has a density that approximates $p\left(\theta | Y^{data}\right)$ well.

  - Marginal posterior distribution of any element of θ displayed as the histogram of the corresponding element $\{\theta(i), i = 1, .., N\}$

Fernandez-Villaverde, Rubio-Ramirez, Sargent Result

- Use the state space, observer representation to derive DSGE implication for VAR.

- System (ignoring constant terms and measurement error):

$$\xi_t = F\xi_{t-1} + D\epsilon_t,\ D = \begin{pmatrix} B \\ 0 \\ I \end{pmatrix},$$

$$Y_t = H\xi_t.$$

- Substituting:

$$Y_t = HF\xi_{t-1} + HD\epsilon_t$$

- Suppose HD is square and invertible. Then

$$\epsilon_t = (HD)^{-1} Y_t - (HD)^{-1} HF\xi_{t-1}.$$

- Substitute latter into the state equation:

$$\begin{aligned} \xi_t &= F\xi_{t-1} + D(HD)^{-1} Y_t - D(HD)^{-1} HF\xi_{t-1} \\ &= \left[I - D(HD)^{-1} H\right] F\xi_{t-1} + D(HD)^{-1} Y_t, \end{aligned}$$

so

$$\xi_t = M\xi_{t-1} + D(HD)^{-1} Y_t,\ M = \left[I - D(HD)^{-1} H\right] F.$$

- If eigenvalues of M are less than unity,

$$\xi_t = D(HD)^{-1} Y_t + MD(HD)^{-1} Y_{t-1} + M^2 D(HD)^{-1} Y_{t-2} + \ldots$$

- Then,

Fernandez-Villaverde, Rubio-Ramirez, Sargent Result ...

$$\begin{aligned}\epsilon_t &= (HD)^{-1} Y_t \\ &\quad - (HD)^{-1} HF \left[D(HD)^{-1} Y_{t-1} + MD(HD)^{-1} Y_{t-2} + M^2 D(HD)^{-1} Y_{t-3} + ... \right]\end{aligned}$$

or,

$$Y_t = B_1 Y_{t-1} + B_2 Y_{t-2} + ... + u_t,$$

where

$$\begin{aligned}u_t &= HD\epsilon_t \\ B_j &= HFM^{j-1} D(HD)^{-1},\ j = 1, 2, ...\end{aligned}$$

- The latter is the VAR representation.
  - Note: ϵ_t is ‘invertible’ because it lies in space of current and past Y_t’s
  - Note: VAR is infinite-ordered.