Homework #9

Economics D11-1

First three questions due Monday, November 30. Last question due Friday, December 4.

Christiano

1. (Dynamic Inefficiency in OG Models). Consider the overlapping generations model in which the utility of the generation born at t is

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t).$$

The young supply one unit of labor inelastically in period zero, and earn the competitive wage rate, w_t. They use their income to purchase the outstanding stock of capital, and when old they finance their consumption from the earnings of the accumulated capital. Thus, their budget constraint is

$$c_t^t + k_{t+1} \leq w_t, \; c_{t+1}^t \leq r_{t+1} k_{t+1}.$$

Note that capital depreciates completely in one period. Firms are competitive in the output market and hire capital and labor in competitive factor markets where the prices are r_t and w_t, respectively.

   (a) Define a sequence of markets equilibrium. Provide expressions for w_t and r_t in terms of k_t.

   (b) Consider a steady state equilibrium in which the aggregate stock of capital, the consumption of each period's young, and the consumption of each period's old are all constants. Time starts up in period 0, with the initial old generation owning the capital stock, which they sell to the period 0 young. Show that the equilibrium rate of return on capital is

$$r_{k,t} = \frac{\alpha}{1+\alpha} \frac{1+\beta}{\beta}, \text{ for all } t.$$

   Interpret this expression. Why is the interest rate infinite if $\beta = 0$? Why is it zero if $\alpha = 0$?

(c) Show that, for parameter values where $r_{k,t} < 1$, the competitive equilibrium is inefficient. That is, prove the following: it is possible to deviate from the equilibiurm consumption allocations by reallocating consumption between each period's old and the same period's young in a way that is compatible with the resource constraint and which makes everyone (i.e., the first generation, the second generation, the third, etc.) better off. How might the result be affected if there were a last date in the economy?

2. (Recent Jones-Manuelli manuscript, exploring links between the economy's growth rate and the volatility of its business cycle fluctuations.) Consider an economy in which the representative agent has the following preferences:

$$E\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma}\frac{(1-n_t)^{1+\psi}}{1+\psi},\ \psi > 0,\ \sigma \neq 1,$$

where σ and ψ are such that the utility function is concave, which positive marginal utility to consumption, c_t, and negative marginal utility to labor effort, n_t. Consumption is restricted to be non-negative, and $0 \leq n_t \leq 1$. Output, y_t, is produced as follows:

$$y_t = s_t A k_t n_t^{1-\alpha},$$

where k_t is the beginning-of-period t stock of capital. Also, s_t is an exogenous shock to productivity, with

$$\begin{aligned} s_t &= 1+\varepsilon \text{ with probability } 1/2 \\ &= 1-\varepsilon \text{ with probability } 1/2. \end{aligned}$$

The shock is independently distributed over time. Capital depreciates completely in one period, so that the aggregate resource constraint is:

$$c_t + k_{t+1} \leq y_t.$$

The efficient allocations are sequences of consumption, labor and capital that maximize utility subject to the various technology constraints. When $\sigma < 1$, we suppose that:

$$\beta A^{1-\sigma} < 1.$$

When $\sigma > 1$, we suppose that $k_{t+1} = k_t$ is feasible.

(a) Let $\varepsilon = 0$, so that there is no uncertainty. Let $v(k_0)$ denote the maximized value of discounted utility under the efficient allocations. Explain clearly why the following statements are true:

   i. $v(k_0) = k_0^{(1-\sigma)} w$, where $-\infty < w < \infty$. (Hint: scale the economy using the capital stock.)

   ii. $k_{t+1} = \lambda k_t$, and $n_t = n$, constant.

(b) Let $\varepsilon > 0$. Write out the Euler equations for n_t and k_{t+1}. Verify that these are satisfied for $k_{t+1} = \phi y_t$ and $n_t = n$ for $t = 0, 1, \ldots$.

(c) Derive a simple expression for y_{t+1}/y_t in terms of s_{t+1} and ϕ.

(d) Let $\sigma > 1$ and suppose there is an increase in ε. Use the Euler equations to predict what happens to n, ϕ, and the average growth rate of the economy with this increase in uncertainty. What is the intuition behind the result?

3. The purpose of this question is to give you a 'hands on' acquaintance with linearization as a strategy for solving a model.

   Consider a model in which the preferences of the representative agent are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \; u(c,n) = \frac{\left[c\,(1-n)^{\psi}\right]^{1-\sigma}}{1-\sigma}, \; \beta = 1.03^{-.25},$$

   where $\psi, \sigma > 0$, and satisfy the other restrictions needed for strict concavity (see the handout on bringing hours worked into the neoclassical growth model.) Suppose the resource constraint is given by

$$c_t + k_{t+1} - (1-\delta)k_t \leq k_t^{\alpha}\left[z_t \exp(s_t) n_t\right]^{1-\alpha} \equiv f(k_t, n_t, z_t, s_t),$$

   where $\delta = 0.012$, $\alpha = 0.36$, $z_t = \exp(0.015) z_{t-1}$, and s_t has the following statistical representation:

$$s_t = 0.95 s_{t-1} + \varepsilon_t,$$

   where ε_t is a zero mean random variable that is independently distributed over time. Note that the mean of s_t is zero. Here, $k_0 > 0$ is given and the restrictions are n_t, k_{t+1}, $1 - n_t$, $c_t \geq 0$, for $t = 0, 1, 2, \ldots$.

(a) Compute the nonstochastic steady state values of n, c/y, k/y, k/z_{-1}, c/z, y/z.

(b) Compute u_c, u_{cc}, u_{cn}, u_n, u_{nn}, $F = y + (1-\delta)k$, F_k, F_{kk}, F_n, F_{nn}, F_z, F_{nz}, F_{kz}. These are the derivatives of the various functions, evaluated in nonstochastic steady state. Here, F is $f + (1-\delta)k$, where f is the function defined above. Also it is convenient to define the function $w(c,n) = -u_n(c,n)/u_c(c,n)$. Compute w, w_c, w_n. Again, these are evaluated in the nonstochastic steady state.

(c) Write the Euler equations as:

$$\begin{aligned} E\left[R(\tilde{k}_t, \tilde{k}_{t+1}, \tilde{k}_{t+2}, n_t, n_{t+1}, s_t, s_{t+1})|s_t\right] &= 0 \\ h(\tilde{k}_t, \tilde{k}_{t+1}, n_t, s_t) &= 0, \end{aligned}$$

where $\tilde{k}_{t+1} = k_{t+1}/z_t$. Obtain the coefficients in the first order Taylor series expansion of R and h about the non-stochastic, steady state values of the arguments. Use the linearized h to solve for n_t in terms of $\tilde{k}_t$, $\tilde{k}_{t+1}$, s_t. Use this expression to substitute out for n_t, n_{t+1} in the linearized version of R. The resulting expression can be written:

$$E\left[g_k(\tilde{k}_t - \tilde{k}) + g_{k'}(\tilde{k}_{t+1} - \tilde{k}) + g_{k''}(\tilde{k}_{t+2} - \tilde{k}) + g_s s_t + g_{s'} s_{t+1}|s_t\right] = 0.$$

Report the values for g_i, $i = \tilde{k}, \tilde{k}', \tilde{k}'', s, s'$. Here, $\tilde{k}$ is the nonstochastic steady state value of the scaled capital stock.

(d) Posit the following policy rule for capital:

$$\tilde{k}_{t+1} = \tilde{k} + \alpha_0(\tilde{k}_t - \tilde{k}) + \alpha_1 s_t.$$

Find the values of $|\alpha_0| < 1$ and α_1 which solve the linearized Euler equation. Using the expression for n_t in terms of $\tilde{k}_t$, $\tilde{k}_{t+1}$, s_t and the policy rule for capital to find β_0 β_1 in:

$$n_t = n + \beta_0(\tilde{k}_t - \tilde{k}) + \beta_1 s_t.$$

(e) Suppose $\varepsilon_1 = 1$, and $\varepsilon_t = 0$ for $t \neq 1$. Set $s_0 = 1$ and compute s_1, s_2, s_3, s_4. Set k_1 equal to steady state k/z_{-1}, and set z_{-1}. Compute c_t, n_t, I_t, for $t = 1, 2, 3, 4$, and calculate the percent difference between these and what they would have been if ε_t had been held constant at zero.

4. Suppose a typical household solves

$$\max_{c,n} u(c,n), \text{ subject to } c \leq (1-\tau)wn, \ c,n \geq 0,$$

where w is the wage rate, τ is the labor tax rate, n is hours worked, and c is consumption. Suppose the production function is $f(n) = n$, so that the equilibrium wage rate is $w = 1$. Also, suppose the utility function has the following form:

$$u(c,n) = c - \frac{n^2}{2}$$

Suppose there is a government which has to finance a fixed level of expenditures, $g \leq \frac{1}{4}$, which are tossed into the ocean. The government's task is to choose $0 \leq \tau \leq 1$ to maximize the utility of the representative agent, subject to its budget constraint:

$$g \leq \tau wn.$$

(a) Suppose the government commits itself to a value for τ before the households make their decisions (i.e., it solves the 'Ramsey problem'). What is the set of τ consistent with the government satisfying its budget constraint? What range of household utilities is associated with the elements in this set? What is the tax rate and utility level that solves the Ramsey problem?

(b) Suppose the government selects a value for τ after households have committed themselves to a level of employment. How many sustainable equilibria are there? What are the associated utility levels?

(c) Replace the government budget constraint by $g = T$, where T represents a lump-sum tax. Also, replace the household's budget constraint by

$$c \leq wn - T.$$

What is the welfare gain from going to the lump sum tax economy from the Ramsey problem?