

Notes on a Model With Calvo-Sticky Prices, Wages and Portfolios
 Christiano, Eichenbaum and Evans

1. Households

There is a continuum of households, indexed by $h \in (0, 1)$. The h^{th} household's preferences are:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t^h) + f(L_t^h) + v\left(\frac{Q_t^h}{P_t}\right) \right],$$

where c_t^h and L_t^h denote the h^{th} household's levels of consumption and market work effort, respectively. Also, Q_t^h/P_t is the real value of the part of its financial wealth that the household chooses to hold in non-interest bearing form. The household faces two forms of uncertainty. There is aggregate uncertainty that stems from uncertainty in the growth rate, μ_t , of nominal money balances:

$$\mu_t = \frac{\bar{M}_{t+1}}{\bar{M}_t},$$

where $\bar{M}_t$ is the beginning of period aggregate, per capita, stock of money. In addition, the household faces two forms of idiosyncratic uncertainty. Being a monopoly supplier of its own labor, it sets its wage rate. However, it can only adjust its wage at exogenously and randomly determined times. Similarly, the household can only adjust Q_t^h at exogenously determined, random times. In modeling this, we follow Calvo. Thus, we seek to understand the consequences of frictions in the adjustment of these variables, without modeling explicitly what the source of those frictions is. A complete analysis would of course do so. Such an analysis would recognize that the timing of household Q and wage decisions is endogenous. We hope that the outcome of our analysis is not distorted too much by our decision not to model this. In any case, our modeling choices are made by a desire to study, in an integrated framework, the operation of a variety of frictions studied in the literature. The Calvo approach has become the standard for modeling frictions in decisions.

We further restrict the analysis by making assumptions which guarantee that the frictions do not cause households to become heterogeneous in ways that are analytically difficult to model and (hopefully!) not important from the point of view of the phenomena we seek to understand. We do this by allowing households to enter into insurance markets against the outcomes of these frictions. Again, here we follow standard practice.

The timing of household decisions is like this. At the beginning of the period, before any uncertainty is realized, the households purchase consumption insurance. Then it is determined whether households can adjust their Q and their wage. Households who can

adjust Q do so before the current period realization of μ . Households who can adjust their wage do so after the current realization of μ . Households also make an investment decision. This is made before the realization of all uncertainty.

1.1. Consumption Insurance

Insurance markets open at the beginning of the period, before that period's uncertainty has been realized. Let $s = (s^q, s^w)$, where $s^q = 1$ indicates the household can change the current period value of Q , and $s^q = 0$ indicates that it cannot. Also, s^w indicates whether it can or cannot change its posted nominal wage rate. We assume that the s_t 's are independent over time, with each other, and with all other random variables, with probability function, $\phi(s)$. We let

$$\xi_l = \text{prob}[s_t^l = 0], \quad l = q, w.$$

We imagine that insurance against h is offered by perfectly competitive insurance companies. For each possible value of $\mu = \bar{\mu}$ and $s = \bar{s}$, there is a representative, competitive insurance company. This company sells contracts which pay \$1 if $\bar{\mu}$ and $\bar{s}$ are realized and nothing otherwise. The quantity of such contracts purchased by the h^{th} household is $B^h(\bar{s}, \bar{\mu})$. Regardless of the value of s that is realized by the h^{th} household, the company charges the h^{th} household a premium, $\delta(\bar{s}, \bar{\mu})B^h(\bar{s}, \bar{\mu})$ if $\mu = \bar{\mu}$ and zero for all other values of μ . We work with symmetric equilibria, in which all households buy the same amount of insurance, i.e.,

$$B^h(\bar{s}, \bar{\mu}) = B^j(\bar{s}, \bar{\mu}) = B(\bar{s}, \bar{\mu}) \text{ for all } h, j,$$

each $\bar{s}, \bar{\mu}$.¹ Then, the payout of the $\bar{s}, \bar{\mu}$ insurance company is

$$\begin{aligned} \phi(\bar{s})B(\bar{s}, \bar{\mu}), \quad \mu &= \bar{\mu} \\ 0, \quad \mu &\neq \bar{\mu}. \end{aligned}$$

The insurance company receives

$$\begin{aligned} \delta(\bar{s}, \bar{\mu})B(\bar{s}, \bar{\mu}) \text{ if } \mu &= \bar{\mu}, \\ 0 \text{ if } \mu &\neq \bar{\mu}. \end{aligned}$$

The zero profit condition on the insurance company is equivalent with

$$\delta(s, \mu) = \phi(s), \text{ for all } \mu.$$

¹Below, we verify that a symmetric equilibrium actually exists.

1.2. The Household's Decisions

At the beginning of the period, before s or μ are realized, the household makes its decision about $B^h(s, \mu)$. Then, s is realized. If the realization of s indicates that the household may change Q^h , then it does so. Otherwise, it goes with $Q_{-1}^h \bar{\mu}$, where Q_{-1}^h denotes the previous period's value of Q^h and $\bar{\mu}$ is the mean money growth rate. Either way, it splits up M^h into two components:

$$M^h - Q^h, \quad Q^h,$$

where $M^h - Q^h$ goes to the financial intermediary, M^h is the h^{th} household's beginning of period money balances and Q^h is sent to the consumption goods market. It now goes to the goods market

The asset evolution equation of the h^{th} household conditional on a realization of μ and s is:

$$\begin{aligned} M^{h,\prime}(\mu, s) = & R(\mu) \left[M^h - Q^h(h) + (\bar{\mu} - 1)\bar{M} \right] + D(\mu) \\ & + B^h(s, \mu) - \sum_{\tilde{s}} \delta(\tilde{s}, \mu) B^h(\tilde{s}, \mu) \\ & + \left[Q^h(s) + W^h(s, \mu) L^h(s, \mu) - P(\mu) \left(c^h(s, \mu) + k^{h'} - (1 - \delta)k^h \right) \right] + R^k(\mu)k^h, \end{aligned}$$

where $D(\mu)$ denotes dividends received from firms and $\bar{M}$ is the economy-wide stock of money, W^i is the wage rate and P is the price level. In each case, we use the notation to indicate which shock the variable depends on.

We now consider the Euler equation for $B^h(s, \mu)$. This Euler equation allows us to prove that u_c is independent of the realization of s . Fix s_0, μ_0 at $\bar{s}_0, \bar{\mu}_0$. Now, differentiate the criterion with respect to $B_0^i(\bar{h}_0, \bar{\mu}_0)$:

$$\pi_0(\bar{\mu}_0)\phi(\bar{s}_0)\nu_0(\bar{s}_0, \bar{\mu}_0) - \delta(\bar{s}_0, \bar{\mu}_0)\pi_0(\bar{\mu}_0) \sum_{s_0} \phi(s_0)\nu_0(s_0, \bar{\mu}_0) = 0.$$

Here, $\nu_0(\bar{s}_0, \bar{\mu}_0)$ is the marginal value of an extra dollar at the end of period 0 conditional on the realized value of s and of money growth being as indicated. That is, it is the multiplier on the household's period 0 budget constraint in the Lagrangian formulation of its problem. Also, $\pi_0(\bar{\mu}_0)$ is the probability that the realization of money growth is $\bar{\mu}_0$ at time zero. We have exploited the fact that $B_0^i(\bar{s}_0, \bar{\mu}_0)$ appears as a cost in each state of nature, while it appears as a benefit in only the state of nature, $\bar{s}_0, \bar{\mu}_0$. Note that $\pi_0(\bar{\mu}_0)$ appears on both sides of the minus sign, so it can be cancelled out. Substituting this and the zero-profit condition for insurance companies into this expression,

$$v_0(\bar{s}_0, \bar{\mu}_0) = \sum_{s_0} \phi(s_0)\nu_0(s_0, \bar{\mu}_0).$$

Since the right hand side is only a function of $\bar{\mu}_0$, it follows that the left hand side is only a function of $\bar{\mu}_0$ too. In particular, it is not a function of $\bar{s}_0$:

$$v_0(s_0, \mu_0) = v_0(\mu_0), \text{ for all } s_0, \mu_0. \quad (1.1)$$

Under additive separability, this implies that consumption is independent of s_0 . (It is easy to verify from the Lagrangian representation of the household's problem that $u_c/P = v$.)

There is a simple (standard) explanation of why u_c is independent of the realization of s . Optimization by the household implies that it equates the marginal rate of substitution in consumption between two different s states of nature to the associated relative price. The marginal utility of consumption is the probability of the realization of s , times u_c for that state. At the same time, the price of consumption in that state is the probability of that state. Equating marginal rate of substitution to relative price there is a cancellation of the probability and price terms, leaving the implication that $u_c(s')/u_c(s) = 1$.

We now differentiate the household's Lagrangian problem with respect to $M_{t+1}^i(s_t, \mu_t)$. The notation here reflects that you can pick end-of-period t money, M_{t+1}^i , contingent on the realized period t uncertainty. Fix $\bar{s}_t, \bar{\mu}_t$ and differentiate with respect to $M_{t+1}^i(\bar{s}_t, \bar{\mu}_t)$:

$$\frac{u_{c,t}}{P_t} = \beta E_t \frac{u_{c,t+1}}{P_{t+1}} R_{t+1}. \quad (1.2)$$

We can derive this equation using a variational argument based on acquisition of state-contingent insurance. Thus, suppose the household buys one extra unit of insurance which pays off in state $\bar{s}_t, \bar{\mu}_t$. That is, $B_t^h(\bar{s}_t, \bar{\mu}_t)$ is increased by 1. The cost of this is $\delta(\bar{s}_t, \bar{\mu}_t)$, to be paid at the end of the period if $\mu_t = \bar{\mu}_t$, regardless of the value of s_t . In utility terms this is $\delta(\bar{s}_t, \bar{\mu}_t) \left[\pi_t(\bar{\mu}_t) \sum_{s_t} \phi(s_t) \frac{u_{c,t}}{P_t} \right] = \delta(\bar{s}_t, \bar{\mu}_t) \pi_t(\bar{\mu}_t) u_{c,t}/P_t$. The benefit is that the extra dollar of insurance payoff that occurs in state $\bar{s}_t, \bar{\mu}_t$ can be invested in the next period's financial market for a return of R_{t+1} per dollar invested, which is valued in utility terms in the amount, $\pi_t(\bar{\mu}_t) \phi(\bar{s}_t) E_t u_{c,t+1} R_{t+1}/P_{t+1}$. The result follows from cancelling $\pi_t(\bar{\mu}_t)$ in the cost and benefit, as well as from the observation (proved above) that $\delta(\bar{s}_t, \bar{\mu}_t) = \phi(\bar{s}_t)$.

There is a different way to gain intuition into (1.2), based on thinking of consumption as a credit good. Suppose you have a dollar at the end of period t , and you're thinking about it at the beginning of period t . You could use the dollar to purchase $1/P_t$ goods today. Or, you could invest the dollar in the morning of period $t+1$, earn R_{t+1} dollars by then end of period $t+1$ and then use those dollars to finance the purchase of R_{t+1}/P_{t+1} consumption goods during $t+1$.

The Euler equation for Q is:

$$E_t \sum_{j=0}^{\infty} (\beta \xi_q)^j \left\{ v' \left(\frac{\tilde{Q}_t \bar{\pi}^j}{P_t \pi_{t+1} \cdot \pi_{t+j}} \right) - u_{c,t+j} [R_{t+j} - 1] \right\} \frac{\bar{\pi}^j}{\pi_{t+1} \cdot \pi_{t+j}} = 0, \quad (1.3)$$

where $\tilde{Q}_t$ is the value of Q chosen by households who have the ability to choose at time t . It is understood that when $j = 0$, $\pi_{t+1} \cdot \pi_{t+j} = 1$. Households that cannot change Q must set it as follows:

$$Q_t = \bar{\pi} Q_{t-1}.$$

This Euler equation has a simple interpretation. Consider first the case where $\xi_q = 0$ and there is no uncertainty at all. In this case, the Euler equation says that you want to equate the marginal utility of an extra dollar spent on Q (this operates via v and via the additional money this gives you to augment your money next period) with the marginal utility of the extra dollars you would have at the end of the period if you had instead deposited the money with the financial intermediary. With uncertainty about s^w and μ , then this first order condition holds in expectation. Now, let $\xi_q > 0$. Then the marginal dollar in q is committed probabilistically over many periods, and the Euler equation reflects this.

To understand this, remember that consumption is a credit good in this model. So, the gross earnings of a dollar in the bank during period 0 can be spent in the period 0 goods market. That's one thing you can do with a dollar. That's on the right side of the equation. Alternatively, you can hang the dollar on your rear view mirror and enjoy utility, while spending it at the same time. That's what's on the left side of the equation. Notice that when u_c is high, then q will be low (because $v'' < 0$). That's because if you care a lot about consumption, you'll maximize that by keeping your money in the bank so as to get as much money as possible. This seems like an unhappy feature of the model. It really draws attention to the fact that this is not a transactions-based model of money demand. If you really want to consume a lot, you'll take that cash off the rear view mirror and stick it in the bank. Having the cash on the rear view mirror is interfering with your ability to enjoy consumption. In a transactions-based model, the more cash you hold on, the easier it is to have consumption. Interestingly, the sign of the income elasticity of q demand is 'right'. That is, the higher the level of consumption, the greater will be q . That's because, with a higher consumption the marginal utility of consumption is lower, so you don't mind having the cash hanging on your rear view mirror.

The household is a monopoly supplier of its own (differentiated) labor. It sells labor to a firm which transforms household labor into a homogeneous input good, X . That producer's production function is:

$$X = \left[\int_0^1 (L^h)^{\frac{1}{\lambda_w}} dh \right]^{\lambda_w}, \quad 1 \leq \lambda_w < \infty. \quad (1.4)$$

The producer's problem is to maximize (1.4) subject to:

$$W_t X_t - \int_0^1 W_t^h L_t^h dh,$$

where W_t is the price of X_t . The first order condition associated with this problem, which is the household's demand for labor curve, is:

$$L_t^h = \left[\frac{W_t^h}{W_t} \right]^{\frac{\lambda_w}{1-\lambda_w}} X_t. \quad (1.5)$$

The household's Euler equation for setting W_t^h , when it is allowed to do so, is:

$$0 = E_{t,w} \sum_{j=0}^{\infty} (\xi_w \beta)^j \tilde{L}_{t+j} \left[\frac{\tilde{W}_t}{W_t} \frac{W_t \bar{\pi}^j}{P_t \pi_{t+1} \cdots \pi_{t+j}} \frac{u_{c,t+j}}{\lambda_w} + f_{L,t+j} \right], \quad (1.6)$$

where $\tilde{W}_t$ is the value of W_t^h that it chooses. Also,

$$\pi_{t+1} = \frac{P_{t+1}}{P_t}.$$

To interpret (1.6), consider first the case, $\xi_w = 0$. In this case, the above expression reduces to:

$$\frac{1}{\lambda_w} \frac{\tilde{W}_t}{P_t} u_{c,t} + f_{L,t}^h = 0.$$

Another way to write this is:

$$-\lambda_w f_{L,t}^h = \frac{\tilde{W}_t}{P_t} u_{c,t}.$$

The right side looks like the average revenue, in units of utility, associated with selling labor. You set that equal to a constant markup, λ_w , over the marginal cost of producing the good, i.e., the marginal utility of leisure, $-f_L^h$. With $\xi_w > 0$, you get a weighted average of these terms over each date.

This is the investment Euler equation:

$$E_{t-1} u_{c,t} = \beta E_{t-1} u_{c,t+1} [r_{t+1}^k + 1 - \delta], \quad (1.7)$$

where

$$r_{t+1}^k = \frac{R_{t+1}^k}{P_{t+1}}.$$

2. Firms

There are three types of firms. One hires labor from the households and transforms it into a homogeneous input good, denoted X . This was discussed in the previous section. The other type of firm buys X and rents capital, and produces an intermediate good which it sells to a final goods producer. There is a continuum of these intermediate goods producers, each of which is a monopoly supply of its own good and is competitive in the markets for inputs.

2.1. Final Good Firms

Production function of representative final good firm:

$$Y_t = \left[\int_0^1 Y_{it}^{\frac{1}{\lambda_f}} di \right]^{\lambda_f}, \quad 1 \leq \lambda_f < \infty \quad (2.1)$$

Func:

$$\left(\frac{P_t}{P_{it}} \right)^{\frac{\lambda_f}{\lambda_f - 1}} = \frac{Y_{it}}{Y_t}. \quad (2.2)$$

Price of final Goods:

$$P_t = \left[\int_0^1 P_{it}^{\frac{1}{1-\lambda_f}} di \right]^{(1-\lambda_f)}. \quad (2.3)$$

2.2. Intermediate Good Firms

Production function of i^{th} , $i \in (0, 1)$, intermediate good firm:

$$Y_{it} = K_{it}^\alpha X_{it}^{1-\alpha}, \quad 0 < \alpha < 1. \quad (2.4)$$

2.2.1. Non-Price Euler Equations

Suppose the i^{th} firm's price, P_{it} , is somehow given (see next subsection for how this is determined). Then the firm must produce Y_{it} in (2.2). Firm solves:

$$\min_{K_i, X_i} WRX_i + R^k K_i + \zeta \left[Y_i - K_i^\alpha X_i^{1-\alpha} \right],$$

where ζ is the Lagrange multiplier. Solving:

$$\zeta = \left(\frac{1}{1-\alpha} \right)^{(1-\alpha)} \left(\frac{1}{\alpha} \right)^\alpha (R^k)^\alpha (WR)^{1-\alpha}$$

$$\begin{aligned} WR &= \zeta F_X \\ R^k &= \zeta F_K, \end{aligned}$$

where ζ is marginal cost, MC . Let s denote real marginal cost (the inverse of the markup):

$$s \equiv \frac{MC}{P} = \frac{\zeta}{P}.$$

Then, we get the usual fons:

$$\begin{aligned} \frac{W}{P} &= \frac{sF_X}{R} \\ \frac{R^k}{P} &= sF_K. \end{aligned}$$

The first of these is the ‘X-input demand’ equation. In a graph with W/P on the vertical axis and X on the horizontal, demand is downward-sloping. A fall in R or a rise in s shifts labor demand to the right. Note:

$$\begin{aligned} R^k &= WR \frac{\alpha}{1-\alpha} \frac{X_i}{K_i} = WR \frac{\alpha}{1-\alpha} \frac{X}{K} \\ Y_i &= \left(\frac{X}{K} \right)^\alpha X_i, \end{aligned}$$

all $i \in (0, 1)$, where

$$X = \int_0^1 X_i di, \quad K = \int_0^1 K_i di.$$

Substituting out for R^k in the marginal cost expression,

$$s_t = \frac{1}{1-\alpha} \left(\frac{X_t}{K_t} \right)^\alpha \frac{W_t}{P_t} R_t.$$

2.2.2. Price Euler Equation

Randomly, $1 - \xi_p$ firms get to set price, and ξ_p must set $P_{it} = \bar{\pi} P_{i,t-1}$. The ones that get to change their price, $\tilde{P}_t$, do so to solve:

$$E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j v_{t+j} Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} \left[\left(\tilde{P}_t \bar{\pi}^j \right)^{1-\frac{\lambda_f}{\lambda_f-1}} - MC_{t+j} \left(\tilde{P}_t \bar{\pi}^j \right)^{-\frac{\lambda_f}{\lambda_f-1}} \right] \quad (2.5)$$

Fonc:

$$E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j v_{t+j} Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} \frac{(\tilde{P}_t \bar{\pi}^j)^{\frac{-\lambda_f}{\lambda_f-1}-1}}{\lambda_f - 1} [\tilde{P}_t \bar{\pi}^j - \lambda_f MC_{t+j}] \bar{\pi}^j = 0. \quad (2.6)$$

or, after rearranging:

$$E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j (v_{t+j} P_{t+j}) Y_{t+j} \left(\frac{P_{t+j}}{\tilde{P}_t \bar{\pi}^j} \right)^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{P}_t \bar{\pi}^j}{P_{t+j}} - \lambda_f \frac{MC_{t+j}}{P_{t+j}} \right],$$

Define:

$$\begin{aligned} \tilde{p}_t &= \frac{\tilde{P}_t}{P_t}, \quad s_t = \frac{MC_t}{P_t} = \frac{1}{1-\alpha} \left(\frac{L_t}{K_t} \right)^\alpha w_t R_t, \quad \pi_{t+1} = \frac{P_{t+1}}{P_t}, \quad \tilde{v}_t = v_t P_t, \\ X_{t,j} &= \begin{cases} \prod_{k=1}^j \frac{\bar{\pi}}{\pi_{t+k}} & j > 0 \\ 1 & j = 0 \end{cases}. \end{aligned}$$

After multiplying by $P_t^{\lambda_f/(\lambda_f-1)}$ and rearranging:

$$E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j \tilde{v}_{t+j} Y_{t+j} (X_{t,j})^{\frac{-\lambda_f}{\lambda_f-1}} [\tilde{p}_t X_{t,j} - \lambda_f s_{t+j}]. \quad (2.7)$$

2.3. Final Good Price

$$P_t = \left[(1 - \xi_p) \tilde{P}_t^{\frac{1}{1-\lambda_f}} + \xi_p (\bar{\pi} P_{t-1})^{\frac{1}{1-\lambda_f}} \right]^{1-\lambda_f},$$

or

$$1 = \left[(1 - \xi_p) \tilde{p}_t^{\frac{1}{1-\lambda_f}} + \xi_p \left(\frac{\bar{\pi}}{\pi_t} \right)^{\frac{1}{1-\lambda_f}} \right]^{1-\lambda_f}$$

3. Monetary Policy

We will consider two types of policies. In one, monetary policy is an exogenous stochastic process for money growth,

$$\mu_t = \frac{\bar{M}_{t+1}}{\bar{M}_t}.$$

Here is the time series representation we will assume for this:

$$\mu_t = \phi\mu_{t-1} + \varepsilon_t + \omega_1\varepsilon_{t-1} + \omega_2\varepsilon_{t-2}. \quad (3.1)$$

We will also consider policies like this:

$$R_t = a_0 + a_\pi E_t \pi_{t+1} + a_y y_t, \quad (3.2)$$

where $y_t = \frac{Y_t - Y}{Y}$ denotes the percent deviation of output from steady state ('trend').

4. Loan Market Clearing

Financial intermediaries receive $M_t - Q_t$ from households and a transfer, $(1 - \mu_t)M_t$ from the monetary authority. They lend the full amount out to the intermediate good firms which use the funds to pay for X . Loan market clearing requires

$$W_t X_t = \mu_t M_t - Q_t. \quad (4.1)$$

5. Resource Constraint

We want to develop an expression that relates aggregate consumption and investment to aggregate employment and the aggregate stock of capital. It turns out that we get close to such an expression. An extra term having to do with the distribution of prices and wages, enters too.

Let Y^* denote the unweighted average of output in each sector:²

$$\begin{aligned} Y^* &= \int_0^1 Y(f) df = \int_0^1 F(K(f), X(f)) df \\ &= \int_0^1 F\left(\frac{K}{X}, 1\right) X(f) df, \end{aligned}$$

²Here, we're assuming more than we need. For example, if the production function were linear homogeneous in K and $X - X^0$, where X^0 is a fixed cost, then all firms would use the same ratio of K to X . To see this, note that they all face the same rental rate on capital, r^k and the same wage rate, W :

$$F_K = r^k, \quad F_X = W.$$

Then, the ratio,

$$\frac{r^k}{W} = \frac{F_K}{F_X} = f\left(\frac{K}{X - X^0}\right),$$

so everyone has the same K/X .

where

$$K = \int_0^1 K(f)df, \quad X = \int_0^1 X(f)df,$$

and X is the total output of the labor contractor (the new line here is that the labor contractor hires the labor effort of individual households to produce a homogeneous good used by the intermediate good producers). Now, each intermediate good firm confronts the same factor prices, so they hire capital and labor in the same proportions. That's why the capital/labor ratio for each individual firm is identical, and identical to the aggregate. Then,

$$Y^* = F\left(\frac{K}{X}, 1\right) \int_0^1 X(f)df = F\left(\frac{K}{X}, 1\right)X = F(K, X).$$

So, the unweighted sum of output of the intermediate goods producers can be represented as a function of aggregate capital and the aggregate output of the labor contractor. There are two reasons why this expression falls short. First, the unweighed sum of output, Y^* , does not correspond to anything in this model. Second, we want to relate output to the unweighted sum of labor, not X , because that is how the labor input is measured in the data.

Note,

$$\begin{aligned} Y^* &= \int_0^1 Y(f)df = \int_0^1 \left[\frac{P}{P(f)} \right]^{\frac{\lambda_f}{\lambda_f-1}} Y df \\ &= Y P^{\frac{\lambda_f}{\lambda_f-1}} (P^*)^{\frac{\lambda_f}{1-\lambda_f}}, \end{aligned}$$

where Y is aggregate output of the final good sector, as defined above, and P^* is the indicated weighted average of the individual prices, where the weights differ from what they are in P . So,

$$Y^* = \left(\frac{P}{P^*} \right)^{\frac{\lambda_f}{\lambda_f-1}} Y.$$

Now, actual output is Y , and this is what is available to be divided into consumption and investment. So, we write the resource constraint as follows:

$$C + K' - (1 - \delta)K \leq \left(\frac{P^*}{P} \right)^{\frac{\lambda_f}{\lambda_f-1}} F(K, X).$$

We also want a measure of 'employment' for our economy (this useage of L is inconsistent with the useage of L way up above...but we intend to go with this new useage):

$$L = \int_0^1 L(h)dh$$

Now

$$\begin{aligned}
L &= \int_0^1 \left[\frac{W(h)}{W} \right]^{\frac{\lambda_w}{1-\lambda_w}} X dh \\
&= X W^{\frac{\lambda_w}{\lambda_w-1}} (W^*)^{\frac{\lambda_w}{1-\lambda_w}} \\
&= X \left(\frac{W}{W^*} \right)^{\frac{\lambda_w}{\lambda_w-1}},
\end{aligned}$$

where

$$W^* = \left[\int_0^1 W(h)^{\frac{\lambda_w}{1-\lambda_w}} dh \right]^{\frac{1-\lambda_w}{\lambda_w}}.$$

Then, the resource constraint can be written as a function of aggregate capital and labor as follows:

$$C + K' - (1 - \delta)K \leq \left(\frac{P^*}{P} \right)^{\frac{\lambda_f}{\lambda_f-1}} F \left[K, \left(\frac{W^*}{W} \right)^{\frac{\lambda_w}{\lambda_w-1}} L \right]$$

This is our aggregate resource constraint.

The analysis is *still* slightly incomplete, however, because we have to discuss the determination of P^* and W^* .

The same logic that delivers expressions for P^* and W^* delivers expressions for these things too:

$$W_t^* = \left[(1 - \xi_w) \left(\tilde{W}_t \right)^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w \left(\bar{\pi} W_{t-1}^* \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w}}$$

Divide both sides by W_t :

$$\frac{W_t^*}{W_t} = \left[(1 - \xi_w) \left(\frac{\tilde{W}_t}{W_t} \right)^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w \left(\bar{\pi} \frac{W_{t-1}}{W_t} \frac{W_{t-1}^*}{W_{t-1}} \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w}}$$

or,

$$w_t^* = \left[(1 - \xi_w) (\tilde{w}_t)^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w \left(\frac{\bar{\pi}}{\pi_t} \frac{w_{t-1}}{w_t} w_{t-1}^* \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w}},$$

where

$$w_t^* = \frac{W_t^*}{W_t}, \quad w_t = \frac{W_t}{P_t}.$$

6. Solving the Model

We now turn to solving the model. We do so by first linearizing the model's Euler equations and other constraints around steady state. The first section computes the steady state. The second subsection presents the linearization formulas. The third section shows how we solved the model.

6.1. Steady State

These are the nine steady state variables we seek: $w, r^k, k, L, R, \pi, q, c, m$. (In steady state, $X = L$). The investment equation is:

$$1 = \beta \left[r^k + (1 - \delta) \right], \quad (6.1)$$

where $r^k = (MC/P) \times F_k$ implies:

$$r^k = \frac{\alpha}{1 - \alpha} w R \frac{L}{K} \quad (6.2)$$

The M' equation is:

$$1 = \beta \frac{R}{\pi}. \quad (6.3)$$

The money equation is:

$$v'(q) = u_c(R - 1). \quad (6.4)$$

The labor decision is:

$$w \frac{u_c}{\lambda_w} + f_L = 0. \quad (6.5)$$

The marginal cost equation is:

$$1 = \lambda_f \left(\frac{1}{1 - \alpha} \right)^{(1-\alpha)} \left(\frac{1}{\alpha} \right)^\alpha (r^k)^\alpha (wR)^{1-\alpha}. \quad (6.6)$$

The money market clearing condition is:

$$wL = \mu m - q. \quad (6.7)$$

The resource constraint is:

$$c + \delta K = K^\alpha L^{1-\alpha}. \quad (6.8)$$

Also, constancy of m implies:

$$\mu = \pi. \quad (6.9)$$

We solve for the nine variables like this. First, compute r^k from (6.1). Then, use (6.6) to obtain wR . Next, use (6.3) to compute R and then get w from $w = wR/R$. Then, use (6.2) to get L/K . Now we use (6.5) and (6.8) to solve for c and L . By (6.8), we get:

$$c = \left[\left(\frac{K}{L} \right)^\alpha - \delta \frac{K}{L} \right] L. \quad (6.10)$$

Letting the utility function be

$$u(c) = \frac{c^{1-\sigma_c}}{1-\sigma_c}, \quad f(L) = \frac{-L^{1+\sigma_L}}{1+\sigma_L} \psi_L, \quad v(q) = \psi_q \frac{q^{1-\sigma_q}}{1-\sigma_q}$$

Writing out (6.5):

$$\frac{wc^{-\sigma_c}}{\lambda_w} = L^{\sigma_L} \psi_L. \quad (6.11)$$

Combining (6.10) and (6.11), we obtain:

$$\frac{w \left\{ \left[\left(\frac{K}{L} \right)^\alpha - \delta \frac{K}{L} \right] L \right\}^{-\sigma_c}}{\lambda_w} = L^{\sigma_L} \psi_L.$$

Since w and K/L have already been determined, this is one equation in the one unknown, L . With L and K/L , we compute K . We get c from (6.10) and q from (6.4). Finally, we get m from (6.7).

6.2. Linearizing the Equations

We will now linearize the various equations.

6.2.1. Linearizing the Price Equation

The Euler equation of a firm that changes its price in period t is given in (2.7). Letting

$A_{t,j} = u_{c,t+j} Y_{t+j} X_{t,j}^{\frac{-\lambda_f}{\lambda_f-1}}$, we can write this as follows:

$$\begin{aligned} & A_{t,0} [\tilde{p}_t X_{t,0} - \lambda_f s_t] + \beta \xi_p A_{t,1} [\tilde{p}_t X_{t,1} - \lambda_f s_{t+1}] \\ & + (\beta \xi_p)^2 A_{t,2} [\tilde{p}_t X_{t,2} - \lambda_f s_{t+2}] + (\beta \xi_p)^3 A_{t,3} [\tilde{p}_t X_{t,3} - \lambda_f s_{t+3}] + \dots \end{aligned}$$

Differentiate this with respect to $\tilde{p}_t$ and evaluate the result in steady state

$$\frac{u_c Y}{1 - \beta \xi_p},$$

where we have used the fact that in steady state, $A_{t,j} = u_c Y$. Differentiate with respect to s_{t+j} , $j \geq 0$ and evaluate the result in steady state:

$$- (\beta \xi_p)^j u_c Y.$$

Now differentiate with respect to π_{t+j} , $j > 0$:

$$- (\beta \xi_p)^j A_{t,j} \tilde{p}_t \frac{X_{t,j}}{\pi_{t+j}} - (\beta \xi_p)^{j+1} A_{t,j+1} \tilde{p}_t \frac{X_{t,j+1}}{\pi_{t+j+1}} - \dots$$

or, in steady state ($\tilde{p} = 1$):

$$- (\beta \xi_p)^j \frac{1}{\bar{\pi}} \frac{u_c Y}{1 - \beta \xi_p}.$$

Using this information to take the Taylor series expansion of (2.7) about steady state, we get:

$$\begin{aligned} 0 &= \frac{u_c Y}{1 - \beta \xi_p} \tilde{p} \hat{\tilde{p}}_t - \lambda_f u_c Y s \sum_{j=0}^{\infty} (\beta \xi_p)^j \hat{s}_{t+j} \\ &\quad - \frac{u_c Y}{1 - \beta \xi_p} \tilde{p} \sum_{j=1}^{\infty} (\beta \xi_p)^j \hat{\pi}_{t+j}, \end{aligned}$$

or, ($s \lambda_f = 1$),

$$\hat{\tilde{p}}_t = (1 - \beta \xi_p) \sum_{j=0}^{\infty} (\beta \xi_p)^j \hat{s}_{t+j} + \sum_{l=1}^{\infty} (\beta \xi_p)^l \hat{\pi}_{t+l}$$

From the formula for the aggregate price level, and using the Calvo trick:

$$P_t = \left[(1 - \xi_p) \left(\tilde{P}_t \right)^{\frac{1}{1-\lambda_f}} + \xi_p \left(\bar{\pi} P_{t-1} \right)^{\frac{1}{1-\lambda_f}} \right]^{1-\lambda_f},$$

or, after dividing by P_t :

$$1 = \left[(1 - \xi_p) \left(\tilde{p}_t \right)^{\frac{1}{1-\lambda_f}} + \xi_p \left(\frac{\bar{\pi}}{\pi_t} \right)^{\frac{1}{1-\lambda_f}} \right]^{1-\lambda_f}.$$

Linearizing

$$\hat{p}_t = \frac{\xi_p}{1 - \xi_p} \hat{\pi}_t.$$

Substituting,

$$\hat{\pi}_t = \frac{(1 - \xi_p)(1 - \beta\xi_p)}{\xi_p} \sum_{j=0}^{\infty} (\beta\xi_p)^j \hat{s}_{t+j} + \frac{1 - \xi_p}{\xi_p} \sum_{l=1}^{\infty} (\beta\xi_p)^l \hat{\pi}_{t+l}.$$

Then,

$$\hat{\pi}_t = \beta\xi_p \hat{\pi}_{t+1} + \frac{(1 - \xi_p)(1 - \beta\xi_p)}{\xi_p} \hat{s}_t + \frac{1 - \xi_p}{\xi_p} \beta\xi_p \hat{\pi}_{t+1}$$

or, after combining:

$$\hat{\pi}_t = \beta \hat{\pi}_{t+1} + \frac{(1 - \xi_p)(1 - \beta\xi_p)}{\xi_p} \hat{s}_t.$$

Expanding the expression for r^k :

$$r^k \hat{r}_t^k = r^k \hat{w}_t + r^k \hat{R}_t + r^k \hat{L}_t - r^k \hat{K}_t,$$

so that,

$$\hat{r}_t^k = \hat{w}_t + \hat{R}_t + \hat{L}_t - \hat{K}_t$$

6.2.2. Linearizing the Q Equation

Rewriting (1.3):

$$E_t \sum_{j=0}^{\infty} (\beta\xi_q)^j \{v'(\tilde{q}_t X_{t,j}) - u_{c,t+j} [R_{t+j} - 1]\} \frac{1}{X_{t,j}} = 0.$$

Writing this out:

$$\begin{aligned} 0 &= v'(\tilde{q}_t) - u_{c,t}(R_t - 1) + \beta\xi_q \{v'(\tilde{q}_t X_{t,1}) - u_{c,t+1} [R_{t+1} - 1]\} \frac{1}{X_{t,1}} \\ &+ (\beta\xi_q)^2 \{v'(\tilde{q}_t X_{t,2}) - u_{c,t+2} [R_{t+2} - 1]\} \frac{1}{X_{t,2}} + \dots \end{aligned}$$

Differentiate with respect to $\tilde{q}_t$:

$$v_t'' + \beta\xi_q v_{t+1}'' + (\beta\xi_q)^2 v_{t+2}'' + \dots$$

or, in steady state:

$$\frac{v''}{1 - \beta \xi_q}.$$

Differentiating with respect to R_{t+j} , $j \geq 0$, and evaluating in steady state:

$$- (\beta \xi_q)^j u_c.$$

Differentiating with respect to $u_{c,t+j}$ and evaluating that in steady state:

$$- (\beta \xi_q)^j (R - 1).$$

Differentiating with respect to π_{t+j} , $j > 0$:

$$- (\beta \xi_q)^j v''_{t+j} \tilde{q}_{t+j} \frac{1}{\pi_{t+j}} - (\beta \xi_q)^{j+1} v''_{t+j+1} \tilde{q}_{t+j+1} \frac{1}{\pi_{t+j+1}} - \dots$$

or, in steady state:

$$\frac{-(\beta \xi_q)^j v'' \tilde{q} \frac{1}{\pi}}{1 - \beta \xi_q}.$$

Putting all this together:

$$\frac{v'' \tilde{q}}{1 - \beta \xi_q} \hat{\tilde{q}}_t - \frac{v'' \tilde{q}}{1 - \beta \xi_q} \sum_{j=1}^{\infty} (\beta \xi_q)^j \hat{\pi}_{t+j} - u_c (R - 1) \sum_{j=0}^{\infty} (\beta \xi_q)^j \left[\frac{R}{R - 1} \hat{R}_{t+j} + \hat{u}_{c,t+j} \right] = 0$$

Rewriting

$$\hat{\tilde{q}}_t = \sum_{j=1}^{\infty} (\beta \xi_q)^j \hat{\pi}_{t+j} + (1 - \beta \xi_q) \frac{u_c}{v'' \tilde{q}} (R - 1) \sum_{j=0}^{\infty} (\beta \xi_q)^j \left[\frac{R}{R - 1} \hat{R}_{t+j} + \hat{u}_{c,t+j} \right].$$

Now, note that $u_c/v' = 1/(R - 1)$. Then, $u_c/(v'' \tilde{q}) = (u_c/v') v'/(v'' \tilde{q}) = [1/(R - 1)] v'/(v'' \tilde{q})$.

Also,

$$\sigma_q = - \frac{v'' \tilde{q}}{v'},$$

so that

$$\frac{u_c}{v'' \tilde{q}} = \frac{1}{R - 1} \frac{v'}{v'' \tilde{q}} = - \frac{1}{(R - 1) \sigma_q}.$$

Substituting this into the expression for $\hat{\tilde{q}}_t$, we get

$$\hat{\tilde{q}}_t = \sum_{j=1}^{\infty} (\beta \xi_q)^j \hat{\pi}_{t+j} - (1 - \beta \xi_q) \frac{1}{(R - 1) \sigma_q} (R - 1) \sum_{j=0}^{\infty} (\beta \xi_q)^j \left[\frac{R}{R - 1} \hat{R}_{t+j} + \hat{u}_{c,t+j} \right],$$

or

$$\hat{q}_t = \sum_{j=1}^{\infty} (\beta \xi_q)^j \hat{\pi}_{t+j} - \frac{1 - \beta \xi_q}{\sigma_q} \sum_{j=0}^{\infty} (\beta \xi_q)^j \left[\frac{R}{R-1} \hat{R}_{t+j} + \hat{u}_{c,t+j} \right]$$

Then, multiplying by $(1 - \beta \xi_q L^{-1})$, where $L^j x_t \equiv x_{t-j}$, for integer values of j :

$$\hat{q}_t = \beta \xi_q \hat{q}_{t+1} + \beta \xi_q \hat{\pi}_{t+1} - \frac{1 - \beta \xi_q}{\sigma_q} \left(\hat{u}_{c,t} + \frac{R}{R-1} \hat{R}_t \right)$$

The aggregate Q is:

$$Q_t = (1 - \xi_q) \tilde{Q}_t + \xi_q \bar{\pi} Q_{t-1}$$

Dividing by P_t :

$$q_t = (1 - \xi_q) \tilde{q}_t + \xi_q \frac{\bar{\pi}}{\pi_t} q_{t-1}.$$

Linearizing:

$$\hat{q}_t = (1 - \xi_q) \hat{\tilde{q}}_t + \xi_q \hat{q}_{t-1} - \xi_q \hat{\pi}_t.$$

or,

$$(1 - \xi_q) \hat{\tilde{q}}_t = \hat{q}_t - [\xi_q \hat{q}_{t-1} - \xi_q \hat{\pi}_t].$$

Now, multiplying the q equation by $(1 - \xi_q)$:

$$\begin{aligned} & (\hat{q}_t - [\xi_q \hat{q}_{t-1} - \xi_q \hat{\pi}_t]) \\ = & \beta \xi_q (\hat{q}_{t+1} - [\xi_q \hat{q}_t - \xi_q \hat{\pi}_{t+1}]) + (1 - \xi_q) \beta \xi_q \hat{\pi}_{t+1} \\ & - (1 - \xi_q) \frac{1 - \beta \xi_q}{\sigma_q} \left(\hat{u}_{c,t} + \frac{R}{R-1} \hat{R}_t \right) \end{aligned}$$

Simplifying,

$$\begin{aligned} & (\hat{q}_t - \xi_q \hat{q}_{t-1}) \\ = & \beta \xi_q (\hat{q}_{t+1} - \xi_q \hat{q}_t) + \xi_q (\beta \hat{\pi}_{t+1} - \hat{\pi}_t) \\ & - (1 - \xi_q) \frac{1 - \beta \xi_q}{\sigma_q} \left(\hat{u}_{c,t} + \frac{R}{R-1} \hat{R}_t \right) \end{aligned}$$

6.2.3. Linearizing the Wage Equation

The household's Euler equation associated with the wage rate is (1.6). Letting $\tilde{w}_t = \tilde{W}_t/W_t$ and $w_t = W_t/P_t$, we rewrite (1.6). First, note that:

$$\begin{aligned}\tilde{L}_{t+j} &= \left(\frac{\tilde{W}_t \bar{\pi}^j}{\tilde{W}_{t+j}} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} = \left(\frac{\tilde{W}_t \bar{\pi}^j}{P_{t+j} w_{t+j}} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} \\ &= \left(\frac{\tilde{W}_t \bar{\pi}^j}{w_{t+j} P_t \pi_{t+1} \cdot \pi_{t+j}} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} = \left(\frac{\tilde{w}_t W_t \bar{\pi}^j}{w_{t+j} P_t \pi_{t+1} \cdot \pi_{t+j}} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} \\ &= \left(\frac{\tilde{w}_t w_t \bar{\pi}^j}{w_{t+j} \pi_{t+1} \cdot \pi_{t+j}} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j}\end{aligned}$$

So, (1.6) can be written as:

$$0 = E_t \sum_{j=0}^{\infty} (\beta \xi_w)^j \tilde{L}_{t+j} \left[\tilde{w}_t w_t X_{t,j} \frac{u_{c,t+j}}{\lambda_w} + f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+j}} X_{t,j} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} \right) \right].$$

Writing this out, term by term:

$$\begin{aligned}0 &= \tilde{L}_t \left[\tilde{w}_t w_t \frac{u_{c,t}}{\lambda_w} + f_L \left(\tilde{w}_t^{\frac{\lambda_w}{1-\lambda_w}} L_t \right) \right] \\ &\quad + \beta \xi_w \tilde{L}_{t+1} \left[\tilde{w}_t w_t X_{t,1} \frac{u_{c,t+1}}{\lambda_w} + f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+1}} X_{t,1} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+1} \right) \right] \\ &\quad + (\beta \xi_w)^2 \tilde{L}_{t+2} \left[\tilde{w}_t w_t X_{t,2} \frac{u_{c,t+2}}{\lambda_w} + f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+2}} X_{t,2} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+2} \right) \right] \\ &\quad + \dots\end{aligned}$$

In steady state, $\tilde{w} = 1, \pi_t = \bar{\pi}$, $w \frac{u_c}{\lambda_w} + f_L = 0$. The latter implies that there is no need to worry about differentiating the term outside the square brackets in the wage-setting problem.

Differentiate with respect to $\tilde{w}_t$, and evaluate the result in steady state:

$$\frac{1}{1 - \beta \xi_w} L \left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right].$$

Differentiate with respect to w_t , and evaluate that in steady state:

$$\begin{aligned} & L \frac{u_c}{\lambda_w} + \frac{\beta \xi_w}{1 - \beta \xi_w} L \left[\frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} \frac{1}{w} L \right] \\ &= \frac{1}{1 - \beta \xi_w} L \left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \frac{1}{w} - f_{LL} \frac{\lambda_w}{1 - \lambda_w} \frac{1}{w} L^2. \end{aligned}$$

Differentiate with respect to w_{t+j} , $j > 0$, and evaluate in steady state:

$$- (\beta \xi_w)^j L f_{LL} \frac{\lambda_w}{1 - \lambda_w} \frac{1}{w} L.$$

Differentiate with respect to π_{t+j} , $j > 0$ and evaluate that in steady state:

$$- \frac{(\beta \xi_w)^j}{1 - \beta \xi_w} L \left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \frac{1}{\pi}.$$

Do the same for L_{t+j} (aggregate employment) and $u_{c,t+j}$:

$$(\beta \xi_w)^j L f_{LL}, \quad (\beta \xi_w)^j L w \frac{1}{\lambda_w}$$

Collecting terms

$$\begin{aligned} 0 &= \frac{1}{1 - \beta \xi_w} L \left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] (\hat{w}_t + \hat{w}_t) - f_{LL} \frac{\lambda_w}{1 - \lambda_w} \frac{1}{w} L^2 \hat{w}_t \\ &\quad - L f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \sum_{j=1}^{\infty} (\beta \xi_w)^j \hat{w}_{t+j} - \frac{L}{1 - \beta \xi_w} \left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \sum_{j=1}^{\infty} (\beta \xi_w)^j \hat{\pi}_{t+j} \\ &\quad + L^2 f_{LL} \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{L}_{t+j} + \frac{L w u_c}{\lambda_w} \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{u}_{ct+j} \end{aligned}$$

After dividing by L , this expression is written:

$$\begin{aligned} 0 &= \frac{1}{1 - \beta \xi_w} \left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] [\hat{w}_t + \hat{w}_t] - \frac{\left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right]}{1 - \beta \xi_w} \sum_{l=1}^{\infty} (\beta \xi_w)^l \hat{\pi}_{t+l} \\ &\quad + w \frac{u_c}{\lambda_w} \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{u}_{c,t+j} - f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \sum_{l=0}^{\infty} (\beta \xi_w)^l \hat{w}_{t+l} \\ &\quad + f_{LL} L \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{L}_{t+j} \end{aligned}$$

Denoting $\left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1-\lambda_w} L\right] = \left[-f_L + f_{LL} \frac{\lambda_w}{1-\lambda_w} L\right] \equiv \tilde{\sigma}_L$, $w \frac{u_c}{\lambda_w} = -f_L$ we obtain:

$$\begin{aligned} & \frac{1}{1-\beta\xi_w} \tilde{\sigma}_L [\hat{w}_t + \hat{w}_t] - \frac{\tilde{\sigma}_L}{1-\beta\xi_w} \sum_{l=1}^{\infty} (\beta\xi_w)^l \hat{\pi}_{t+l} - f_L \sum_{j=0}^{\infty} (\beta\xi_w)^j \hat{u}_{c,t+j} \\ & - f_{LL} L \frac{\lambda_w}{1-\lambda_w} \sum_{l=0}^{\infty} (\beta\xi_w)^l \hat{w}_{t+l} + f_{LL} L \sum_{j=0}^{\infty} (\beta\xi_w)^j \hat{L}_{t+j} = 0 \end{aligned}$$

or,

$$\begin{aligned} & \frac{1}{1-\beta\xi_w} \tilde{\sigma}_L [\hat{w}_t + \hat{w}_t] = \frac{\beta\xi_w}{1-\beta\xi_w} \tilde{\sigma}_L [\hat{w}_{t+1} + \hat{w}_{t+1}] + \frac{\tilde{\sigma}_L \beta\xi_w}{1-\beta\xi_w} \hat{\pi}_{t+1} + f_L \hat{u}_{c,t} \\ & + f_{LL} L \frac{\lambda_w}{1-\lambda_w} \hat{w}_t - f_{LL} L \hat{L}_t \end{aligned}$$

Define

$$\sigma_L = \frac{f_{LL} L}{f_L}.$$

Note, that $\tilde{\sigma}_L = -f_L + f_{LL} \frac{\lambda_w}{1-\lambda_w} L = f_L \left[\frac{\lambda_w}{1-\lambda_w} \frac{f_{LL} L}{f_L} - 1 \right] = f_L \left[\frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right]$, so dividing expression above by f_L

$$\begin{aligned} \frac{1}{1-\beta\xi_w} \left[\frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right] [\hat{w}_t + \hat{w}_t] &= \frac{\beta\xi_w}{1-\beta\xi_w} \left[\frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right] [\hat{w}_{t+1} + \hat{w}_{t+1}] + \\ & \frac{1}{1-\beta\xi_w} \left[\frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right] \beta\xi_w \hat{\pi}_{t+1} + \hat{u}_{c,t} \\ & + \sigma_L \frac{\lambda_w}{1-\lambda_w} \hat{w}_t - \sigma_L \hat{L}_t \end{aligned}$$

The aggregate wage looks like this:

$$W_t = \left[\int_0^1 (W_t(i))^{\frac{1}{1-\lambda_w}} di \right]^{1-\lambda_w}.$$

There are two types of firms: $1 - \xi_w$ have the new wage, $\tilde{W}_t$, this time, because they got tapped. The ξ_w others are stuck with the wage they had last period. Because the ones that are stuck are randomly selected, the distribution of stuck wages is exactly the same as

the overall distribution of wages last time. The difference is that there is a smaller overall measure. Then,

$$W_t = \left[(1 - \xi_w) (\tilde{W}_t)^{\frac{1}{1-\lambda_w}} + \xi_w (\bar{\pi} W_{t-1})^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w}$$

Dividing this by P_t , we get:

$$w_t = \left[(1 - \xi_w) (\tilde{w}_t w_t)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\bar{\pi}}{\pi_t} w_{t-1} \right)^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w}.$$

Now, differentiate (ignoring the constant terms, which cancel):

$$w \hat{w}_t = (1 - \lambda_w) w^{\frac{-\lambda_w}{1-\lambda_w}} \left[\frac{1 - \xi_w}{1 - \lambda_w} w^{\frac{1}{1-\lambda_w}-1} (\hat{w}_t + \hat{\tilde{w}}_t) + \frac{\xi_w}{1 - \lambda_w} (\hat{w}_{t-1} - \hat{\pi}_t) \right] w^{\frac{1}{1-\lambda_w}}$$

Now, simplify:

$$\hat{w}_t = (1 - \xi_w) (\hat{w}_t + \hat{\tilde{w}}_t) + \xi_w (\hat{w}_{t-1} - \hat{\pi}_t)$$

or,

$$(1 - \xi_w) \hat{\tilde{w}}_t = \xi_w \hat{w}_t - \xi_w (\hat{w}_{t-1} - \hat{\pi}_t)$$

Combining this with the wage equation:

$$\begin{aligned} & \frac{1}{1-\beta\xi_w} \left[\frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right] \frac{1}{(1-\xi_w)} [\hat{w}_t - \xi_w (\hat{w}_{t-1} - \hat{\pi}_t)] = \\ & \frac{\beta\xi_w}{1-\beta\xi_w} \left[\frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right] \frac{1}{(1-\xi_w)} [\hat{w}_{t+1} - \xi_w (\hat{w}_t - \hat{\pi}_{t+1})] + \\ & \frac{1}{1-\beta\xi_w} \left[\frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right] \beta\xi_w \hat{\pi}_{t+1} + \hat{u}_{c,t} + \sigma_L \frac{\lambda_w}{1-\lambda_w} \hat{w}_t - \sigma_L \hat{L}_t \end{aligned}$$

Note, that when we set $\xi_w = 0$ this expression, becomes:

$$\left[\frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] \hat{w}_t = \hat{u}_{c,t} + \sigma_L \frac{\lambda_w}{1 - \lambda_w} \hat{w}_t - \sigma_L \hat{L}_t$$

or

$$\hat{u}_{c,t} - \sigma_L \hat{L}_t + \hat{w}_t = 0,$$

which is just the static Euler quation we would have started out with if we had set ξ_w at the very beginning. Note that λ_w is not present here. Its only influence on the system is on the steady state.

6.2.4. Intertemporal Consumption Euler Equation

The Euler equation for M is given in (1.2). It can be rewritten

$$u_{c,t} = \beta u_{c,t+1} \frac{R_{t+1}}{\pi_{t+1}}.$$

Let

$$\sigma_c = \frac{-u_{cc}c}{u_c}.$$

Then,

$$\hat{c}_t = \hat{c}_{t+1} - \frac{1}{\sigma_c} (\hat{R}_{t+1} - \hat{\pi}_{t+1}).$$

Note that this is a slightly weird equation. It's got tomorrow's interest rate in it. That reflects that consumption is a credit good.

6.2.5. Investment Euler Equation

We now linearize the investment Euler equation, (1.7):

$$u_{cc}c\hat{c}_t = \beta u_{cc}[r^k + 1 - \delta]c\hat{c}_{t+1} + \beta u_c r^k \hat{r}_{t+1}^k$$

But, $\beta[r^k + 1 - \delta] = 1$, and, letting $\sigma_c = -u_{cc}c/u_c$:

$$\hat{c}_t = \hat{c}_{t+1} - \frac{1}{\sigma_c} \beta r^k \hat{r}_{t+1}^k,$$

or,

$$\hat{c}_t = \hat{c}_{t+1} - \frac{1}{\sigma_c} (1 - \beta(1 - \delta)) \hat{r}_{t+1}^k.$$

Previously, we derived an expression for $\hat{r}_{t+1}^k$:

$$\hat{r}_{t+1}^k = \hat{w}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1}.$$

Substituting this into the linearized intertemporal Euler equation:

$$\hat{c}_t = \hat{c}_{t+1} - \frac{1}{\sigma_c} (1 - \beta(1 - \delta)) (\hat{w}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1}).$$

6.2.6. Resource Constraint

Log-linearizing:

$$\hat{w}_t^* = \frac{1 - \lambda_w}{\lambda_w} \left\{ \frac{\lambda_w}{1 - \lambda_w} (1 - \xi_w) \hat{w}_t + \xi_w \frac{\lambda_w}{1 - \lambda_w} \left[-\hat{\pi}_t - \hat{w}_t + \hat{w}_{t-1} + \hat{w}_{t-1}^* \right] \right\},$$

or

$$\hat{w}_t^* = (1 - \xi_w) \hat{w}_t + \xi_w \left[-\hat{\pi}_t - \hat{w}_t + \hat{w}_{t-1} + \hat{w}_{t-1}^* \right]$$

From before

$$(1 - \xi_w) \hat{w}_t = \xi_w \hat{w}_t - \xi_w (\hat{w}_{t-1} - \hat{\pi}_t)$$

Substituting this into the $\hat{w}_t^*$ equation, we get:

$$\hat{w}_t^* = \xi_w \hat{w}_{t-1}^*$$

Now we'll figure out P^*/P . The following expression holds for P^* :

$$P_t^* = \left[(1 - \xi_f) \left(\tilde{P}_t \right)^{\frac{\lambda_f}{1 - \lambda_f}} + \xi_f \left(\bar{\pi} P_{t-1}^* \right)^{\frac{\lambda_f}{1 - \lambda_f}} \right]^{\frac{1 - \lambda_f}{\lambda_f}}.$$

Dividing by P_t :

$$p_t^* = \left[(1 - \xi_f) \left(\tilde{p}_t \right)^{\frac{\lambda_f}{1 - \lambda_f}} + \xi_f \left(\frac{\bar{\pi}}{\pi_t} p_{t-1}^* \right)^{\frac{\lambda_f}{1 - \lambda_f}} \right]^{\frac{1 - \lambda_f}{\lambda_f}}.$$

Taking the log-linear approximation:

$$\hat{p}_t^* = \frac{1 - \lambda_f}{\lambda_f} \left\{ \frac{\lambda_f}{1 - \lambda_f} (1 - \xi_f) \hat{\tilde{p}}_t + \frac{\lambda_f}{1 - \lambda_f} \xi_f \left[-\hat{\pi}_t + \hat{p}_{t-1}^* \right] \right\}$$

or,

$$\hat{p}_t^* = (1 - \xi_f) \hat{\tilde{p}}_t + \xi_f \left[-\hat{\pi}_t + \hat{p}_{t-1}^* \right]$$

Substituting out for $\hat{\tilde{p}}_t$ in terms of $\hat{\pi}_t$, we get

$$\hat{p}_t^* = \xi_f \hat{p}_{t-1}^*$$

The message here is that $\hat{p}_t^* = 0$ if you start from steady state. This means that the variations in P^*/P and W^*/W will be nill in the linear approximation.

We now linearize the resource constraint:

$$c_t + K_{t+1} - (1 - \delta)K_t \leq \left(\frac{P_t^*}{P_t}\right)^{\frac{\lambda_f}{\lambda_f - 1}} F(K_t, \left(\frac{W_t^*}{W_t}\right)^{\frac{\lambda_w}{\lambda_w - 1}} L_t).$$

Log-linearizing this: $\frac{c}{Y^*}\hat{c}_t + \frac{K}{Y^*}[\hat{K}_{t+1} - (1 - \delta)\hat{K}_t] = \frac{F_K K}{Y^*}\hat{K}_t + \frac{F_X L}{Y^*}\hat{L}_t$. Let

$$s_c = \frac{c}{Y}, \quad s_k = \frac{K}{Y}, \quad \alpha_k = \frac{F_K K}{Y^*}, \quad \alpha_L = \frac{F_X L}{Y^*},$$

so that

$$\hat{c}_t + \frac{s_k}{s_c}[\hat{K}_{t+1} - (1 - \delta)\hat{K}_t] = \frac{\alpha_k}{s_c}\hat{K}_t + \frac{\alpha_L}{s_c}\hat{L}_t$$

6.2.7. Loan Market Clearing

Dividing the loan market clearing condition, (4.1), by P_t , we obtain

$$w_t X_t = \mu_t m_t - q_t.$$

Here, $m_t = M_t/P_t$ evolves as follows:

$$m_{t+1} = \frac{\mu_t}{\pi_{t+1}} m_t.$$

Linearizing these expressions:

$$wX(\hat{w}_t + \hat{X}_t) = \mu m(\hat{\mu}_t + \hat{m}_t) - q\hat{q}_t$$

and

$$\hat{m}_{t+1} = \hat{\mu}_t + \hat{m}_t - \hat{\pi}_{t+1}.$$

6.2.8. Collecting the Linearized Equations

Price equation

$$(1)E_t \left\{ \xi_p \beta \hat{\pi}_{t+1} + (1 - \xi_p)(1 - \beta \xi_p) [\alpha \hat{L}_t - \alpha \hat{K}_t + \hat{w}_t - \hat{\pi}_t + \hat{R}_t] - \xi_p \hat{\pi}_t \right\} = 0,$$

where $\hat{s}_t = \alpha(\hat{X}_t - \hat{K}_t) + \hat{w}_t + \hat{R}_t$ and $\hat{w}_t = \hat{\bar{w}}_t - \hat{\pi}_t$.

This is the Q equation, after adopting the change of variable, $\hat{q}_t = \hat{\bar{q}}_t - \hat{\pi}_t$:

$$\begin{aligned} (2)0 &= E_{t-1}\{\beta\xi_q(\hat{\bar{q}}_{t+1} - \xi_q\hat{\bar{q}}_t) + \xi_q(\beta\xi_q - 1)\hat{\pi}_t \\ &\quad - (1 - \xi_q)\frac{1 - \beta\xi_q}{\sigma_q}\left(-\sigma_c\hat{c}_t + \frac{R}{R-1}\hat{R}_t\right) \\ &\quad - \hat{\bar{q}}_t + \hat{\pi}_t + \xi_q(\hat{\bar{q}}_{t-1} - \hat{\pi}_{t-1})\}, \end{aligned}$$

where $\hat{u}_{c,t} = -\sigma_c\hat{c}_t$. This change of variable was adopted because we want to think of $\hat{\bar{q}}_t$ as the choice variable, not $\hat{q}_t$. The latter is Q_t divided by P_t , but we are thinking of Q_t as chosen before P_t is observed. So, a given choice of $\hat{q}_t$ corresponds to a random choice of Q_t , conditional on the realization of P_t .

This is the wage equation

$$\begin{aligned} 0 &= -\frac{1}{1-\beta\xi_w}[\lambda_w\sigma_L - (1 - \lambda_w)]\frac{1}{(1-\xi_w)}[\hat{w}_t - \xi_w(\hat{w}_{t-1} - \hat{\pi}_t)] + \\ &\quad E_t\left\{\frac{\beta\xi_w}{1-\beta\xi_w}[\lambda_w\sigma_L - (1 - \lambda_w)]\frac{1}{(1-\xi_w)}[\hat{w}_{t+1} - \xi_w(\hat{w}_t - \hat{\pi}_{t+1})] + \right. \\ &\quad \left. [\lambda_w\sigma_L - (1 - \lambda_w)]\frac{\beta\xi_w}{1-\beta\xi_w}\hat{\pi}_{t+1} - (1 - \lambda_w)\sigma_c\hat{c}_t + \sigma_L\lambda_w\hat{w}_t - (1 - \lambda_w)\sigma_L\hat{L}_t\right\} \end{aligned}$$

or,

$$(3)E_t\left\{\eta_0\hat{w}_{t-1} + \eta_1\hat{w}_t + \eta_2\hat{w}_{t+1} + \eta_3\hat{\pi}_t + \eta_4\hat{\pi}_{t+1} + \eta_5\hat{L}_t + \eta_6\hat{c}_t\right\} = 0$$

where

$$\begin{aligned} \eta &= \frac{1}{1-\beta\xi_w}[\lambda_w\sigma_L - (1 - \lambda_w)]\frac{\xi_w}{(1-\xi_w)} \\ &\quad - \frac{1}{1-\beta\xi_w}[\lambda_w\sigma_L - (1 - \lambda_w)]\frac{1}{(1-\xi_w)}[1 + \beta\xi_w^2] \\ &\quad - \frac{\beta\xi_w}{1-\beta\xi_w}[\lambda_w\sigma_L - (1 - \lambda_w)]\frac{1}{(1-\xi_w)} + \sigma_L\lambda_w\beta\xi_w \\ &\quad - \frac{1}{1-\beta\xi_w}[\lambda_w\sigma_L - (1 - \lambda_w)]\frac{\xi_w}{(1-\xi_w)} \\ &\quad - \frac{\beta\xi_w}{1-\beta\xi_w}[\lambda_w\sigma_L - (1 - \lambda_w)]\frac{1}{(1-\xi_w)} \\ &\quad - \sigma_L(1 - \lambda_w) \\ &\quad - \sigma_c(1 - \lambda_w) \end{aligned}$$

Now, we'd actually like to think of $\hat{\bar{w}}_t$ instead of $\hat{w}_t$, where $\bar{w}_t = W_t/P_{t-1}$. This will allow us to consider situations in which W_t is chosen before the current realization of P_t . Now, $\hat{w}_t = \hat{\bar{w}}_t - \hat{\pi}_t$.

$$E_t\left\{\eta_0(\hat{\bar{w}}_{t-1} - \hat{\pi}_{t-1}) + \eta_1(\hat{\bar{w}}_t - \hat{\pi}_t) + \eta_2(\hat{\bar{w}}_{t+1} - \hat{\pi}_{t+1}) + \eta_3\hat{\pi}_t + \eta_4\hat{\pi}_{t+1} + \eta_5\hat{L}_t + \eta_6\hat{c}_t\right\} = 0,$$

or

$$(3')E_t\left\{\eta_0\hat{\bar{w}}_{t-1} + \eta_1\hat{\bar{w}}_t + \eta_2\hat{\bar{w}}_{t+1} - \eta_0\hat{\pi}_{t-1} + (\eta_3 - \eta_1)\hat{\pi}_t + (\eta_4 - \eta_2)\hat{\pi}_{t+1} + \eta_5\hat{L}_t + \eta_6\hat{c}_t\right\} = 0,$$

$$(4)E_t \left\{ \hat{c}_{t+1} - \frac{1}{\sigma_c}(\hat{R}_{t+1} - \hat{\pi}_{t+1}) - \hat{c}_t \right\} = 0.$$

$$(5)E_{t-1} \left\{ \hat{c}_{t+1} - \frac{1 - \beta(1 - \delta)}{\sigma_c} (\hat{w}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1}) - \hat{c}_t \right\} = 0.$$

In the alternative case,

$$(5')E_{t-1} \left\{ \hat{c}_{t+1} - \frac{1 - \beta(1 - \delta)}{\sigma_c} (\hat{\bar{w}}_{t+1} - \hat{\pi}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1}) - \hat{c}_t \right\} = 0.$$

$$(6)\hat{c}_t + \frac{s_k}{s_c} [\hat{K}_{t+1} - (1 - \delta)\hat{K}_t] - \left[\frac{\alpha_k}{s_c} \hat{K}_t + \frac{\alpha_L}{s_c} \hat{L}_t \right] = 0$$

$$(7)\mu m (\hat{\mu}_t + \hat{m}_t) - q (\hat{\bar{q}}_t - \hat{\pi}_t) - wX (\hat{w}_t + \hat{L}_t) = 0$$

In the alternative case,

$$(7')\mu m (\hat{\mu}_t + \hat{m}_t) - q (\hat{\bar{q}}_t - \hat{\pi}_t) - wX (\hat{\bar{w}}_t - \hat{\pi}_t + \hat{L}_t) = 0$$

$$(8)\hat{\mu}_{t-1} + \hat{m}_{t-1} - \hat{\pi}_t - \hat{m}_t = 0.$$

The final equation is the one that characterizes monetary policy. It is either (3.1), or

$$(9)E_t [R\hat{R}_t - a_0 - a_\pi \pi \hat{\pi}_{t+1} - a_y \hat{Y}_t] = 0. \quad (6.12)$$

If it is (3.1), then things need to be simplified a little first. Let

$$\theta_t = \rho \theta_{t-1} + e_t,$$

where

$$\theta_t = \begin{pmatrix} \mu_t \\ \varepsilon_t \\ \varepsilon_{t-1} \\ \mu_{t-1} \end{pmatrix}, \quad \rho = \begin{bmatrix} \phi & \omega_1 & \omega_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad e_t = \begin{pmatrix} \varepsilon_t \\ \varepsilon_t \\ 0 \\ 0 \end{pmatrix}.$$

We must remember to distinguish between q and $\bar{q}$:

$$q_t = \frac{Q_t}{P_t}, \quad \bar{q}_t = \frac{Q_t}{P_{t-1}},$$

so

$$q = \bar{q}/\bar{\pi}.$$

6.2.9. Solving the Equations

Let z_t denote a vector of the model variables determined at time t . The Euler equations can be written in the following format:

$$\mathcal{E}_t \{ \alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 \theta_{t+1} + \beta_1 \theta_t \} = 0,$$

where α_i are 8×8 matrices, β_i is 8×2 , $i = 0, 1$. Also, $\mathcal{E}_t$ is the expectation operator, which reflects the timing assumptions on the expectations in (1)-(8). A solution is an 8×8 matrix A and an $8 \times n$ matrix B , where $z_t = Az_{t-1} + Bs_t$, which satisfies the Euler equations and the information constraints. Also, $s_t = \theta_t$ when the information set for all expectations includes all date t shocks and $s_t = (\theta_t, \theta_{t-1})'$ when the information set is lagged for one or more equations. We have $n = 2$ in the first case and $n = 4$ in the second.

6.2.10. Benchmark case

In the benchmark case, money has the exogenous representation,

$$z_t = \begin{pmatrix} \hat{\pi}_t \\ \hat{q}_t \\ \hat{w}_t \\ \hat{m}_t \\ \hat{K}_{t+1} \\ \hat{c}_t \\ \hat{L}_t \\ \hat{R}_t \end{pmatrix}.$$

In the Euler equation

$$\alpha_0 = \begin{matrix} & \hat{\pi}_t & \hat{q}_t & \hat{w}_t & \hat{m}_t & \hat{K}_{t+1} & \hat{c}_t & \hat{L}_t & \hat{R}_t \\ \begin{bmatrix} \xi_p \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta \xi_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_4 - \eta_2 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sigma_c} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{\sigma_c} \\ \frac{1-\beta(1-\delta)}{\sigma_c} & 0 & -\frac{1-\beta(1-\delta)}{\sigma_c} & 0 & 0 & 1 & -\frac{1-\beta(1-\delta)}{\sigma_c} & -\frac{1-\beta(1-\delta)}{\sigma_c} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$\alpha_1(1 : 4) = \begin{bmatrix} \hat{\pi}_t & \hat{q}_t & \hat{w}_t & \hat{m}_t & \hat{K}_{t+1} & \hat{c}_t & \hat{L}_t & \hat{R}_t \\ \begin{bmatrix} -(a_p + \xi_p) & 0 & a_p & 0 \\ 1 + \xi_q(\beta\xi_q - 1) & -(\beta\xi_q^2 + 1) & 0 & 0 \\ \eta_3 - \eta_1 & 0 & \eta_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ q + wL & -q & -wL & \mu m \\ -1 & 0 & 0 & -1 \end{bmatrix} \end{bmatrix},$$

where $a_i = (1 - \xi_i)(1 - \beta\xi_i)$.

$$\alpha_1(5 : 8) = \begin{bmatrix} \hat{\pi}_t & \hat{q}_t & \hat{w}_t & \hat{m}_t & \hat{K}_{t+1} & \hat{c}_t & \hat{L}_t & \hat{R}_t \\ \begin{bmatrix} 0 & 0 & \alpha a_p & -\frac{a_p}{(R-1)\sigma_q} \\ 0 & a_q \frac{\sigma_c}{\sigma_q} & 0 & 0 \\ 0 & \eta_6 & \eta_5 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1-\beta(1-\delta)}{\sigma_c} & -1 & 0 & 0 \\ \frac{s_k}{s_c} & 1 & -\frac{\alpha_L}{s_c} & 0 \\ 0 & 0 & -wL & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\alpha_1 = [\alpha_1(1 : 4), \alpha_1(5 : 8)]$$

$$\alpha_2 = \begin{bmatrix} \hat{\pi}_t & \hat{q}_t & \hat{w}_t & \hat{m}_t & \hat{K}_{t+1} & \hat{c}_t & \hat{L}_t & \hat{R}_t \\ \begin{bmatrix} 0 & 0 & 0 & 0 & -\alpha a_p & 0 & 0 & 0 \\ -\xi_q & \xi_q & 0 & 0 & 0 & 0 & 0 & 0 \\ -\eta_0 & 0 & \eta_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(s_k(1-\delta) + \alpha_k)}{s_c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\beta_0 = \mathbf{0}_{8 \times 4}, \quad \beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$