Expectation Traps and Discretion: the Markov Case

1. The Markov Equilibrium and the Monetary Authority's Problem

Following is the state, s, from the perspective of the monetary authority:

$$s = (p_{-1}, z, \zeta_1).$$

Here, p_{-1} denotes the price level set last period by those who set prices then, scaled by last period's end-of-period money stock, M (i.e., *this* period's beginning of period stock); ζ_1 is a sunspot variable; and z is a shock to technology.

The state, s_2, at the time private agents make their decisions is:

$$s_2 = (s, \mu, \zeta_2),$$

where μ is the money growth rate selected by the monetary authority in the current period:

$$\mu = \frac{M'}{M}.$$

Here, M is the beginning of current period stock of money, and M' is the beginning of next period's stock of money. Also, ζ_2 is another sunspot variable.

Let

$$\sigma(s),\ c(s_2;\sigma),\ l(s_2;\sigma),\ p(s_2;\sigma),$$

denote functions characterizing the monetary authority's money growth decision, competitive consumption and employment allocations, and the price (scaled by M) set by current period price setters. The latter three are contingent upon the current monetary action, μ (via s_2) and the view that the monetary action in the future will be the output of the function, σ.

The law of motion for s is:

$$s' = \left(\frac{p(s_2;\sigma)}{\mu}, z', \zeta_1' \right).$$

Consider the following function:

$$V(s;\sigma) = E\left[u\left(c\left(s, \sigma(s), \zeta_2; \sigma\right), l\left(s, \sigma(s), \zeta_2; \sigma\right)\right) + \beta V(s';\sigma) | s; \sigma \right]. \tag{1.1}$$

This is the utility value of the equilibrium allocations associated with following the policy rule, σ, forever, given that the current state, s, has been observed.

The monetary authority takes future policy, σ, as given and selects the current policy action, μ. Let $\tilde{\sigma}(s;\sigma)$ denote this action, expressed as a function of s:

$$\tilde{\sigma}(s;\sigma) = \arg\max_{\mu} E\left[u\left(c(s_2;\sigma), l(s_2;\sigma)\right) + \beta V(s';\sigma) | s; \sigma \right]. \tag{1.2}$$

A Markov equilibrium is a $\sigma(s)$ such that:

$$\sigma(s) = \tilde{\sigma}(s;\sigma). \tag{1.3}$$

2. The Private Economy

We now proceed to characterize the objects, $c(s_2;\sigma)$, $l(s_2;\sigma)$, $p(s_2;\sigma)$. From here on, for notational simplicity, we delete the argument, σ.

2.1. Firms

Final Good Firms

Output is produced by final goods firms and intermediate goods firms. Final goods are produced by a perfectly competitive firm that combines a continuum of intermediate goods, indexed by $i \in (0,1)$, using the following technology:

$$c(s_2) = \left[\int_0^1 (y_i(s_2))^\lambda \, di\right]^{\frac{1}{\lambda}} = \left[\frac{1}{2}y(s_2)^\lambda + \frac{1}{2}y_{-1}(s_2)^\lambda\right]^{\frac{1}{\lambda}}. \tag{2.1}$$

Here, y denotes the output of intermediate good firms setting their prices in the current period and y_{-1} denotes the output of firms who set their prices in the previous period, and $0 < \lambda < 1$.

The final good producer's problem is:

$$\max_{c(s_2),\{y_i(s_2)\}} \bar{p}(s_2)c(s_2) - \int_0^1 p_i(s_2)y_i(s_2)di, \tag{2.2}$$

subject to (2.1). Here, $\bar{p}(s_2)$ denotes the price of the final good, and $p_i(s_2)$ denotes the price of the i^{th} intermediate good, where each has been scaled by the beginning of period stock of money. Problem (2.2) gives rise to the following input demand functions:

$$y_i(s_2) = c(s_2) \left(\frac{\bar{p}(s_2)}{p_i(s_2)} \right)^{\frac{1}{1-\lambda}}. \tag{2.3}$$

In conjunction with (2.1), this implies:

$$\bar{p}(s_2) = \left[\int_0^1 p_i(s_2)^{\frac{\lambda}{\lambda-1}} di \right]^{\frac{\lambda-1}{\lambda}} = \left[\frac{1}{2} p(s_2)^{\frac{\lambda}{\lambda-1}} + \frac{1}{2} p_{-1}^{\frac{\lambda}{\lambda-1}} \right]^{\frac{\lambda-1}{\lambda}}. \tag{2.4}$$

Each of these price functions are functions of σ.

Intermediate Good Firms

The i^{th} intermediate good firm has the following production function:

$$y_i(s_2) = z l_i(s_2).$$

It is a monopolist in the provision of its good. It sets price, and then supplies whatever demand materializes at that price. It sets its price at a constant value for two periods. Each period, half the firms do so. The firms setting prices in the current period do so after observing s_2. Profits, scaled by the beginning of period money stock, for firm i in the current period are:

$$\pi_i(s_2; p_i) = p_i(s_2) y_i(s_2) - \left[\frac{R(s_2) w(s_2)}{z} \right] y_i(s_2),$$

where the term in square brackets is marginal cost. According to the demand curve, $p_i y_i = c\left(\bar{p}\right)^{1/(1-\lambda)}\left(p_i\right)^{\lambda/(\lambda-1)}$ and $y_i = c\left(\bar{p}\right)^{1/(1-\lambda)}\left(p_i\right)^{-1/(1-\lambda)}$. Substituting:

$$\pi_i(s_2;p_i) = c(s_2)\left(\bar{p}(s_2)\right)^{1/(1-\lambda)}\left\{(p_i(s_2))^{\frac{\lambda}{\lambda-1}} - \left[\frac{R(s_2)w(s_2)}{z}\right](p_i(s_2))^{\frac{1}{\lambda-1}}\right\}$$

The only decision made by firms is to set their own price, and they do this every other period. Consider the situation of a firm setting its price, given that the state is s_2. The firm is owned by the households, and it must weigh profits in different states of nature in a way that is consistent with this. The valuation of profits must take into account that a current period's profits cannot be spent by the household until the next period, after s_2' is realized. A dollar next period is worth

$$\pi(s_2'|s_2)\frac{\beta u_c(s_2')}{M\mu\bar{p}(s_2')},$$

where $\pi(s_2'|s_2)$ denotes the probability of s_2', conditional on $s_2 = (s, \mu, \zeta_2)$, and M is the beginning-of-period stock of money. Then, $M\mu\bar{p}(s_2')$ is the next period money price of a unit of the consumption good. It follows that the value to the household of a dollar at the end of a period when the state is s_2 is $q(s_2)/M$, where

$$q(s_2) = \sum_{s_2'|s_2}\pi(s_2'|s_2)\frac{\beta u_c(s_2')}{\mu\bar{p}(s_2')}.$$

Then, if the state is s_2 and the beginning of period stock of money is M, then the value of $M\pi(s_2)$ dollars of profits is $M\pi(s_2)q(s_2)/M = \pi(s_2)q(s_2)$. The firm's objective is:

$$\max_{p_i(s_2)} \left\{ \pi(s_2)q(s_2) + \beta \sum_{s_2'|s_2} \pi(s_2'|s_2)q(s_2')\pi(s_2') \right\},$$

subject to:

$$\pi_i(s_2'; p_i) = c(s_2')\left(\bar{p}(s_2')\right)^{1/(1-\lambda)} \left\{ \left(\frac{p_i(s_2)}{\mu}\right)^{\frac{\lambda}{\lambda-1}} - \left[\frac{R(s_2')w(s_2')}{z'}\right] \left(\frac{p_i(s_2)}{\mu}\right)^{\frac{1}{\lambda-1}} \right\},\ s_2' = (s', \sigma(s'), \zeta_2').$$

The presence of $p_i(s_2)/\mu$ in this expression reflects that the formula requires the firm's next period price to be scaled in terms of next period's beginning of period money stock, M'. This leads to the following first order condition for price setting firms:

$$p\left(s_2\right) = \frac{q(s_2)\bar{p}(s_2)^{\frac{2-\lambda}{1-\lambda}} \frac{w}{\bar{p}}(s_2)\frac{R(s_2)}{z}c(s_2) + \frac{\beta}{\mu}\sum_{s_2'|s_2} \pi(s_2'|s_2)q(s_2')\left(\bar{p}(s_2')\mu\right)^{\frac{2-\lambda}{1-\lambda}} \frac{w}{\bar{p}}(s_2')\frac{R(s_2')}{z'}c(s_2')}{\lambda\left(q(s_2)\bar{p}(s_2)^{\frac{1}{1-\lambda}}c(s_2) + \frac{\beta}{\mu}\sum_{s_2'|s_2} \pi(s_2'|s_2)q(s_2')\left(\bar{p}(s_2')\mu\right)^{\frac{1}{1-\lambda}} c(s_2')\right)}. \tag{2.5}$$

2.2. Households

Household optimization requires:

$$\frac{w}{\bar{p}}(s_2) = -\frac{u_l(s_2)}{u_c(s_2)},$$

and

$$R(s_2) = \frac{c(s_2)u_c(s_2)}{\beta E\left[\frac{c(s_2')u_c(s_2')}{\sigma(s')}|s_2\right]}.$$

2.3. Equilibrium Relationships

We now derive the following equilibrium relationship:

$$c(s_2) = zl(s_2)\frac{g(s_2)}{h(s_2)},$$

where

$$g(s_2) = \left[\frac{1}{2} + \frac{1}{2}\left(\frac{p(s_2)}{p_{-1}}\right)^{\frac{\lambda}{1-\lambda}}\right]^{\frac{1}{\lambda}}$$

$$h(s_2) = \frac{1}{2} + \frac{1}{2}\left(\frac{p(s_2)}{p_{-1}}\right)^{\frac{1}{1-\lambda}}.$$

First,

$$\frac{c(s_2)}{y_{-1}(s_2)} = \left(\frac{\bar{p}(s_2)}{p_{-1}(s_2)}\right)^{\frac{1}{\lambda-1}} = \left[\frac{1}{2} + \frac{1}{2}\left(\frac{p(s_2)}{p_{-1}(s_2)}\right)^{\frac{\lambda}{\lambda-1}}\right]^{\frac{1}{\lambda}},$$

where the first equality corresponds to the final good producer's euler equation, and the second equality is a manipulation on the expression for $\bar{p}$. Second,

$$y_{-1}(s_2) = \left(\frac{1}{2}y(s_2) + \frac{1}{2}y_{-1}(s_2)\right)\frac{1}{\frac{1}{2} + \frac{1}{2}\frac{y(s_2)}{y_{-1}(s_2)}} = \frac{zl(s_2)}{\frac{1}{2} + \frac{1}{2}\left(\frac{p_{-1}}{p(s_2)}\right)^{\frac{1}{1-\lambda}}},$$

where the second equality involves dividing the final good firm's euler equations: $y/c = (\bar{p}/p)^{1/(1-\lambda)}$ and $y_{-1}/c = (\bar{p}/p_{-1})^{1/(1-\lambda)}$. Combining the previous two results:

$$c(s_2) = \frac{zl(s_2)\left[\frac{1}{2} + \frac{1}{2}\left(\frac{p(s_2)}{p_{-1}}\right)^{\frac{\lambda}{\lambda-1}}\right]^{\frac{1}{\lambda}}}{\frac{1}{2} + \frac{1}{2}\left(\frac{p_{-1}}{p(s_2)}\right)^{\frac{1}{1-\lambda}}}\;\frac{\left(\frac{p(s_2)}{p_{-1}}\right)^{\frac{1}{1-\lambda}}}{\left(\frac{p(s_2)}{p_{-1}}\right)^{\frac{1}{1-\lambda}}} = zl(s_2)\frac{g(s_2)}{h(s_2)}$$

Also,

$$l(s_2) = \frac{1}{2}l(\text{firms setting prices now}) + \frac{1}{2}l(\text{firms with pre-set prices}).$$

The amount by which the object, g/h, deviates from unity measures the degree of resource misallocation in the economy. It is useful to understand how this object varies with $x = p/p_{-1}$. Thus:

$$\frac{d(g/h)}{dx} = \frac{g}{h}\left[\frac{1}{g}\frac{dg}{dx} - \frac{1}{h}\frac{dh}{dx}\right].$$

But,

$$\frac{dh}{dx} = \frac{1}{2}\frac{1}{1-\lambda}x^{\frac{\lambda}{1-\lambda}},\ \frac{dg}{dx} = \frac{1}{2}\frac{1}{1-\lambda}g^{(1-\lambda)}x^{\frac{2\lambda-1}{1-\lambda}}.$$

Then,

$$\begin{aligned}\frac{d(g/h)}{dx} &= \frac{1}{1-\lambda}\frac{g}{h}x^{\frac{\lambda}{1-\lambda}}\left[\frac{1}{1+x^{\frac{\lambda}{1-\lambda}}}\frac{1}{x} - \frac{1}{1+x^{\frac{1}{1-\lambda}}}\right] \\ &= \kappa(1-x),\end{aligned}$$

where

$$\kappa = \frac{(g/h)\,x^{\frac{2\lambda-1}{1-\lambda}}}{(1-\lambda)(1+x^{\frac{\lambda}{1-\lambda}})(1+x^{\frac{1}{1-\lambda}})}.$$

Evidently, for positive inflation ($x > 1$), then $d(g/h)/dx < 0$.

The household cash constraint and the loan market clearing condition imply:

$$\bar{p}(s_2)c(s_2) = \mu.$$

3. Computations For A Deterministic Model[1]

Here, we consider the special case, $z = 1$, $\zeta_1 = \zeta_2 = 0$, so that $s = (p_{-1})$, $s_2 = (p_{-1}, \mu)$. The computational strategy involves looking for three functions: the policy function, $\sigma(p_{-1}) : P \to R$, the pricing function, $p(p_{-1}, \mu) : P \times \Lambda \to R$, and the value function, $V(p_{-1}) : P \to R$, in finite parameter function spaces. The parameters of these functions are denoted by the column vectors, $\gamma_\sigma, \gamma_p, \gamma_V$, respectively. The basic strategy selects these parameter values to mimick certain properties satisfied by the exact solution. In particular, in the exact solution, the three functions satisfy three functional equations: the Euler equation associated with the monetary authority's problem, (1.2), the equilibrium pricing equation, (2.5), and the functional equation, (1.1), that must be satisfied by $V(p_{-1})$. So, our computational strategy selects values for $\gamma_\sigma, \gamma_p, \gamma_V$ so that these functional equations are approximately satisfied. If the finite parameter function spaces we worked with contained the exact solution, then we could expect to find values for $\gamma_\sigma, \gamma_p, \gamma_V$ so that the functional equations are satisfied exactly. We choose a function space so that, if the exact functions of interest satisfied the appropriate smoothness conditions, then by making the vectors, $\gamma_\sigma, \gamma_p, \gamma_V$, long enough we can get arbitrarily close to the exact solution.

3.1. General Strategy

We begin by defining precisely the functional equations we work with. To start, we assume a value of γ_σ is given, that is, $\sigma(p_{-1})$ is initially treated as known.

[1] For other game-theoretic models, see McGuire and Pakes (Rand Journal a couple of years ago), Mariel Miranda (Judd will send references), Kotlikov, Shoven and Spivak ('Saving Book') and Judd will send us his work on this. Our work is related to work on "differential games", and contrasts with McGuire and Pakes' which focusses on jump processes.

1. In the following we define the equations used to pin down γ_p, given γ_σ.

   1. For each (p_{-1}, μ) belonging to a grid of points in $(P \times \Lambda)$, evaluate the following expressions in $(x, y) \in R^2$, for $(x, y) = (x_i, y_i)$, $i = 0, 1, 2$, with

$$(x_0, y_0) = (p_{-1}, \mu), \ (x_i, y_i) = \left[\frac{p(x_{i-1}, y_{i-1})}{y_{i-1}}, \sigma\left(\frac{p(x_{i-1}, y_{i-1})}{y_{i-1}} \right) \right], \ i = 1, 2.$$

We assume that the functions have the property that $(x_i, y_i) \in (P \times \Lambda)$, $i = 0, 1, 2$.

The expressions are:

$$\bar{p}(x, y) = \left[\frac{1}{2} p(x, y)^{\frac{\lambda}{\lambda-1}} + \frac{1}{2} x^{\frac{\lambda}{\lambda-1}} \right]^{\frac{\lambda-1}{\lambda}},$$

$$c(x, y) = \frac{y}{\bar{p}(x, y)},$$

$$g(x, y) = \left[\frac{1}{2} + \frac{1}{2} \left(\frac{p(x, y)}{x} \right)^{\frac{\lambda}{1-\lambda}} \right]^{\frac{1}{\lambda}},$$

$$h(x, y) = \frac{1}{2} + \frac{1}{2} \left(\frac{p(x, y)}{x} \right)^{\frac{1}{1-\lambda}},$$

$$l(x, y) = \frac{h(x, y)}{g(x, y)} c(x, y), \tag{3.1}$$

$$\frac{w}{\bar{p}}(x, y) = -\frac{u_l}{u_c}(x, y),$$

$$cu_c(x, y) = c(x, y) u_c(x, y)^{(1-\sigma)} \left[1 - l(x, y) \right]^{(1-\sigma)\psi}.$$

The last two expressions require taking a stand on preferences. One possibility is a unit elasticity of substitution between consumption and leisure. In this case

$$\frac{w}{\bar{p}}(x, y) = \frac{\psi c(x, y)}{1 - l(x, y)},$$

and

$$c u_c(x, y) = c(x, y)^{(1-\sigma)} \left[1 - l(x, y)\right]^{(1-\sigma)\psi}.$$

A potential problem with this utility function is that the constraint, $0 \leq l \leq 1$, may be binding, in the sense that for an arbitrary γ_σ and γ_p, (3.1) may imply a value of l in excess of 1. Restricting γ_σ and γ_p to guarantee $l \leq 1$ is likely to be computationally cumbersome. An alternative utility function that does not have this property is the zero income effect on leisure utility function:

$$u(c, l) = \begin{cases} \frac{\left[c - \psi_0 \frac{l^{1+\psi}}{1+\psi}\right]^{(1-\sigma)}}{1-\sigma}, \text{ for } \sigma \neq 1 \\ \log\left(c - \psi_0 \frac{l^{1+\psi}}{1+\psi}\right), \text{ for } \sigma = 1. \end{cases}$$

However, with this utility function, the requirement that marginal utility be well defined requires $c \geq \psi_0 \frac{l^{1+\psi}}{1+\psi}$, which imposes some restrictions on γ_σ and γ_p. Another utility function:

$$u(c, l) = \begin{cases} \frac{c^{(1-\sigma)}}{1-\sigma} - \psi_0 \frac{l^{1+\psi}}{1+\psi}, \text{ for } \sigma \neq 1 \\ \log(c) - \psi_0 \frac{l^{1+\psi}}{1+\psi}, \text{ for } \sigma = 1. \end{cases}.$$

This utility function looks attractive, because it helps makes the algorithm described below 'bullet proof' by making things well defined for any γ_σ and γ_p.

2. Evaluate the following objects for the indicated values of (x, y) only:

$$q(x_i, y_i) = \frac{\beta u_c\left(x_{i+1}, y_{i+1}\right)}{y_i \bar{p}\left(x_{i+1}, y_{i+1}\right)}, \;\; i = 0, 1.$$

$$R(x_0, y_0) = \frac{1}{\beta} \frac{\sigma\left(x_1\right) c u_c(x_0, y_0)}{c u_c\left(x_1, y_1\right)}$$

$$a_1(x_0, y_0) = q(x_0, y_0) \bar{p}(x_0, y_0)^{\frac{2-\lambda}{1-\lambda}} \frac{w}{\bar{p}}(x_0, y_0) R(x_0, y_0) c(x_0, y_0),$$

$$a_2(x_0, y_0) = q(x_0, y_0) \bar{p}(x_0, y_0)^{\frac{1}{1-\lambda}} c(x_0, y_0).$$

$$a_3(x_0, y_0) = \frac{\beta}{y_0} q(x_1, y_1) \left(\bar{p}(x_1, y_1)\mu\right)^{\frac{2-\lambda}{1-\lambda}} \frac{w}{\bar{p}}(x_1, y_1) R(x_1, y_1) c(x_1, y_1)$$

$$a_4(x_0, y_0) = \frac{\beta}{\mu} q(x_1, y_1) \left(\bar{p}(x_1, y_1)\mu\right)^{\frac{1}{1-\lambda}} c(x_1, y_1)$$

3. Finally:

$$F_p(p_{-1}, \mu; \gamma_p | \gamma_\sigma) = p\left(x_0, y_0\right) - \frac{a_1(x_0, y_0) + a_3(x_0, y_0)}{\lambda\left(a_2(x_0, y_0) + a_4(x_0, y_0)\right)}$$

Note, $F_p : (P \times \Lambda) \to R$. If we somehow managed to have in hand the exact function, $\sigma(p_{-1})$, and the function space in which we looked for $p(p_{-1}, \mu)$ contained the exact function, $p(p_{-1}, \mu)$, then it would be possible to select γ_p so that $F :$ $(P \times \Lambda) \to 0$. Presumably we can't do this, but we'll select γ_p to come as 'close' to this as possible instead.

2. In the following we define the equations used to select γ_V. These calculations presume values for γ_p and γ_σ are available. We find γ_V so that

$$F_V(p_{-1}; \gamma_V | \gamma_p, \gamma_\sigma) = V(p_{-1}) - u\left(c\left(p_{-1}, \sigma(p_{-1})\right), l\left(p_{-1}, \sigma(p_{-1})\right)\right) - \beta V\left(\frac{p\left(p_{-1}, \sigma(p_{-1})\right)}{\sigma(p_{-1})}\right),$$

is small $p_{-1} \in P$.

As noted before, the values of γ_V and γ_p that satisfy the above equations, do so conditional on a presumed value for γ_σ. To find γ_σ, we select a value for it so that

$$F_\sigma(p_{-1};\gamma_\sigma) = \hat{\sigma}(p_{-1};\gamma_\sigma) - \arg\max_{\mu}\left[u\left(c\left(p_{-1},\mu\right), l\left(p_{-1},\mu\right)\right) + \beta V\left(\frac{p\left(p_{-1},\mu\right)}{\mu}\right)\right],$$

is small for $p_{-1} \in P$. Here, for each γ_σ, F_σ is evaluated by first obtaining values of γ_p and γ_V, as discussed above. The parameter vector γ_p determines the functions c and l, while γ_V determines V.

To summarize, a 'solution' to our problem is a set of parameters, $\gamma_V, \gamma_p, \gamma_\sigma$, such that:

$$\begin{aligned}
&F_\sigma(p_{-1};\gamma_\sigma) \text{ small for } p_{-1} \in P \\
&F_p(p_{-1},\mu;\gamma_p|\gamma_\sigma) \text{ small for } (p_{-1},\mu) \in (P \times \Lambda) \qquad (3.2)\\
&F_V(p_{-1};\gamma_V|\gamma_p,\gamma_\sigma) \text{ small for } p_{-1} \in P.
\end{aligned}$$

3.2. Details About the Strategy

Three things need to be addressed: nature of the sets, P, Λ; the metric for determining that $F_\sigma(p_{-1};\gamma_\sigma)$, $F_V(p_{-1};\gamma_V|\gamma_p,\gamma_\sigma)$, and $F_p(p_{-1},\mu;\gamma_p|\gamma_\sigma)$ are small; and a space of functions for $\sigma(p_{-1})$, $p(p_{-1},\mu), V(p_{-1})$. Let p_{-1}^l and p_{-1}^u define the lower and upper boundaries of P, and let μ^l and μ^u define the lower and upper boundaries of Λ. We discuss the problem of finding γ_σ first.

3.2.1. The Function, F_σ

The strategy we will adopt in solving (3.2) is to convert it into a problem in solving three equations in three sets of parameters, $\gamma_\mu, \gamma_p, \gamma_\sigma$. These equations could be solved simultaneously. However, we will in effect substitute the second two into the first. Thus, we set up the problem as one of finding γ_σ to make F_σ small. The other two equations will then be used to define F_σ. Let $\hat{\sigma}(p_{-1}; \gamma_\sigma)$ denote the function we use to approximate $\sigma(p_{-1})$, and let n_σ denote the number of elements in the column vector, γ_σ. To discuss the computational issues relevant to solving our problem, it is useful to write it in weighted residual form:

$$\int_{p_{-1}^l}^{p_{-1}^u} F_\sigma(p_{-1}; \gamma_\sigma) w_i(p_{-1}) dp_{-1} = 0, \ i = 1, ..., n_\sigma, \tag{3.3}$$

where the choice of weighting functions, $w_i(p_{-1})$, operationalizes the notion of 'small'. The three issues that have to be confronted are evident in the weighted residual representation: a choice of $\hat{\sigma}(p_{-1}; \gamma_\sigma)$ is need, we require a set of n_σ weighting functions, and P must be constructed. The menu is as follows.

There are two types of finite parameter functions: spectral and finite-element. In spectral functions (e.g., polynomials) each element of γ_σ has a global impact on the function, $\hat{\sigma}(p_{-1}; \gamma_\sigma)$. In finite-element functions (e.g., sequences of straight lines, or polynomials) each element of γ_σ controls $\hat{\sigma}(p_{-1}; \gamma_\sigma)$ only for p_{-1} belonging to a strict subset of P (an important text on finite element methods is Reddy.) For functions whose behavior looks very different in some regions than others, finite-element approximations are presumably efficient.

There are two major types of weighting functions, among others. These correspond to the Galerkin and Collocation methods for approximating functions.

1. Galerkin. Here, the weighting functions, w_i, are constructed from the basis functions generating $\hat{\sigma}(p_{-1};\gamma_\sigma)$.

2. Collocation. Here, the weighting functions are delta functions, so that the weighted residual form just reduces to $F_\sigma(p_{-1};\gamma_\sigma)=0$ for n values of $p_{-1}\in P$.

Regardless of which method is used, there is an excellent accuracy check available, based in part on comparing the graph of $F_\sigma(p_{-1};\gamma_\sigma)$ for $p_{-1}\in P$, with the zero line. This cannot be the only check, however. That's because the function F_σ is itself only an approximation, since it is constructed using approximations to $p(p_{-1},\mu)$ and $V(p_{-1})$ (see below). Further checks, which focus on derivatives of (2.5) and the first order condition associated with the maximization problem in (1.2) are also important.

3.2.2. The Spectral-Galerkin Method

We will now describe a Spectral-Galerkin method, which uses Chebyshev polynomials to approximate $\hat{\sigma}(p_{-1};\gamma_\sigma)$. This method has been advocated by Judd for solving dynamic, economic problems. The policy function is:

$$\hat{\sigma}(p_{-1};\gamma_\sigma)=\mu^l+\frac{\mu^u-\mu^l}{1+\exp\left[-\gamma_\sigma' T\left(\varphi_p\left(\log(p_{-1})\right)\right)\right]}, \tag{3.4}$$

where

$$\gamma_\sigma=\begin{bmatrix}\gamma_{\sigma,1}\\ \gamma_{\sigma,2}\\ \vdots\\ \gamma_{\sigma,n_\sigma}\end{bmatrix},\ T(x)=\begin{bmatrix}T_0(x)\\ T_1(x)\\ \vdots\\ T_{n_\sigma-1}(x)\end{bmatrix},\ x\in[-1,1] \tag{3.5}$$

and

$$\varphi_p(x) = 2\frac{x - \underline{x}}{\overline{x} - \underline{x}} - 1,\ \varphi_p : [\underline{x}, \overline{x}] \rightarrow [-1, 1].$$

Here,

$$\underline{x} = \log(p_{-1}^l),\ \overline{x} = \log(p_{-1}^u).$$

Note:

$$\varphi_p^{-1}(r) = \underline{x} + \frac{1}{2}(1 + r)(\overline{x} - \underline{x}).$$

Also, the Chebyshev polynomials, $T_i : [-1, 1] \rightarrow [-1, 1]$, are defined as follows:

$$T_0(r) = 1,\ T_1(r) = r,\ T_i(r) = 2rT_{i-1}(r) - T_{i-2}(r),\ i \geq 2. \tag{3.6}$$

The form of the policy function is restricted so that a money growth rate less than μ^l is impossible. To illustrate (3.4), it is useful to consider the case $n_\sigma = 2$:

$$\begin{aligned} \hat{\sigma}(p_{-1}; \gamma_\sigma) &= \mu^l + \frac{\mu^u - \mu^l}{1 + \exp\left\{-\gamma_{\sigma,1} - \gamma_{\sigma,2}\varphi_p\left(\log(p_{-1})\right)\right\}} \\ &= \mu^l + \frac{\mu^u - \mu^l}{1 + \exp\left\{-\left[\gamma_{\sigma,1} - \gamma_{\sigma,2}\left(\frac{\overline{x}+\underline{x}}{\overline{x}-\underline{x}}\right)\right] - \frac{2\gamma_{\sigma,2}}{\overline{x}-\underline{x}}\log(p_{-1})\right\}} \end{aligned}$$

Note that, by construction, $\hat{\sigma}(p_{-1}; \gamma_\sigma) : P \rightarrow \Lambda$. Also, as $\gamma_\sigma \rightarrow +\infty$, $\hat{\sigma} \rightarrow \mu^u$ and as $\gamma_\sigma \rightarrow -\infty$, $\hat{\sigma} \rightarrow \mu^l$ for all $p_{-1} \in P$.

Next, compute $r_1, ..., r_{m_\sigma}$, the m_σ roots of $T_{m_\sigma}(r)$, $m_\sigma \geq n_\sigma$, and the associated values of p_{-1}:

$$r_i = \cos\left(\frac{\pi\left(i - \frac{1}{2}\right)}{m_\sigma}\right),\ p_{-1,i} = \exp\left[\varphi_p^{-1}\left(r_i\right)\right],\ i = 1, ..., m_\sigma,$$

and form the $n \times m$ matrix and $m \times 1$ vector:

$$A_\sigma = \begin{bmatrix} T_0(r_1) & T_0(r_2) & \cdots & T_0(r_{m_\sigma}) \\ T_1(r_1) & T_1(r_2) & \cdots & T_1(r_{m_\sigma}) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n_\sigma-1}(r_1) & T_{n_\sigma-1}(r_2) & \cdots & T_{n_\sigma-1}(r_{m_\sigma}) \end{bmatrix}, \; F_\sigma\left(\gamma_\sigma\right) = \begin{bmatrix} F_\sigma(p_{-1,1};\gamma_\sigma) \\ F_\sigma(p_{-1,2};\gamma_\sigma) \\ \vdots \\ F_\sigma(p_{-1,m_\sigma};\gamma_\sigma) \end{bmatrix}.$$

A property of the matrix A_σ is that each row is orthogonal to all the others. That is, $A_\sigma A'_\sigma$ is a diagonal matrix. Form the following nonlinear system of n_σ equations in the n_σ unknowns, γ_σ:

$$A_\sigma F_\sigma\left(\gamma_\sigma\right) = 0. \tag{3.7}$$

This can be solved using standard computational techniques. Solving these equations amounts to solving (3.3) for a particular weighting function, using m_σ-point Guass-Chebyshev quadrature to evaluate the integral. When $n_\sigma = m_\sigma$, solving (3.7) is equivalent to simply setting $F_\sigma\left(\gamma_\sigma\right) = 0$, and the method corresponds to Collocation instead of Galerkin.

3.2.3. The Other Functions

As noted previously, to define the set of equations, (3.7), we need to approximate the functions $p(p_{-1},\mu)$ and $V(p_{-1})$. We do this using the Spectral-Galerkin method described above. For this, suppose a function, $\sigma(p_{-1})$ is available. This is the case, for example, if we're in the middle of evaluating (3.7) for some particular value of γ_σ.

We compute approximations to $p(p_{-1},\mu)$ and $V(p_{-1})$ sequentially. Denote these approximations by $\hat{p}(p_{-1},\mu;\gamma_p)$ and $\hat{V}(p_{-1};\gamma_V)$, respectively.

The Function, $p(p_{-1}, \mu)$

We denote the number of elements in the column vector, γ_p, by n_p, and we assume $(n_p)^{1/2}$ is an integer. For $\hat{p}(p_{-1}, \mu; \gamma_p)$, we adopt the tensor product basis for degree $(n_p)^{1/2} - 1$ Chebyshev polynomial functions in 2 variables:

$$\Phi = \left\{ T_i(x) T_j(y) : i, j = 0, 1, ..., n_p^{1/2} - 1 \right\},$$

where $(x, y) \in [-1, 1] \times [-1, 1]$ and the T_i's are defined in (3.6). Note that Φ contains n_p elements, corresponding to the n_p possible values of (i, j). Denote these by the functions,

$$\phi_l(x, y) : [-1, 1] \times [-1, 1] \to [-1, 1] \times [-1, 1], \; l = 1, ..., n_p.$$

To construct these functions, we use the Kronecker tensor product. Thus, define, as in (3.5),

$$T(x) = \begin{bmatrix} T_0(x) \\ T_1(x) \\ \vdots \\ T_{n_p^{1/2}-1}(x) \end{bmatrix}, \; T(y) = \begin{bmatrix} T_0(y) \\ T_1(y) \\ \vdots \\ T_{n_p^{1/2}-1}(y) \end{bmatrix}, \; x, y \in [-1, 1].$$

Then,[2]

$$\phi(x,y) = T(x) \otimes T(y),\ \ \phi(x,y) : [-1,1]^2 \to [-1,1]^{n_p}.$$

That is, for any $x, y \in [-1,1]^2$, $\phi(x,y)$ is an $n_p \times 1$ vector of numbers, each of which belongs to $[-1,1]$. Our approximating function is:

$$\hat{p}(p_{-1}, \mu; \gamma_p) = \mu \left\{ p_{-1}^l + \frac{p_{-1}^u - p_{-1}^l}{1 + \exp\left[-\gamma_p' \phi\left(\varphi_p\left(\log(p_{-1})\right), \varphi_\mu\left(\mu\right)\right)\right]} \right\}, \tag{3.8}$$

where φ_p is defined above and

$$\varphi_\mu(x) = 2\frac{\mu - \mu^l}{\mu^u - \mu^l} - 1,\ \ \varphi_\mu : [\mu^l, \mu^u] \to [-1,1].$$

To illustrate (3.8), it is useful to consider the case $n_p = 4$:

$$\begin{aligned}
\hat{p}(p_{-1}, \mu; \gamma_p) &= \mu \left\{ p_{-1}^l + \frac{p_{-1}^u - p_{-1}^l}{1 + \exp\left[-\gamma_{p,1} - \gamma_{p,2}\varphi_\mu\left(\mu\right) - \gamma_{p,3}\varphi_p\left(\log(p_{-1})\right) - \gamma_{p,4}\varphi_p\left(\log(p_{-1})\right)\varphi_\mu\left(\mu\right)\right]} \right\} \\
&= \mu \left\{ p_{-1}^l + \frac{p_{-1}^u - p_{-1}^l}{1 + \exp\left[-\psi_1 - \psi_2\mu - \psi_3 \log(p_{-1}) - \psi_4 \mu \log(p_{-1})\right]} \right\}
\end{aligned}$$

[2] Recall that the Kronecker product, $\otimes$, of the $n_a \times m_a$ matrix a and the $n_b \times m_b$ matrix b is the $(n_a n_b) \times (m_a m_b)$ matrix c, where

$$c = \begin{bmatrix} a_{11}b & a_{12}b & \cdots & a_{1m_a}b \\ a_{21}b & a_{22}b & \cdots & a_{2m_a}b \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_a}b & a_{n_a}b & \cdots & a_{n_a m_a}b \end{bmatrix}.$$

where

$$
\begin{aligned}
\psi_1 &= \gamma_{p,1} - \gamma_{p,2}\left(\frac{\mu^u+\mu^l}{\mu^u-\mu^l}\right) - \gamma_{p,3}\left(\frac{\overline{x}+\underline{x}}{\overline{x}-\underline{x}}\right) + \gamma_{p,4}\left(\frac{\mu^u+\mu^l}{\mu^u-\mu^l}\right)\left(\frac{\overline{x}+\underline{x}}{\overline{x}-\underline{x}}\right) \\
\psi_2 &= \frac{2}{\mu^u-\mu^l}\left[\gamma_{p,2} - \gamma_{p,4}\left(\frac{\overline{x}+\underline{x}}{\overline{x}-\underline{x}}\right)\right] \\
\psi_3 &= \frac{2}{\overline{x}-\underline{x}}\left[\gamma_{p,3} - \gamma_{p,4}\left(\frac{\mu^u+\mu^l}{\mu^u-\mu^l}\right)\right] \\
\psi_4 &= \frac{4\gamma_{p,4}}{\left(\overline{x}-\underline{x}\right)\left(\mu^u-\mu^l\right)}.
\end{aligned}
$$

Next, compute $r_1, ..., r_{m_p^{1/2}}$, the $m_p^{1/2}$ roots of $T_{m_p^{1/2}}(x)$, $m_p \geq n_p$. Let the $m_p \times 2$ matrix $\bar{r}$ be defined as follows:

$$
\bar{r} = \begin{bmatrix}
r_1 & r_1 \\
r_1 & r_2 \\
\vdots & \vdots \\
r_1 & r_{m_p^{1/2}} \\
r_2 & r_1 \\
r_2 & r_2 \\
\vdots & \vdots \\
r_2 & r_{m_p^{1/2}} \\
\vdots & \vdots \\
\vdots & \vdots \\
r_{m_p^{1/2}} & r_1 \\
r_{m_p^{1/2}} & r_2 \\
\vdots & \vdots \\
r_{m_p^{1/2}} & r_{m_p^{1/2}}
\end{bmatrix}
$$

Thus, $\bar{r}_i = (\bar{r}_{i,1}, \bar{r}_{i,2})$ is a 1×2 vector, $i = 1, ..., m_p$. Let

$$s_{i,1} = \exp\left[\varphi_p^{-1}\left(\bar{r}_{1,i}\right)\right],\ s_{i,2} = \varphi_\mu^{-1}(\bar{r}_{2,i}),\ \ i = 1, ..., m_p,$$

so that $s_i = (s_{i,1}, s_{i,2})$ is the value of $(p_{-1}, \mu) \in P \times \Lambda$ associated with $\bar{r}_i$, $i = 1, ..., m_p$. Form the $n_p \times m_p$ matrix A_p based on the $n_p^{1/2} \times m_p^{1/2}$ matrix A :

$$A_p = A \otimes A,\ \ A = \begin{bmatrix} T_0(r_1) & T_0(r_2) & \cdots & T_0(r_{m_p^{1/2}}) \\ T_1(r_1) & T_1(r_2) & \cdots & T_1(r_{m_p^{1/2}}) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n_p^{1/2}-1}(r_1) & T_{n_p^{1/2}-1}(r_2) & \cdots & T_{n_p^{1/2}-1}(r_{m_p^{1/2}}) \end{bmatrix}.$$

Also, form the $m^p \times 1$ vector:

$$F_p\left(\gamma_p\right) = \begin{bmatrix} F_p(s_1; \gamma_\sigma) \\ F_p(s_2; \gamma_\sigma) \\ \vdots \\ F_p(s_{m^p}; \gamma_\sigma) \end{bmatrix}.$$

Form the following nonlinear system of n_p equations in the n_p unknowns, γ_p:

$$A_p F_p\left(\gamma_p\right) = 0. \tag{3.9}$$

A value for γ_p may be found by solving these equations using standard methods. Note that each row of A_p is orthogonal to all the others, since $A_p A_p'$ is a diagonal matrix.

The Function, $V(p_{-1})$

We find the approximating function to $V(p_{-1})$, $\hat{V}(p_{-1}; \gamma_V)$, using the same strategy used for approximating $\sigma(p_{-1})$. Thus, we approximate $V(p_{-1})$ as follows:

$$\hat{V}(p_{-1}; \gamma_V) = \gamma_V' T \left(\varphi_p \left(\log(p_{-1}) \right) \right),$$

where n_V is the number of elements in the column vector, γ_V. Conditional on values for γ_σ and γ_p, we solve for γ_V by solving the appropriately modified version of equations (3.7).