Perturbation Method

- Example:

$$h(x, y) = 0.$$

- Policy Rule:

$$y = g(x).$$

- Functional Equation:

$$R(x; g) = h(x, g(x)) = 0$$

- Know:

$$g(x^*) = g^*, \text{ some } x^*$$

$$R^{(n)}(x) = 0, \text{ for } n = 1, 2, ...$$

- First order perturbation:

$$R^{(1)}(x^*) \; = \; h_1(x, g^*) + h_2(x, g^*) g'(x^*) = 0$$

Solve this for $g'(x^*)$.

- Second order perturbation:

$$\begin{aligned} R^{(2)}\left(x^*\right) &= h_{11}\left(x^*, g^*\right) + 2h_{12}\left(x^*, g^*\right) g'\left(x^*\right) \\ & h_{22}\left(x^*, g^*\right)\left[g'\left(x^*\right)\right]^2 + h_2\left(x^*, g^*\right) g''\left(x^*\right). \end{aligned}$$

Solve this for $g''\left(x^*\right)$.

- Recursively, Solve $R^{(n)}\left(x^*\right)$ for $g^{(n)}\left(x^*\right)$, conditional on $g^{(n-i)}\left(x^*\right)$, $i = 1, 2, ..., n$.

- Finally, for n^{th} order perturbation:

$$g\left(x\right) \approx g^* + g'\left(x^*\right) \times \left(x - x^*\right) + \frac{1}{2} g''\left(x^*\right) \times \left(x - x^*\right)^2 + ... + \frac{1}{n!} g^{(n)}\left(x^*\right) \times \left(x - x^*\right)^n$$

Model

- Preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

- Technology:

$$c_t + \exp(k_{t+1}) \leq f(k_t, a_t),\ t = 0, 1, 2, \ldots.$$

$$a_t \ = \ \rho a_{t-1} + \varepsilon_t.$$

$$f(k_t, a_t) \ = \ \exp(\alpha k_t) \exp(a_t) + (1-\delta)\exp(k_t).$$

  – Euler Equation:

$$E_t[u'(f(k_t, \rho a_{t-1} + \varepsilon_t) - \exp(k_{t+1}))$$

$$-\beta u'\left(f\left(k_{t+1}, \rho^2 a_{t-1} + \rho\varepsilon_t + \varepsilon_{t+1}\right) - \exp(k_{t+2})\right) f_K\left(k_{t+1}, \rho^2 a_{t-1} + \rho\varepsilon_t + \varepsilon_{t+1}\right)] \ = \ 0.$$

Model ...

- Seek Policy Rule:

$$k_{t+1} = g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right).$$

- Must Satisfy:

$$R\left(k_t, a_{t-1}, \varepsilon_t, \sigma; \{\varepsilon_{t+1}\}\right) \equiv E_t[u'\left(f\left(k_t, \rho a_{t-1} + \varepsilon_t\right) - g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right)\right)$$

$$-\beta u'\left(f\left(g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right), \rho^2 a_{t-1} + \rho\varepsilon_t + \sigma\varepsilon_{t+1}\right) - g\left(g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right), a_t, \sigma\varepsilon_{t+1}, \sigma\right)\right)$$

$$\times f_K\left(g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right), \rho^2 a_{t-1} + \rho\varepsilon_t + \sigma\varepsilon_{t+1}\right)]$$

$$= 0.$$

Model ...

- Value of the function is Known At a Particular Point:

$$k = g\left(k, 0, 0, 0\right),$$

$$k = \log\left\{\left[\frac{\alpha\beta}{1-\left(1-\delta\right)\beta}\right]^{\frac{1}{1-\alpha}}\right\}.$$

- Second Order Perturbation:

$$\begin{aligned}
&\hat{g}\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right) \\
= \;& k + g_k\left(k_t - k\right) + g_a a_{t-1} + g_\varepsilon \varepsilon_t + g_\sigma \sigma \\
&+\frac{1}{2}[g_{kk}\left(k_t - k\right)^2 + g_{aa} a_{t-1}^2 + g_{\varepsilon\varepsilon}\varepsilon_t^2 + g_{\sigma\sigma}\sigma^2] \\
&+g_{ka}\left(k_t - k\right) a_{t-1} + g_{k\varepsilon}\left(k_t - k\right)\varepsilon_t + g_{k\sigma}\left(k_t - k\right)\sigma \\
&+g_{a\varepsilon} a_{t-1}\varepsilon_t + g_{a\sigma} a_{t-1}\sigma + g_{\varepsilon\sigma}\varepsilon_t \sigma
\end{aligned}$$

- Error function:

$$
\begin{aligned}
&R\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right) \\
&\equiv E_t[u'\left(f\left(k_t, \rho a_{t-1} + \varepsilon_t\right) - g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right)\right) \\
&-\beta u'(f\left(g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right), \rho^2 a_{t-1} + \rho\varepsilon_t + \sigma\varepsilon_{t+1}\right) \\
&-g\left(g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right), \rho a_{t-1} + \varepsilon_t, \sigma\varepsilon_{t+1}, \sigma\right)) \\
&\times f_k\left(g\left(k_t, a_{t-1}, \varepsilon_t, \sigma\right), \rho^2 a_{t-1} + \rho\varepsilon_t + \sigma\varepsilon_{t+1}\right)] = 0.
\end{aligned}
$$

Model ...

- Differentiating R and evaluating at $k_t = k$, $a_{t-1} = 0$, $\varepsilon_t = 0$, $\sigma = 1$, $E_t \varepsilon_{t+1} = 0$:

$$R_k = u'' (f_k - g_k) - \beta u'' \left(f_k g_k - g_k^2\right) f_k - \beta u' f_{kk} g_k = 0$$

$$R_a = u'' (f_2 \rho - g_a) - \beta u'' \left(f_k g_a + f_2 \rho^2 - g_k g_a - g_a \rho\right) f_k - \beta u' \left(f_{kk} g_a + f_{k2} \rho^2\right) = 0$$

$$R_\varepsilon = u'' (f_2 - g_\varepsilon) - \beta u'' (f_k g_\varepsilon + f_2 \rho - g_k g_\varepsilon - g_a) f_k - \beta u' (f_{kk} g_\varepsilon + f_{k2} \rho)$$

$$R_\sigma = -\left[u'' + \beta u'' (f_k - g_k - 1) + \beta u' f_{kk}\right] g_\sigma$$

- Finding g_k :

$$g_k^2 - \left[1 + \frac{1}{\beta} + \beta \frac{u'}{u''} f_{kk}\right] g_k + \frac{1}{\beta} = 0, \; f_{kk} \frac{u'}{u''} = \alpha (1 - \alpha) k^{\alpha - 2} \frac{c}{\gamma}.$$