

Notes on the Perturbation Method

I Introduction

These notes provide a brief introduction to the use of perturbation methods for solving the policy rule in an equilibrium model.* The example used is the neoclassical growth model with a consumption/saving decision and one source of uncertainty. Briefly, perturbation methods use the implicit function theorem to compute various derivatives of the function of interest. Those derivatives can then be used to construct the Taylor series approximation to the equilibrium function of interest. For example, suppose we have a model in which there is an exogenous variables, x , and an endogenous variable, y , and an equilibrium condition which links the two:

$$h(x, y) = 0.$$

The equilibrium policy rule, $y = g(x)$ has the property:

$$R(x) = h(x, g(x)) = 0.$$

Since $R(x) = 0$ for all x , we know that $R^{(n)}(x) = 0$ for $n = 1, 2, \dots$. Suppose the necessary differentiability conditions are satisfied. Then,

$$\begin{aligned} R^{(1)}(x) &= h_1(x, g(x)) + h_2(x, g(x)) g'(x) \\ R^{(2)}(x) &= h_{11}(x, g(x)) + 2h_{12}(x, g(x)) g'(x) \\ &\quad + h_{22}(x, g(x)) [g'(x)]^2 + h_2(x, g(x)) g''(x), \end{aligned}$$

and so on, for $R^{(n)}$, $n = 1, 2, \dots$. Suppose we know the value of the function, g , at a particular point, say x^* . That is, the value of $g(x^*) \equiv g$ is known. Let

$$h_i \equiv h_i(x^*, g), \quad i = 1, 2$$

$$h_{ij} \equiv h_{ij}(x^*, g), \quad i, j = 1, 2.$$

The objects, h_i and h_{ij} , $i, j = 1, 2$, are computable because g and the function, h , are known.

We can now compute g' , g'' using the fact, $R^{(n)} = 0$ for $n = 1, 2$:

$$\begin{aligned} g' &= -\frac{h_1}{h_2} \\ g'' &= -\frac{h_{11} + 2h_{12}g' + h_{22}[g']^2}{h_2}. \end{aligned}$$

This strategy can obviously be used to compute $g^{(n)}$, for $n = 3, 4, 5, \dots$. We can use these derivatives to construct an approximation to $g(x)$. For example, they can be used to construct the Taylor series expansion. According to Taylor's theorem, if $g^{(i)}(x)$ is continuous on $[a, b]$, for $i = 1, \dots, n-1$, and $g^{(n)}(x)$ is continuous on (a, b) , then there exists an $\tilde{x} \in (a, b)$ such that

$$\begin{aligned} g(x) &= g + g' \times (x - x^*) + \frac{1}{2}g'' \times (x - x^*)^2 + \dots + \frac{1}{(n-1)!}g^{(n-1)} \times (x - x^*)^{n-1} \\ &\quad + \frac{1}{n!}g^{(n)}(\tilde{x}) \times (x - x^*)^n \end{aligned}$$

An implication is that as long as g is differentiable and $g^{(n)}$ is small relative to $n!$, the Taylor series expansion is highly accurate for x over a potentially very large range.[†]

II Neoclassical Growth Model Without Labor

Consider a version of the neoclassical growth model with preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

and with resource constraint:

$$c_t + \exp(k_{t+1}) \leq f(k_t, a_t), \quad t = 0, 1, 2, \dots$$

where

$$\begin{aligned} a_t &= \rho a_{t-1} + \varepsilon_t. \\ u(c_t) &= \frac{c_t^{1-\gamma}}{1-\gamma}, \quad f(k_t, a_t) = \exp(\alpha k_t) \exp(a_t) + (1-\delta) \exp(k_t). \end{aligned}$$

Here, k_{t+1} denotes the log of the capital stock at the beginning of time $t+1$.

The equilibrium condition for this model is, after combining the resource constraint, the law of motion of the shock and the intertemporal Euler equation:

$$\begin{aligned} &E_t[u'(f(k_t, \rho a_{t-1} + \varepsilon_t) - \exp(k_{t+1})) \\ &-\beta u'(f(k_{t+1}, \rho^2 a_{t-1} + \rho \varepsilon_t + \varepsilon_{t+1}) - \exp(k_{t+2})) f_K(k_{t+1}, \rho^2 a_{t-1} + \rho \varepsilon_t + \varepsilon_{t+1})] = 0. \end{aligned}$$

Here, f_K denotes the marginal physical product of capital (not, the log of capital!). At date t , the variable, ε_t , is observed and future realizations of ε_t are as yet unknown. We suppose the latter are realizations of a random variable with variance V_ε .

We find it convenient to consider a sequence of problems, in which the variance of ε_{t+1} is σV_ε , for σ ranging from zero to unity. Let the policy rule which solves the model for given σ be:

$$k_{t+1} = g(k_t, a_{t-1}, \varepsilon_t, \sigma).$$

The function, g , satisfies the following functional equation:

$$\begin{aligned} R(k_t, a_{t-1}, \varepsilon_t, \sigma; \varepsilon_{t+1}) &\equiv E_t[u'(f(k_t, \rho a_{t-1} + \varepsilon_t) - g(k_t, a_{t-1}, \varepsilon_t, \sigma)) \\ &-\beta u'(f(g(k_t, a_{t-1}, \varepsilon_t, \sigma), \rho^2 a_{t-1} + \rho \varepsilon_t + \sigma \varepsilon_{t+1}) - g(g(k_t, a_{t-1}, \varepsilon_t, \sigma), a_t, \sigma \varepsilon_{t+1}, \sigma)) \\ &\times f_K(g(k_t, a_{t-1}, \varepsilon_t, \sigma), \rho^2 a_{t-1} + \rho \varepsilon_t + \sigma \varepsilon_{t+1})] = 0. \end{aligned}$$

Note that $R(k_t, a_{t-1}, \varepsilon_t) = 0$, so that all the derivatives of this function are zero as well.

The policy rule, g , is the analog of the function, g , in the previous section. The value of this function, at the non-stochastic steady state is known. That is, if k denotes the non-stochastic steady state value of the log of capital, then,

$$k = g(k, 0, 0, 0).$$

We compute k by imposing $f_K \beta = 1$:

$$k = \log \left\{ \left[\frac{\alpha \beta}{1 - (1 - \delta) \beta} \right]^{\frac{1}{1-\alpha}} \right\}.$$

We seek

$$\begin{aligned} (1) \quad & \hat{g}(k_t, a_{t-1}, \varepsilon_t, \sigma) \\ &= k + g_k(k_t - k) + g_a a_{t-1} + g_\varepsilon \varepsilon_t + g_\sigma \sigma \\ & \quad + \frac{1}{2} [g_{kk}(k_t - k)^2 + g_{aa} a_{t-1}^2 + g_{\varepsilon\varepsilon} \varepsilon_t^2 + g_{\sigma\sigma} \sigma^2] \\ & \quad + g_{ka}(k_t - k) a_{t-1} + g_{k\varepsilon}(k_t - k) \varepsilon_t + g_{k\sigma}(k_t - k) \sigma \\ & \quad + g_{a\varepsilon} a_{t-1} \varepsilon_t + g_{a\sigma} a_{t-1} \sigma + g_{\varepsilon\sigma} \varepsilon_t \sigma \end{aligned}$$

where $\hat{g}$ is the second-order Taylor series expansion of g about $k_t = k$, $a_{t-1} = \varepsilon_t = \sigma = 0$. Here, as everywhere, a function whose arguments are missing is assumed to be evaluated in nonstochastic steady state.

$$\begin{aligned} R(k_t, a_{t-1}, \varepsilon_t, \sigma) &\equiv E_t[u'(f(k_t, \rho a_{t-1} + \varepsilon_t) - g(k_t, a_{t-1}, \varepsilon_t, \sigma))] \\ & - \beta u'(f(g(k_t, a_{t-1}, \varepsilon_t, \sigma), \rho^2 a_{t-1} + \rho \varepsilon_t + \sigma \varepsilon_{t+1}) - g(g(k_t, a_{t-1}, \varepsilon_t, \sigma), \rho a_{t-1} + \varepsilon_t, \sigma \varepsilon_{t+1}, \sigma)) \\ & \quad \times f_k(g(k_t, a_{t-1}, \varepsilon_t, \sigma), \rho^2 a_{t-1} + \rho \varepsilon_t + \sigma \varepsilon_{t+1}))] = 0. \end{aligned}$$

Differentiating R and evaluating at $k_t = k$, $a_{t-1} = 0$, $\varepsilon_t = 0$, $\sigma = 1$, $E_t \varepsilon_{t+1} = 0$:

$$\begin{aligned} R_k &= u''(f_k - g_k) - \beta u''(f_k g_k - g_k^2) f_k - \beta u' f_{kk} g_k = 0 \\ R_a &= u''(f_2 \rho - g_a) - \beta u''(f_k g_a + f_2 \rho^2 - g_k g_a - g_a \rho) f_k - \beta u'(f_{kk} g_a + f_{k2} \rho^2) = 0 \\ R_\varepsilon &= u''(f_2 - g_\varepsilon) - \beta u''(f_k g_\varepsilon + f_2 \rho - g_k g_\varepsilon - g_a) f_k - \beta u'(f_{kk} g_\varepsilon + f_{k2} \rho) \\ R_\sigma &= -[u'' + \beta u''(f_k - g_k - 1) + \beta u' f_{kk}] g_\sigma \end{aligned}$$

To solve this, note that R_k is a function of g_k only. The value of g_k that we want satisfies $|g_k| < 1$ (see Stokey-Lucas on this) and

$$g_k^2 - \phi g_k + \frac{1}{\beta} = 0,$$

where

$$\begin{aligned}\phi &= 1 + \frac{1}{\beta} + \beta \frac{u'}{u''} f_{kk}, \\ f_{kk} \frac{u'}{u''} &= \alpha(1 - \alpha) k^{\alpha-2} \frac{c}{\gamma}.\end{aligned}$$

Note that given g_k , we can solve linearly for g_a and g_ε . Note that if $\rho = 0$, then $g_a = 0$. Note, too, that $g_\sigma = 0$. The latter implies that when a linear perturbation about steady state is done, there is no need to adjust the constant term in the policy rule, to accommodate uncertainty.

To obtain the coefficients in the second-order part of the expansion, we compute all the second derivatives of R and in each case we evaluate the result at $k_t = k$, $a_{t-1} = 0$, $\varepsilon_t = 0$, $\sigma = 0$. Let R_{ji} denote the cross derivative of R with respect to j and i , $j, i = k, a, \varepsilon, \sigma$, obtained in this way. The objects, R_{ji} , are functions of all the coefficients in $\hat{g}$ in (1).

There is considerable structure on R_{ji} . The most important is that, given coefficients on the linear terms in (1), R_{ji} is a linear function of the coefficients on the cross terms. This greatly simplifies the problem of determining the value of these coefficients which set $R_{ji} = 0$, for $j, i = k, a, \varepsilon, \sigma$. There is additional structure, which simplifies this problem even more. First, it is easy to show that $g_{j\sigma}$ must be zero for $j = k, a, \varepsilon$. That is, in a second order expansion of g about steady state there is no need to adjust the slope terms to accommodate uncertainty. To see this, note that the derivative of the error function, $R_j(k_t, a_{t-1}, \varepsilon_t; \sigma)$, has the following form, for $j = k, a, \varepsilon$:

$$\begin{aligned}R_j(k_t, a_{t-1}, \varepsilon_t; \sigma) &= \int h[k_t, a_{t-1}, \sigma \varepsilon_{t+1}, g(k_t, a_{t-1}, \varepsilon_t, \sigma), \\ &\quad g(g(k_t, a_{t-1}, \varepsilon_t, \sigma), \rho a_{t-1} + \varepsilon_t, \sigma \varepsilon_{t+1}, \sigma), \\ &\quad g_j(k_t, a_{t-1}, \varepsilon_t, \sigma), g_j(g(k_t, a_{t-1}, \varepsilon_t, \sigma), \rho a_{t-1} + \varepsilon_t, \sigma \varepsilon_{t+1}, \sigma)] dF(\varepsilon_{t+1}),\end{aligned}$$

where F denotes the distribution function associated with ε_{t+1} . In the above expression, we have isolated the various ways that σ enters $R_j(k_t, a_{t-1}, \varepsilon_t; \sigma)$. Differentiating this expression

with respect to σ and evaluating the result at $k_t = k$, $a_{t-1} = 0$, $\varepsilon_t = 0$, $\sigma = 0$:

$$\begin{aligned}
R_j &= \int h_3 \varepsilon_{t+1} dF(\varepsilon_{t+1}) \\
&\quad + \int h_4 g_\sigma dF(\varepsilon_{t+1}) + \int h_5 (g_k g_\sigma + g_\varepsilon \varepsilon_{t+1} + g_\sigma) dF(\varepsilon_{t+1}) \\
&\quad + \int h_6 g_{j\sigma} dF(\varepsilon_{t+1}) + \int h_7 (g_{jk} g_\sigma + g_{j\varepsilon} \varepsilon_{t+1} + g_{j\varepsilon} g_\sigma) dF(\varepsilon_{t+1}) \\
&= h_6 g_{j\sigma}.
\end{aligned}$$

Here, the last terms reflects the facts:

$$g_\sigma = \int \varepsilon_{t+1} dF(\varepsilon_{t+1}) = 0.$$

In numerical experiments, we found that $h_6 \neq 0$. From this we conclude that $g_{j\sigma} = 0$ for $j = k, a, \varepsilon$.

An additional useful feature of $R_{ji} = 0$ is that these equations have a particular recursive structure, which allows us to solve them by solving a sequence of one-dimensional (linear) equations. Thus, the only second-order terms in (1) that enters R_{kk} is g_{kk} . So, we first solve for g_{kk} . Given a value for g_{kk} , the only second order term in (1) that enters R_{ka} is g_{ka} . Given g_{kk} and g_{ka} , R_{ke} is a function only of g_{ke} . The object, $R_{k\sigma}$ is only a function of $g_{k\sigma}$. Given g_{kk} and g_{ka} , R_{aa} is a function only of g_{aa} . Given g_{kk} , g_{ka} , g_{aa} , R_{ae} is a function only of g_{ae} . Given $g_{k\sigma}$, $R_{a\sigma}$ is a function only of $g_{a\sigma}$. Given g_{aa} , g_{ka} , g_{kk} , R_{ee} is only a function of g_{ee} . Given $g_{k\sigma}$, $g_{a\sigma}$, $R_{e\sigma}$ is a function only of $g_{e\sigma}$. Finally, given g_{aa} , g_{ae} , $g_{a\sigma}$, $g_{e\sigma}$, g_{ka} , g_{ke} , g_{kk} , $g_{k\sigma}$, g_{ee} , $R_{\sigma\sigma}$ is only a function of $g_{\sigma\sigma}$. These computations have been programmed in MATLAB program, second.m.