Optimal Monetary Policy
Optimal monetary policy addresses questions like:

- ‘how much inflation volatility should we have?’
- ‘should we have inflation targeting or should we have price level targeting?’
- answer: depends on what gets you closest to the optimal policy benchmark.

Actual policy seems to be gravitating in the direction of ‘optimal policy’.

- In early days of ‘rational expectations revolution’, looked for simple rules with good operating characteristics to impose on central banks.
- Perception was that central banks had done a terrible job.
- Simplicity of the rule supposed to help with monitoring central bank, which was viewed as always having an incentive to deviate from good policy and inflate (‘inflation bias’).
– Inflation bias is nowadays not thought to be a big problem.

* The classic vehicle for the inflation bias idea, the Barro-Gordon model, appeared to be rejected by the data.

* US seems to have tamed inflation without the institutional reform that BG said was required
  (for a challenge to the logic of this argument, see Chari, Christiano, Eichenbaum (JET), Albanesi, Chari, Christiano (RESTUD).)

* Operating without constraints, Fed seems to have done a great job: tamed inflation, performed well in 1987, 2001. (We’ll see what happens in the near future....)

* Thus, assessment of benefits of simple rules (overcoming inflation bias problem) was reduced
– Assessment of costs of simple rules were increased.

* ‘Mother of all rules’, Friedman’s $k$–percent rule, viewed as positively dangerous when velocity fluctuates a lot, as it seems to.

* Something like the Taylor rule seems like a rough guide at best, but does not capture many of the things that central bankers look at.

– The reaction is that central banks are gravitating towards ‘look at everything, optimal’ policy.

* Flexible inflation targeting.

* Flexible inflation targeting requires a certain discipline.
Example #1: Optimal Monetary Policy - Toy Example

• Setup
  – Model
    * One equation characterizing private sector behavior:

\[ \pi_t - \beta \pi_{t+1} - \gamma y_t = 0, \ t = 0, 1, 2, \ldots \]  \hspace{1cm} (1)

* Another equation characterizes policy.

– Want to do optimal policy, so throw away policy equation.

– System is now under-determined: one equation in two variables, \( \pi_t \) and \( y_t \).
Example #1: Optimal Monetary Policy - Toy Example ...

- Optimization delivers the other equations.

* Optimize objective:

\[ \sum_{t=0}^{\infty} \beta^t u(\pi_t, y_t) \]

subject to (1).

- If objective corresponds to social welfare function, this is called Ramsey optimal problem

- Objective may be preferences of policy maker.
Example #1: Optimal Monetary Policy - Toy Example ...

• Lagrangian representation of problem:

\[
\max_{\{\pi_t, y_t; t=0,1,\ldots\}} \sum_{t=0}^{\infty} \beta^t \left\{ u(\pi_t, y_t) + \lambda_t [\pi_t - \beta \pi_{t+1} - \gamma y_t] \right\}
\]

\[
= \max_{\{\pi_t, y_t; t=0,1,\ldots\}} \left\{ u(\pi_0, y_0) + \lambda_0 [\pi_0 - \beta \pi_1 - \gamma y_0] + \beta u(\pi_1, y_1) + \beta \lambda_1 [\pi_1 - \beta \pi_2 - \gamma y_1] + \ldots \right\}
\]

• First order necessary conditions for optimization:

\[
u_{\pi}(\pi_0, y_0) + \lambda_0 = 0 \quad (*)
\]

\[
u_{\pi}(\pi_1, y_1) + \lambda_1 - \lambda_0 = 0
\]

\[
\ldots
\]

\[
u_y(\pi_0, y_0) - \gamma \lambda_0 = 0
\]

\[
u_y(\pi_1, y_1) - \gamma \lambda_1 = 0
\]

\[
\ldots
\]

\[
\pi_0 - \beta \pi_1 - \gamma y_0 = 0
\]

\[
\pi_1 - \beta \pi_2 - \gamma y_1 = 0
\]

\[
\ldots
\]
Example #1: Optimal Monetary Policy - Toy Example ... 

- These equations ‘look’ different than the ones we’ve seen before
  - They are not stationary, (*) is different from the others.
    * reflects that at time 0 there is a constraint ‘missing’
    * no need to respect what people were expecting you to do as of time \(-1\)
    * do need to respect what they expect you to do in the future, because that affects current behavior.
    * that’s the source of the ‘time inconsistency of optimal plans’.

- Can trick the problem into being stationary (see, e.g., Kydland and Prescott (JEDC, 1990s) and Levin, Onatski, Williams, and Williams, Macro Annual, 2005). Then, apply standard log-linearization solution method.
Example #1: Optimal Monetary Policy - Toy Example ...

- Consider:

\[ v(\pi_t, \pi_{t+1}, y_t, \lambda_t, \lambda_{t-1}) = \begin{bmatrix}
  u_\pi(\pi_t, y_t) + \lambda_t - \lambda_{t-1} \\
  u_y(\pi_t, y_t) - \gamma\lambda_t \\
  \pi_t - \beta\pi_{t+1} - \gamma y_t
\end{bmatrix}, \text{ for all } t. \]

- time \( t \) ‘endogenous variables’: \( \lambda_t, \pi_t, y_t \)
- time \( t \) ‘state variable’: \( \lambda_{t-1} \).
- ‘solution’:

\[ \lambda_t = \lambda(\lambda_{t-1}), \pi_t = \pi(\lambda_{t-1}), y_t = y(\lambda_{t-1}), \]

such that

\[ v(\pi(\lambda_{t-1}), \pi(\lambda(\lambda_{t-1})), y(\lambda_{t-1}), \lambda(\lambda_{t-1}), \lambda_{t-1}) = 0, \text{ for all possible } \lambda_{t-1}. \]
Example #1: Optimal Monetary Policy - Toy Example ... 

• In general, solving this problem exactly is intractable.
• But, can log-linearize!

  – **Step 1**: find \( \pi^*, y^*, \lambda^* \) such that following three equations are satisfied:
    \[
    v(\pi^*, \pi^*, y^*, \lambda^*, \lambda^*) = 0_{3 \times 1}.
    \]

  – **Step 2**: log-linearly expand \( v \) about steady state
    \[
    v(\pi_t, \pi_{t+1}, y_t, \lambda_t, \lambda_{t-1}) \simeq v_1 \pi^* \hat{\pi}_t + v_2 \pi^* \hat{\pi}_{t+1} + v_3 y^* \hat{y}_t + v_4 \Delta \hat{\lambda}_t + v_5 \Delta \hat{\lambda}_{t-1},
    \]
    where
    \[
    \Delta \hat{\lambda}_t \equiv \lambda_t - \lambda^* \) (play it safe, don’t divide by something that could be zero!)

  – **Step 3**: Posit
    \[
    \Delta \hat{\lambda}_t = A_\lambda \Delta \hat{\lambda}_{t-1}, \quad \hat{\pi}_t = A_\pi \Delta \hat{\lambda}_{t-1}, \quad \hat{y}_t = A_y \Delta \hat{\lambda}_{t-1},
    \]
    and find \( A_\lambda, A_\pi, A_y \) that solve
    \[
    [v_1 \pi^* A_\pi + v_2 \pi^* A_\pi A_\lambda + v_3 y^* A_y + v_4 A_\lambda + v_5] \Delta \hat{\lambda}_{t-1} = 0_{3 \times 1}
    \]
    for all \( \Delta \hat{\lambda}_{t-1} \).
Example #1: Optimal Monetary Policy - Toy Example ...

• What does the stationary solution have to do with the original non-stationary problem?

  – Do we have a solution to the period 0 problem, (*)?

    \[ u_\pi (\pi_0, y_0) + \lambda_0 = 0. \]

  – Yes! Just pretend that this equation really has the following form:

    \[ u_\pi (\pi_0, y_0) + \lambda_0 - \lambda_{-1} = 0. \]

    Expression (*) does have this form, if we set \( \lambda_{-1} = 0 \). Then,

    \[ \pi_0 = \pi (0), \ y_0 = y (0), \ \lambda_0 = \lambda (0). \]
Example #1: Optimal Monetary Policy - Toy Example ...

- The situation is exactly what it is in the neoclassical model when we want to know what happens when initial capital is away from steady state.

  – Plug $k_0$ into the stationary rule

  \[
  k_1 = g(k_0) .
  \]

- Possible computational pitfall: if $\lambda_{-1} = 0$ is far from $\lambda^*$, then linearized solution might be highly inaccurate (see LOWW).
Example #1: Optimal Monetary Policy - Toy Example ...

- Optimal policy in real time.

- Suppose today is date zero.

  - Solve for $\lambda(\cdot), y(\cdot), \pi(\cdot)$

  - set $\lambda_{-1} = 0$

  - Compute and present in charts:

    \[
    \begin{align*}
    \lambda_0 &= \lambda(\lambda_{-1}), \quad y_0 = y(\lambda_{-1}), \quad \pi_0 = \pi(\lambda_{-1}) \\
    \lambda_1 &= \lambda(\lambda_0), \quad y_1 = y(\lambda_0), \quad \pi_1 = \pi(\lambda_0) \\
    \quad &\ldots \quad \ldots \\
    \lambda_t &= \lambda(\lambda_{t-1}), \quad y_t = y(\lambda_{t-1}), \quad \pi_t = \pi(\lambda_0) \\
    \quad &\ldots \quad \ldots
    \end{align*}
    \]
Example #1: Optimal Monetary Policy - Toy Example ...

• One could take the perspective that date 0 is actually a continuation of optimal policy that started up long ago. Under these assumptions, $\lambda_{-1}$, is an unobserved variable that you have to extract from the data.

• Warning, for this real time optimal policy strategy to make sense, you must ‘always respect your multipliers’.
  – Avoid temptation to set them to zero.
  – Time inconsistency of optimal plans.

• What does ‘always respect your multipliers’ mean in practice?
  – Make sure the charts you present from one meeting to the next are consistent
Example #1: Optimal Monetary Policy - Toy Example...

– Example:

\[ y_0 = y(0), \quad y_1 = y(\lambda(\lambda_{-1})), \quad y_2 = y(\lambda(\lambda(\lambda_{-1}))), \ldots \]

date 0 meeting :  

\[ y_1 = y(\lambda(\lambda_{-1})), \quad y_2 = y(\lambda(\lambda(\lambda_{-1}))), \ldots \]

date 1 meeting :  

YES -  

\[ y_1 = y(\lambda(\lambda_{-1})), \quad y_2 = y(\lambda(\lambda(\lambda_{-1}))), \ldots \]

NO -  

\[ y_1 = y(0), \quad y_2 = y(\lambda_1(0)), \ldots \]

– If Central Bank selects the bad (‘**NO**’) option people will see the temporal inconsistency of policy, and CB will lose credibility.

– Any differences in charts from one meeting to the next must be fully explicable in terms of new information.
Example #2: Optimal Monetary Policy - More General Discussion

- The equilibrium conditions of a model

\[ E_t f (z_{t-1}, z_t, z_{t+1}, s_t, s_{t+1}) = 0, \text{ for all } z_{t-1} \text{ (endogenous), } s_t \text{ (exogenous)} \]

\[ s_t = P s_{t-1} + \varepsilon_t. \]

- Preferences:

\[ E_t \sum_{t=0}^{\infty} \beta^t U (z_t, s_t). \]

- Could include discounted utility in \( f \):

\[ v (z_{t-1}, z_t, s_t) = U (z_t, s_t) + \beta E_t v (z_t, z_{t+1}, s_{t+1}) \]
Example #2: Optimal Monetary Policy - More General Discussion ...

- Optimum problem:

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U(z_t, s_t) + \lambda'_t \begin{pmatrix} E_t f(z_{t-1}, z_t, z_{t+1}, s_t, s_{t+1}) \end{pmatrix}_{1 \times (N-1)} \right\}. 
\]

- \( N \) first order conditions:

\[
\begin{align*}
&U_1(z_t, s_t) + \lambda'_t E_t f_2(z_{t-1}, z_t, z_{t+1}, s_t, s_{t+1}) \\
&+ \beta^{-1} \lambda'_{t-1} f_3(z_{t-2}, z_{t-1}, z_t, s_{t-1}, s_t) \\
&+ \beta \lambda'_{t+1} E_t f_1(z_t, z_{t+1}, z_{t+2}, s_{t+1}, s_{t+2}) = 0
\end{align*}
\]

- Endogenous variables: \( z_t \ (N), \lambda_t \ (N - 1) \)

- Equations: Ramsey optimality conditions \( (N) \), equilibrium condition \( (N - 1) \)
Example #2: Optimal Monetary Policy - More General Discussion ...

- First order conditions of optimum problem have exactly the same form as the type of problem we solved using linearization methods.

- Seem much more cumbersome:
  
  – must differentiate $f$ (includes private first order conditions that have already involved differentiation!)

  – good news: LOWW wrote a program that takes $U, f$ as input and writes Dynare code for solving the system

  – solving policy optimum problem is no harder than solving original problem.
Example #3: Optimal Monetary Policy - Rotemberg Model

- Household preferences:

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left[ \log(C_t) - \frac{\chi}{2} h_t^2 \right]. \]

- Household budget constraint:

\[ P_tC_t + B_t = (1 + R_{t-1}) B_{t-1} + W_t h_t + \Pi_t. \]

- First order conditions:

\[ \chi h_t C_t = \frac{W_t}{P_t}, \quad \frac{1}{1 + R_t} = \beta E_t \frac{P_tC_t}{P_{t+1}C_{t+1}}, \quad t = 0, 1, 2, \ldots \]
Example #3: Optimal Monetary Policy - Rotemberg Model

• Final good firms

production function
\[ Y_t = \left( \int_0^1 Y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}, \varepsilon \geq 1, \]

first order condition (demand curve for \( j^{th} \) intermediate good producer)
\[ Y_{j,t} = \left( \frac{P_{j,t}}{P_t} \right)^{-\frac{1}{\varepsilon}} Y_t \]

• \( j^{th} \) intermediate good firm:

\[
E_t \sum_{l=0}^{\infty} \beta^l \nu_{t+l} [(1 + \nu) P_{j,t+l} Y_{j,t+l} - s_{t+l} P_{t+l} Y_{j,t+l}]
\]

\(- \frac{\phi}{2} \left( \frac{P_{j,t+l}}{P_{j,t+l-1}} - 1 \right)^2 P_{t+l} C_{t+l} \]

production function
\[ Y_{j,t} = A_t h_{j,t}, \quad a_t \equiv \log (A_t), \quad s_t = \frac{W_t}{P_t A_t} = \frac{\chi h_t C_t}{A_t}. \]
Example #3: Optimal Monetary Policy - Rotemberg Model...

• Substitute out $j^{th}$ firm’s demand curve $\nu_t = 1/(P_tC_t)$:

$$\max_{\{P_{j,t}\}^\infty_{i=0}} E_t \sum_{l=0}^{\infty} \beta^l \frac{1}{P_{t+l}C_{t+l}} [(1 + \nu) P_{j,t+l} \left( \frac{P_{j,t+l}}{P_{t+l}} \right)^{-\varepsilon} Y_{t+l}$$

$$- P_{t+l}s_{t+l} \left( \frac{P_{j,t+l}}{P_{t+l}} \right)^{-\varepsilon} Y_{t+l} - \phi \left( \frac{P_{j,t+l}}{P_{j,t+l-1}} - 1 \right)^2 P_{t+l}C_{t+l}$$

• Differentiate with respect to $P_{j,t}$:

$$\left[ (1 + \nu) (1 - \varepsilon) \left( \frac{P_{j,t}}{P_t} \right)^{-\varepsilon} \frac{1}{P_t} + s_t \varepsilon \left( \frac{P_{j,t}}{P_t} \right)^{-\varepsilon-1} \frac{1}{P_t} \right] Y_t$$

$$\frac{1}{C_t} - \phi \left( \frac{P_{j,t}}{P_{j,t-1}} - 1 \right) \frac{1}{P_{j,t-1}}$$

$$+ \beta \phi E_t \left( \frac{P_{j,t+1}}{P_{j,t}} - 1 \right) \frac{P_{j,t+1}}{(P_{j,t})^2} = 0.$$
Example #3: Optimal Monetary Policy - Rotemberg Model ...

- Rearrange firm efficiency condition:

\[ (1 + \nu) \frac{P_{j,t}}{P_t} = \frac{\varepsilon}{\varepsilon - 1} \times \left( \frac{\text{real price received}}{s_t} \right) \]

\[ \text{real price received} \quad \text{markup} \quad \text{real marginal cost (exclusive of price adjustment costs)} \]

if static considerations entail a rise in price, adj costs imply rising by less

\[ + \frac{1}{\varepsilon - 1} \phi \left( \frac{P_{j,t}}{P_t} \right)^\varepsilon \frac{C_t}{Y_t} \]

if contemplating a rise in future price, then raise price today by more

\[ + \beta E_t \left( \frac{P_{j,t+1}}{P_{j,t}} - 1 \right) \left( \frac{P_{j,t+1}}{P_{j,t}} \right) \]

- When \( \phi = 0 \):
  - get ‘normal’ efficiency condition, ‘price equals markup over marginal cost’
  - get ‘normal’ monopoly power correction: \( 1 + \nu = \varepsilon / (\varepsilon - 1) \).
Example #3: Optimal Monetary Policy - Rotemberg Model ...

- Impose \( P_{j,t} = P_{i,t} = P_t \) for all \( i,j \):
  \[
  (\pi_t - 1) \pi_t = \frac{1}{\phi} \left( 1 + \nu - \frac{\varepsilon}{\varepsilon - 1} \right) (1 - \varepsilon) \frac{Y_t}{C_t} + \varepsilon (s_t - 1) \frac{Y_t}{C_t} + \beta E_t (\pi_{t+1} - 1) \pi_{t+1}.
  \]

- Resource constraint:
  \[
  C_t \begin{bmatrix}
    \text{consumption} & \frac{\phi}{2} (\pi_t - 1)^2 \\
    1
  \end{bmatrix}
  \] + total output
  \[
  \frac{\phi}{2} (\pi_t - 1)^2 = A_t h_t = Y_t, \quad a_t = \rho a_{t-1} + u_t, \quad a_t \equiv \log A_t
  \]

- Substitute out the resource constraint:
  \[
  (\pi_t - 1) \pi_t = \frac{1}{\phi} \left[ \left( 1 + \nu - \frac{\varepsilon}{\varepsilon - 1} \right) (1 - \varepsilon) + \varepsilon (s_t - 1) \right] \left[ 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right]
  + \beta E_t (\pi_{t+1} - 1) \pi_{t+1}.
  \]

- Looks ‘sort of’ like Calvo equilibrium relation for inflation.
Example #3: Optimal Monetary Policy - Rotemberg Model ... 

- Log-linearize around ‘efficient steady state’ \((\pi_t = 1, 1 + \nu = \varepsilon/(\varepsilon - 1), s_t = 1)\):

\[
(\pi_t - 1) \pi_t = \frac{1}{\phi} \left[ \left(1 + \nu - \frac{\varepsilon}{\varepsilon - 1}\right)(1 - \varepsilon) + \varepsilon (s_t - 1) \right] \left[ 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right]
+ \beta E_t (\pi_{t+1} - 1) \pi_{t+1}.
\]

\[
d[(\pi_t - 1) \pi_t] = \pi_t d\pi_t + (\pi_t - 1) d\pi_t = \pi d\pi_t \equiv \pi^2 \hat{\pi}_t = \hat{\pi}_t.
\]

- Doing the log-linearization:

\[
\hat{\pi}_t = \frac{\varepsilon}{\phi} \hat{s}_t + \beta E_t \hat{\pi}_{t+1}.
\]

‘marginal cost affects price less the bigger is \(\phi\’

so Rotemberg IS Calvo, up to linear approximation and with a different interpretation of slope on marginal cost.
Example #3: Optimal Monetary Policy - Rotemberg Model

- Summarizing equilibrium conditions:

  – household intertemporal efficiency condition:
    \[ \frac{1}{1 + R_t} = \beta E_t \frac{P_tC_t}{P_{t+1}C_{t+1}} \]
    
  – firm efficiency condition for prices (after rearranging) -
    \[ \left[ \left(1 + \nu - \frac{\varepsilon}{\varepsilon - 1}\right)(1 - \varepsilon) + \varepsilon (s_t - 1) \right] \left(1 + \frac{\phi}{2} (\pi_t - 1)^2\right) - \phi (\pi_t - 1) \pi_t \]
    \[ + \beta \phi E_t (\pi_{t+1} - 1) \pi_{t+1} = 0 \]

  – marginal cost: uses household intratemporal efficiency condition
    \[ s_t = \frac{\chi h_tC_t}{A_t} \]

  – Resource constraint:
    \[ C_t \left[ 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right] = A_th_t, \ a_t = \rho a_{t-1} + u_t \]
Example #3: Optimal Monetary Policy - Rotemberg Model ...  

- Ramsey optimal problem:

$$
\max_{\{\nu,C_t,h_t,\pi_t,R_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{ \left( \log (C_t) - \frac{\chi}{2} h_t^2 \right) + \\
\lambda_{1t} \left( \frac{1}{1 + R_t} - \beta E_t \frac{C_t}{\pi_{t+1} C_{t+1}} \right) + \\
\lambda_{2t} \left[ \left( 1 + \nu - \frac{\varepsilon}{\varepsilon - 1} \right) (1 - \varepsilon) + \varepsilon \left( \frac{\chi h_t C_t}{A_t} - 1 \right) \right] \left( 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right) + \\
- \phi (\pi_t - 1) \pi_t + \beta \phi E_t (\pi_{t+1} - 1) \pi_{t+1} \right] + \\
\lambda_{3t} \left[ C_t \left( 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right) - A_t h_t \right] \}
$$
Example #3: Optimal Monetary Policy - Rotemberg Model ...

- Conjecture about solution to Ramsey problem:

\[ \lambda_{1t} = \lambda_{2t} = 0. \]

- We will implement, and then verify the conjecture formally.

- Intuition:
  
  - Intertemporal equation non-binding from the point of view of maximizing utility, because \( R_t \) (a variable of no direct interest in utility) can always be chosen to enforce intertemporal Euler equation).

  - Price equation non-binding from the point of view of maximizing utility, because \( \nu \) (a variable of no direct interest in utility) can always be chosen to enforce the price equation.
Example #3: Optimal Monetary Policy - Rotemberg Model

• Simplified Ramsey problem ($\nu, R_t$ are a matter of indifference here)

$$\max_{\{C_t, h_t, \pi_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{ \left( \log(C_t) - \frac{\chi}{2} h_t^2 \right) + \lambda_{3t} \left[ C_t \left( 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right) - A_t h_t \right] \}. $$

• – first order necessary condition for $\pi_t$:

$$\lambda_{3t} C_t \phi (\pi_t - 1) = 0 \rightarrow \pi_t = 1 \text{ (obviously, } \lambda_{3t} = 0, \text{ or } C_t = 0 \text{ is not the solution!)}$$

– first order condition for $h_t, C_t$:

$$-\chi h_t - \lambda_{3t} A_t = 0, \quad \frac{1}{C_t} + \lambda_{3t} \left( 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right) = 0 \rightarrow \lambda_{3t} = -\frac{1}{C_t}. $$

or

$$\chi h_t = \frac{1}{C_t} A_t, \quad \rightarrow \underbrace{\chi h_t C_t}_{\text{MRS}} = \underbrace{\lambda_{3t}}_{\text{MPL}} A_t$$

so solution to Ramsey problem achieves ‘first best’:

$$\max_{C_t, h_t} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[ \log(C_t) - \frac{\chi}{2} h_t^2 \right], \text{ subject to } C_t = A_t h_t.$$
Example #3: Optimal Monetary Policy - Rotemberg Model

• Solution to Ramsey problem:

\[ C_t = A_t h_t \]

\[ \chi h_t C_t = A_t \rightarrow h_t^2 = \left[ \frac{1}{\chi} \right]^{1/2}, \]

\[ \pi_t = 1, \]

\[ R_t = \frac{1}{\beta E_t \frac{P_t C_t}{P_{t+1} C_{t+1}}} - 1, \]

\[ 1 + \nu = \frac{\varepsilon}{\varepsilon - 1}. \]

• Notice:
  – this is first best, and there is no time inconsistency problem!
  – treatment of \( \nu \) crucial here (reason that price equation is non-binding)
Example #3: Optimal Monetary Policy - Rotemberg Model

- Ramsey problem with $\nu$ fixed, $\neq \varepsilon / (\varepsilon - 1)$

$$
\max_{\{C_t, h_t, \pi_t, R_t\}} \sum_{t=0}^{\infty} b^n_t \{ \left( \log (C_t) - \frac{\chi}{2} h_t^2 \right) + 
\lambda_{1t} \left( \frac{1}{1 + R_t} - \beta E_{t+1} \frac{C_t}{\pi_{t+1} C_{t+1}} \right) + 
\lambda_{2t} \left[ \left( \left( 1 + \nu - \frac{\varepsilon}{\varepsilon - 1} \right) (1 - \varepsilon) + \varepsilon \left( \frac{\chi h_t C_t}{A_t} - 1 \right) \right) \left( 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right) - \phi (\pi_t - 1) \pi_t + \beta \phi E_t (\pi_{t+1} - 1) \pi_{t+1} \right] + 
\lambda_{3t} \left[ C_t \left( 1 + \frac{\phi}{2} (\pi_t - 1)^2 \right) - A_t h_t \right] \},
$$

- Note: intertemporal condition still non-binding.
Example #3: Optimal Monetary Policy - Rotemberg Model

• First order conditions for Ramsey problem (impose $\lambda_{1t} \equiv 0$, $\beta_n$ is discount rate in planner objective, $\lambda_{i,-1} = 0$) for $h_t, \pi_t, C_t$:

\[-\chi h_t + \lambda_{2t} \varepsilon \frac{C_t}{A_t} \left(1 + \frac{\phi}{2} (\pi_t - 1)^2\right) - \lambda_{3t} A_t = 0.\]

\[\lambda_{2t} \left[ \left(1 + \nu - \frac{\varepsilon}{\varepsilon - 1}\right)(1 - \varepsilon) + \varepsilon \left(\frac{\chi h_t C_t}{A_t} - 1\right) - 1\right] \phi (\pi_t - 1) = 0\]

\[\lambda_{3t} \phi (\pi_t - 1) + \lambda_{2t-1} \beta_n^{-1} \beta \phi [(\pi_t - 1) + \pi_t] + \lambda_{3t} C_t \phi (\pi_t - 1) = 0\]

\[\frac{1}{C_t} + \lambda_{2t} \varepsilon \frac{h_t}{A_t} \left(1 + \frac{\phi}{2} (\pi_t - 1)^2\right) + \lambda_{3t} \left(1 + \frac{\phi}{2} (\pi_t - 1)^2\right) = 0\]

• Plus, intertemporal household equation, price equation and resource constraint yields 6 equations in $R_t, h_t, C_t, \pi_t, \lambda_{2t}, \lambda_{3t}$. 
Example #3: Optimal Monetary Policy - Rotemberg Model ...

- Conclusion of example #3

  - When labor market is treated optimally, policy achieves first best and there is no time consistency problem.

  - When labor market not treated optimally, there may be a time consistency problem, but must solve model numerically to find out.

  - Can use same log-linear methods discussed before!

  - Bad news: Ramsey first order conditions painful to compute in practice.

  - Good news: there exists computer code for deriving the Ramsey first order conditions symbolically (we will explore this in a homework with Dynare).
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model

- Optimal policy problem:
  
  \[ 1 + \nu = \frac{\varepsilon}{\varepsilon - 1}, \]

- Equilibrium conditions \((f)\):
  * intermediate good firm optimality
  * household intra- and inter-temporal conditions
  * aggregate conditions: prices, output and resources.
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model

– Lagrangian:

\[
\max_{\nu_t, p_t^*, N_t, R_t, \pi_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right. \\
+ \lambda_{1t} \left[ \frac{1}{p_t^* N_t} - \frac{A_t \beta}{p_{t+1}^* A_{t+1} N_{t+1} \bar{\pi}_{t+1}} R_t \right] \\
+ \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1 - \theta) \left( \frac{1 - \theta (\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}} \right) \right] \\
+ \lambda_{3t} \left[ 1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t \right] \\
+ \lambda_{4t} \left[ (1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \exp(\tau_t) N_t^{1+\varphi} p_t^* (1 - \psi + \psi R_t) + E_t \bar{\pi}_t^{\varepsilon} \beta \theta K_{t+1} - K_t \right] \\
+ \lambda_{5t} \left[ F_t \left[ \frac{1 - \theta \bar{\pi}_t^{\varepsilon-1}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}} - K_t \right] \}
\]

– The problem looks like a normal public finance problem.

– ‘two degree of freedom’ 7 variables, 5 equilibrium conditions
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

- Optimal inflation rate depends on $\nu_t$.

  - two cases: $\nu_t$ constant, and $\nu_t$ can vary.

- Consider the variable $\nu_t$ case.

  - Conjecture: restrictions 1, 3, 4, 5 nonbinding (i.e., $\lambda_{1t} = \lambda_{3t} = \lambda_{4t} = \lambda_{5t} = 0$)

    * Step 1: Optimize w.r.t. $p_t^*, \bar{\pi}_t, N_t$ ignoring restrictions 1, 3, 4, 5.

    * Step 2: Solve for $\nu_t, R_t, F_t, K_t$, to satisfy restrictions 1, 3, 4, 5.

  - If this can be done, then the conjecture is verified.
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model

- Simplified problem under conjecture:

\[
\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp (\tau_t) \frac{N_t^{1+\phi}}{1+\phi} \right) \right. \\
+ \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1 - \theta) \left( \frac{1 - \theta (\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right) \right)^\frac{\varepsilon}{\varepsilon-1} \right. \\
\left. \left. + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right] \right\}
\]

first order conditions with respect to \( p_t^*, \bar{\pi}_t, N_t \) (after rearranging):

\[
p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^{\varepsilon} = \lambda_{2t}, \quad \bar{\pi}_t = \left[ \frac{(p_{t-1}^*)^{\varepsilon-1}}{1 - \theta + \theta (p_{t-1}^*)^{\varepsilon-1}} \right] \frac{1}{1-\varepsilon}, \quad N_t = \exp \left( -\frac{\tau_t}{\phi + 1} \right)
\]

- Substituting the solution for \( \bar{\pi}_t \) into the law of motion for \( p_t^* \):

\[
p_t^* = \left[ (1 - \theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right] \frac{1}{(\varepsilon-1)}.
\]
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ... 

• Can the other constraints be satisfied?

  – Choose $R_t$ so the intertemporal constraint is satisfied:

  $$R_t = \frac{1}{p_t^n N_t} \frac{A_t \beta}{E_t \frac{p_{t+1}^n A_{t+1} N_{t+1} \bar{\pi}_{t+1}}{p_{t+1}^n A_{t+1} N_{t+1} \bar{\pi}_{t+1}}}.$$ 

  – Remaining constraints: three price-setting conditions.
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

- Price setting conditions:

\[
1 + E_t \bar{\pi}^{\varepsilon-1}_{t+1} \beta \theta F_{t+1} = F_t \tag{1}
\]

\[
(1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \exp(\tau_t) N_t^{1+\varphi} p_t^* (1 - \psi + \psi R_t) + E_t \bar{\pi}^{\varepsilon}_{t+1} \beta \theta K_{t+1} = K_t \tag{2}
\]

\[
F_t \left[ \frac{1 - \theta \bar{\pi}^{\varepsilon-1}_{t}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}} \text{ (making use of the expression for optimal inflation)} = F_t p^*_t = K_t \tag{3}
\]

- Divide (2) by \( p_t^* \), impose (3) and use \( \bar{\pi}_{t+1} = p_t^*/p_{t+1}^* \):

\[
\varepsilon \times (1 - \nu_t) \times \exp(\tau_t) N_t^{1+\varphi} (1 - \psi + \psi R_t) + E_t \bar{\pi}^{\varepsilon-1}_{t+1} \beta \theta F_{t+1} = F_t
\]

- Subtract from (1) (subsidy must cancel markup and interest rate distortion):

\[
(1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} (1 - \psi + \psi R_t) = 1.
\]
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

• Bottom line. Optimality under state-contingent $\nu_t$ implies:

$$p_t^* = \left[(1 - \theta) + \theta \left(\frac{p_{t-1}^*}{p_t^*}\right)^{(\varepsilon - 1)}\right]^{\frac{1}{(\varepsilon - 1)}}$$

$$\bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}$$

$$N_t = \exp\left(-\frac{\tau_t}{1 + \varphi}\right)$$

$$1 - \nu_t = \frac{\varepsilon - 1}{\varepsilon (1 - \psi + \psi R_t)}$$

$$C_t = p_t^* A_t N_t.$$ 

• Ramsey-optimal policy is time consistent (no forward-looking constraints on core problem).

• If $\psi > 0$ and $\nu_t$ not state-contingent must work out Ramsey solution numerically.

(For further discussion, see Christiano-Motto-Rostagno, ‘Two Reasons Why Money Might be Useful in Monetary Policy’, 2007 NBER WP.)
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

- Example - no working capital channel (no lending channel):

\[ \theta = 0.5, \, \varepsilon = 2, \, \beta = 0.99, \, \psi = 0, \, \rho = 0.5, \, \varphi = 1. \]

- In this case:

\[
N_t = 1 + 0.45(\lambda_{1t-1} - \lambda_1) + 0.06(\lambda_{3,t-1} - \lambda_3) + 0.63(\lambda_{4,t-1} - \lambda_4)
\]
\[
r_t = 0.01 - 0.50(\lambda_{1t-1} - \lambda_1) + 0.10(\lambda_{3,t-1} - \lambda_3) - 0.02(\lambda_{4,t-1} - \lambda_4) + 0.25a_{t-1} + 0.51u_t
\]
\[
\pi_t = 1 + 0.07(\lambda_{1t-1} - \lambda_1) + 0.09(\lambda_{3,t-1} - \lambda_3) + 0.31(\lambda_{4,t-1} - \lambda_4) + 0.25(p_{t-1}^* - 1)
\]
\[
\lambda_{1t} = 0,
\]
\[
\lambda_{2,t} = 3.88 + 0.82(\lambda_{1t-1} - \lambda_1) + 1.46(\lambda_{3,t-1} - \lambda_3) + 3.65(\lambda_{4,t-1} - \lambda_4) + 4.13(p_{t-1}^* - 1)
\]
\[
\lambda_{3,t} = 0.05(\lambda_{1t-1} - \lambda_1) + 0.69(\lambda_{3,t-1} - \lambda_3) + 0.12(\lambda_{4,t-1} - \lambda_4)
\]
\[
\lambda_{4,t} = -0.05(\lambda_{1t-1} - \lambda_1) + 0.06(\lambda_{3,t-1} - \lambda_3) + 0.63(\lambda_{4,t-1} - \lambda_4)
\]
\[
\lambda_{5,t} = 0.05(\lambda_{1t-1} - \lambda_1) - 0.06(\lambda_{3,t-1} - \lambda_3) + 0.12(\lambda_{4,t-1} - \lambda_4)
\]
\[
\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \, \lambda_2 = 3.88
\]

- ‘Resetting multipliers’ makes no difference, no time inconsistency problem.
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

- Example with $\psi = 0.7$:

\[
N_t = 1 + 0.50\lambda_{1t-1} + 0.03\lambda_{3,t-1} + 0.40\lambda_{4,t-1} - 0.02a_{t-1} - 0.03u_t \\
r_t = 0.01 - 0.51\lambda_{1t-1} + 0.12\lambda_{3,t-1} + 0.30\lambda_{4,t-1} + 0.24a_{t-1} + 0.49u_t \\
\pi_t = 1 + 0.05\lambda_{1t-1} + 0.10\lambda_{3,t-1} + 0.31\lambda_{4,t-1} + 0.01a_{t-1} + 0.25(p_{t-1}^* - 1) + 0.02u_t \\
p_{t}^* = 1 + .75(p_{t-1}^* - 1) \\
\lambda_{1t} = -0.01\lambda_{1t-1} + 0.04\lambda_{3,t-1} + 0.44\lambda_{4,t-1} - 0.02a_{t-1} - 0.03u_t \\
\lambda_{2,t} = 3.88 + 0.95\lambda_{1t-1} + 1.42\lambda_{3,t-1} + 3.63\lambda_{4,t-1} - 0.09a_{t-1} - 0.18u_t + 4.13(p_{t-1}^* - 1) \\
\lambda_{3,t} = 0.01\lambda_{1t-1} + 0.70\lambda_{3,t-1} + 0.13\lambda_{4,t-1} + 0.02a_{t-1} + 0.04u_t \\
\lambda_{4,t} = -0.01\lambda_{1t-1} + 0.05\lambda_{3,t-1} + 0.62\lambda_{4,t-1} - 0.02a_{t-1} - 0.05u_t \\
\lambda_{5,t} = 0.01\lambda_{1t-1} - 0.05\lambda_{3,t-1} + 0.13\lambda_{4,t-1} + 0.02a_{t-1} + 0.05u_t
\]

- Properties: all multipliers response to $u_t$; optimal plan not time consistent; employment and inflation respond to $u_t$; $r_t$ drops a little less than before (it’s a tax now); $N_t$ falls somewhat because of the interest rate ‘tax’.

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Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

• Experiment:
  
  – Economy is in steady state of optimal plan up to period $t$.

  – A positive shock to technology occurs.

  – Monetary authority computes optimal policy and displays it in a set of charts.

  – Redo charts one period later.
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

- Discussion of the results

  – In the absence of a working capital channel (i.e., $\psi = 0$) it is optimal to cut the interest rate, to encourage households not smooth consumption away from what is optimal.

  – In the presence of a working capital channel, (i.e., $\psi > 0$), the cut in the interest rate reduces the marginal cost of labor and expands output and employment. By reducing marginal cost, inflation drops.

  – The rise in employment and fall in inflation are both costly, and so:

    * it is optimal when $\psi > 0$ to cut the interest rate by less.

    * it is optimal to manage expectations so that the incentive to cut prices in the present is reduced.
      - announce inflation close to zero in the next period
      - announce relatively small interest rate drop in the next period.
Example #4: Optimal Monetary Policy - Clarida-Gali-Gertler Model ...

• Previous example illustrates importance of working capital channel (‘lending channel’)

  – CCG model with Taylor rule illustrates this too

  \[ R_t = 1.5 \pi_t^e + 1 \]

  – With \( \psi = 0 \), obtain standard result that inflation expectations cannot be self-fulfilling.

  – With \( \psi > 0 \), results turn upside down (see CMR, ‘Two Reasons...’)

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Phillips curve

LM

IS(\pi e)
\[ \text{LM} \]

\[ \text{IS}(\pi e') \]
The diagram illustrates the Phillips curve and the IS-LM model in macroeconomics. The Phillips curve is a graphical representation of the trade-off between inflation and unemployment. The IS-LM model shows the interaction between the aggregate demand and aggregate supply in an economy. The curves and lines indicate the different equilibrium points and their implications for economic policy.
Any initial rise in $\pi^e$ would quickly disappear.
now consider the case with a working capital channel
Higher $\pi^e$ confirmed and likely to persist.