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Optimal Lending Contracts and Firm Dynamics

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We develop a general model of lending in the presence of endogenous borrowing constraints. Borrowing constraints arise because borrowers face limited liability and debt repayment cannot be perfectly enforced. In the model, the dynamics of debt are closely linked with the dynamics of borrowing constraints. In fact, borrowing constraints must satisfy a dynamic consistency requirement: the value of outstanding debt restricts current access to short-term capital, but is itself determined by future access to credit. This dynamic consistency is not guaranteed in models of exogenous borrowing constraints, where the ability to raise short-term capital is limited by some prespecified function of debt. We characterize the optimal default-free contract—which minimizes borrowing constraints at all histories—and derive implications for firm growth, survival, leverage and debt maturity. The model is qualitatively consistent with stylized facts on the growth and survival of firms. Comparative statics with respect to technology and default constraints are derived.

1. INTRODUCTION

Borrowing constraints are an important determinant of firm growth and survival.[1] Such constraints may arise in connection to the financing of investment opportunities faced by firms or temporary liquidity needs, sueh as those required to survive a recession. This paper develops a theory of endogenous borrowing constraints and studies its implications for firm dynamics. In our model, debt is constrained by the firm's limited liability and option to default. A lending contract specifies an initial loan size, future financing, and a repayment schedule. The choice of these variables in turn determines future growth, the firm's future borrowing capacity, and its ability and willingness to repay. Hence, borrowing constraints and firm dynamics are jointly determined. We study this dynamic design problem.

Our model builds on Thomas and Worral's (1994) model of foreign direct investment. At time zero a risk neutral borrower (*firm or entrepreneur*) has a project which requires a fixed initial set-up cost. Every period the project yields revenues that increase with the amount of

1. There is considerable evidence suggesting that financing constraints are important determinants of firm dynamics. Gertler and Gilchrist (1994) argue that liquidity constraints may explain why small manufacturing firms respond more to a tightening of monetary policy than do larger manufacturing firms. Perez-Quiros and Timmermann (2000) show that in recessions smaller firms are more sensitive to the worsening of credit market conditions as measured by higher interest rates and default premia. Evans and Jovanovic (1989) show that the liquidity constraints are essential in the decision to become an entrepreneur. Fazzari, Hubbard and Petersen (1988), among others, view financial constraints as an explanation for the dynamic behaviour of aggregate investment, and Cabral and Mata (1996) are able to fit reasonably well the size distribution of Portuguese manufacturing firms by estimating a simple model of financing constraints. For surveys see Hubbard (1998) and Stein (2003).

capital input and a revenue shock, which follows a Markov process. A risk neutral lender (*bank*) finances the initial investment and provides liquidity to support the firm's growth process. At any point in time the project may be liquidated. A lending contract specifies transfers to and payments from the borrower and a liquidation decision, contingent on all past shocks. The firm, has limited commitment and can choose to default at any time. Default gives the firm an outside value which increases with the amount of capital financed and the current revenue shock. We study the contract that maximizes total firm value subject to the no-default and limited liability constraints.

The optimal contract defines a Pareto frontier between the value for the lender (which we call long-term debt) and the value for the entrepreneur (which we call equity). By defaulting, the entrepreneur obtains an outside value but loses its equity. Thus, the firm's ability to expand is constrained by the entrepreneurs entitlement. Equity grows over time as the firm pays off the long-term debt. This weakens borrowing constraints, as the increased equity provides the bonding necessary to accumulate increasing amounts of capital.

Competition by lenders determines an initial long-term debt equal to the initial set-up cost. The equilibrium contract maximizes the entrepreneur's equity value (and total firm value) subject to expected repayment of this set-up cost. A unique debt maturity structure attains this initial equity value. Any other debt maturity either leads to default or a lower initial firm value.

In the optimal lending contract equity grows at the maximum possible rate (the interest rate), eventually reaching a level at which borrowing constraints are no longer binding. Along this path, dividends are zero. As equity grows, so does the size of the firm and its probability of survival. Our model is thus consistent with the firm age and size effects found in the literature on firm dynamics.[2] In addition, it implies that the capital structure is an important determinant of the firm's growth and exit decisions, in accordance with the evidence presented in Zingales (1998). Moreover, we show that investment of a financially constrained firm responds to Tobin's Q as well as the current level of cash flows. Finally, we show that firms with higher market-to-book ratio of assets display a lower ratio of long-term debt to short-term debt, conditional on the revenue shock (*e.g.* Barclay and Smith, 1995). This property arises because conditional on the revenue shock, a firm with higher market-to-book value of assets is also a firm with higher equity entitlement, weaker borrowing constraints and lower long-term debt.

The growth in the firm's equity is state contingent, as the optimal contract must trade-off borrowing constraints across different states. As a consequence, even though the process for firm shocks is first-order Markov, the resulting process for firm size, profits and value exhibits a more complex lag structure. As an example, even when shocks are i.i.d., firms with better histories of shocks will have higher equity and total value. This is not the result of lack of insurance, as the contract we consider is fully state contingent. The dependence of firm equity and size on its past history is analysed in this paper.

The optimal lending contract has some appealing comparative statics. Projects with lower sunk costs, better prospects or growth opportunities can sustain a larger initial debt and size, exhibit higher survival probability, the repayment of long-term debt is faster and borrowing constraints are eliminated sooner. A lower value of default (*e.g.* better outside enforcement or credit rating) implies larger firm size, leverage, and, consistent with Barclay and Smith (1995), more long-term debt. Firms with higher revenues are also predicted to have more leverage and long-term debt. Consistent with this prediction, Titman and Wessels (1988) present evidence that firms with greater sales display higher debt to asset ratios. Higher interest rates lead to a smaller initial sustainable debt and firm size. Though the relationship between risk and borrowing

2. See Evans (1987) and Hall (1987) for evidence on growth properties of firms by age and Dunne, Roberts and Samuelson (1989*a*,*b*) for evidence on entry and exit.

constraints is less clear, our analysis indicates that riskier projects could face tighter constraints. These and other comparative statics questions obviously cannot be addressed by existing models of firm dynamics which assume exogenous borrowing constraints.

A theory of endogenous borrowing constraints must address the following dynamic consistency: equity restricts current access to capital, but is itself determined by future access to credit. This dynamic consistency is not guaranteed in models of exogenous borrowing constraints, where the ability to raise short-term capital is limited by some pre-specified function of equity. Of all dynamically consistent lending plans, the optimal contract is the one that maximizes access to short-term capital and the expected rate of decrease of long-term debt.

Implementation of the optimal contract is straightforward in the deterministic version of our model and can be achieved by an initial long-term debt with a specific maturity structure. In every period the entrepreneur makes payments to the lender and expands its capacity with retained earnings. The outstanding long-term debt decreases over time. In the stochastic case the firm's earnings may not suffice to finance its expansion when confronted with highly productive states, so long-term debt is necessarily state contingent. In parallel to the deterministic case, average debt decreases over time.

The paper that is most related to ours is Thomas and Worral (1994). In their model, shocks are i.i.d., there is no liquidation value and outside value is given by the firm's revenues. Our extensions are important for several reasons. As a framework for the analysis of firm dynamics, the possibility of liquidation/exit and persistent shocks are very relevant. Secondly, by considering a general outside value function, we are able to examine the robustness of the results. Finally, in contrast to their model, the lender has full commitment to the contract. This turns out to simplify the analysis considerably by allowing the use of dynamic programming methods and thus providing a more extensive characterization of the optimal contract.

Our theory of debt is related to Hart and Moore (1994, 1998).[3] In Hart and Moore (1994) the threat of repudiation by the entrepreneur sets a lower bound on the present value obtained by him, which is equivalent to an upper bound on the value of debt. In addition, debt payments are subject to a cash flow constraint. These are also our two main assumptions. While in their set-up debt is either raised or not to fund the project, we let leverage be state contingent and time varying. Furthermore, we let revenues be state dependent and we allow for liquidation of the firm. These features give us added margins to discuss the dynamics of real and financial choices of firms. In our set-up, financial constraints give rise to three types of inefficiencies: (i) projects may not be financially feasible initially, as in Hart and Moore; (ii) firms may be credit constrained and produce below the optimal level, as in Thomas and Worral (1994); (iii) projects may be terminated too soon.

As in Fernández and Rosenthal (1990), our model implies a maximum sustainable long-term debt. In their paper, default constraints put a limit on repayment schedules and in some cases make it infeasible for the borrower to credibly commit to paying back the loans received. In such cases, the lender must forgive a certain fraction of the initial debt. Notice that if the initial investment exceeds this borrowing limit, the project will not be undertaken unless the entrepreneur contributes with its own funds. The feasibility of a project thus depends on the nature of default constraints. Bulow and Rogoff (1989*b*) show that if upon defaulting, the borrower cannot be excluded from saving at the market interest rate and has access to actuarially fair insurance, there is no financially feasible contract. Furthermore, in any feasible contract, total debt is limited by the costs borne by the borrower upon default.

3. For a survey on the corporate capital structure literature see Harris and Raviv (1991).

This paper is also related to the literature on optimal debt financing with incomplete contracts.[4] We briefly refer to the work that is most related to ours. A dynamic model of borrowing and lending with no default constraints was first introduced by Eaton and Gersovitz (1981), in the context of international lending. Kehoe and Levine (1993) present a general equilibrium theory under no default (or participation) constraints. These participation constraints generate endogenous debt limits. Alvarez and Jermann (2000) apply this framework to the analysis of risk sharing and asset pricing with limited commitment. Also related is the work by Marcet and Marimon (1992). Their simulations suggest that economic growth can be substantially impaired by the presence of limited enforcement. Bulow and Rogoff (1989*a*) study a model of international lending with imperfect commitment, where the lender can punish the borrower by means of direct sanctions and contracts can be renegotiated.

Imperfect enforcement is a source of contractual incompleteness that gives rise to a hold-up problem. An obvious way of dealing with this problem is through bonding. In our model, the hold-up problem is gradually resolved over time as the borrower builds up this bond by increasing its claims to future profits. A similar situation arises in the context of repeated insurance contracts when agents cannot commit not to take outside offers in the future. Harris and Hölmstrom (1982) use this mechanism to explain an increasing wage profile, when the ability of workers becomes known over time. Another example is Phelan (1995), that considers a repeated moral hazard model where agents can recontract with outside principals, generating increasing profiles of consumption.

Diamond (1989) studies reputation building in a model with both adverse selection (in project riskiness) and moral hazard (in project choice). There is a sequential equilibrium where interest rates decrease over time as the probability of default decreases. The fall in interest rate increases the value of maintaining a good reputation and thus reduces the incentives to take excessive risk, mitigating the conflict of interest between the borrower and the lender.

This paper is organized as follows. Section 2 provides a simple example of our framework. Section 3 introduces the model. In Section 4 we characterize the optimal contract along several dimensions. We also discuss how to implement the optimal contract using more standard financial instruments. As an alternative formulation, Section 5 presents the problem under study as a constrained growth problem. Section 6 concludes. We leave the proofs and other technical results to the Appendix.

2. AN EXAMPLE

Consider a project that requires an initial investment I_0 and gives revenues $R(k_t)$ in every period, where k_t is the working capital employed in period t. Cash flows are deterministic. Net profits are given by $R(k_t) - (1+r)k_t$, which are maximized at the optimal level of working capital k^*. Suppose that $R(k^*) - (1+r)k^* > rI_0$, so investing in this project is profitable. However, the entrepreneur has no wealth. In absence of enforcement problems, the entrepreneur would initially raise *total debt* of $D_0 = I_0 + k^*$, reinvesting every period k^* from its revenues and making payments to the debt holder. The Modigliani–Miller theorem applies to this set-up, so the specific payments to be made are undetermined. The total value of the project is

$$W = \frac{R(k^*) - (1+r)k^*}{r},$$

which by the above condition exceeds the initial investment.

4. Aghion and Bolton's (1992) seminal paper develops a theory of capital structure based on wealth constraints on the part of the entrepreneur and on the inability of the parties to write contingent contracts. For an excellent survey of the literature, see Hart (1995).

Now suppose that the entrepreneur has an alternative outside opportunity with value k_t and unless the project grants him this value, he would choose to default. According to this outside opportunity the entrepreneur can steal the working capital. It is used here for simplicity of exposition and is a special case of the more general framework used throughout the rest of the paper. Suppose also that this outside option is not contractible and that

$$rI_0 < R(k^*) - (1+r)k^* < r(I_0 + k^*). \tag{1}$$

As we will now show, this condition implies that even though it is efficient to start the project at full scale, a default-free contract must necessarily involve borrowing constraints, *i.e.* $I_0 < D_0 < I_0 + k^*$, so that $k_0 < k^*$. To see this, notice that if the project were carried out in full scale from the start, the value to the entrepreneur would be

$$\begin{aligned} V_0 &= W - I_0 \\ &= \frac{R(k^*) - (1+r)k^*}{r} - I_0, \end{aligned}$$

which from equation (1) implies that $V_0 < k^*$ and thus the entrepreneur would choose to default. This example also illustrates that borrowing constraints would arise even if $I_0 = 0$.

We now derive the optimal no-default lending contract. Let $\{D_t\}$ and $\{V_t\}$ denote the debt and equity values in this contract. Given the outside opportunity, it follows that

$$k_t \leq V_t,$$

with equality when $V_t < k^*$, so as the equity of the firm increases over time, k_t will also increase. It is obvious that while $V_t < k^*$, no dividends will be distributed, so $k_t = V_t = (1+r)V_{t-1} = (1+r)k_{t-1}$. This in turn implies that at time t the lender must receive a payment

$$\tau_t = R(k_{t-1}) - k_t = R(k_{t-1}) - (1+r)k_{t-1}. \tag{2}$$

When k^* is reached, the borrower receives rk^* every period as dividends and the lender gets the remaining cash flows. Letting D_0 be the initial debt and $k_0 = D_0 - I_0$, it follows that k_t will grow at rate r until the optimal level k^* is reached. Letting $W(k_0)$ denote the total value of the project if the initial size is k_0, it follows that the maximum initial debt satisfies $V_0 = W(k_0) - I_0 = k_0$. The initial total debt $D_0 = I_0 + k_0$ together with the maturity structure defined by equation (2) define the optimal lending contract. Notice that the maturity structure and the initial debt are jointly determined: a different maturity structure violates the no-default constraint unless the initial debt is smaller.

In an abstract sense, the optimal debt contract specifies a growth policy for the firm and a sequence of cash flows. One implementation of this contract was described above, involving a single initial loan $I_0 + k_0$ and a repayment plan. An alternative implementation is to consider an initial *long-term debt* $B_0 = I_0$, together with a sequence of short-term loans k_t. Short-term loans are repaid in full at the end of the period. All remaining profits are applied to the long-term loan until equity reaches the point $V = k^*$. At that point, the firm is able to raise the optimal amount of short-term capital. Section 4.5.1 in what follows discusses these implementations in greater detail and generality.

Having reached k^* does not imply that the entrepreneur gets rid of the bank. There could still be positive debt at this point. What is true is that beyond this level if equity keeps increasing, a point can be reached where the entrepreneur does not need the bank: when equity is sufficient to finance the project. When uncertainty is added later on, this statement needs to be qualified to say: under all contingencies.

This example shows how borrowing constraints arise when the no-default constraint binds. It also points to an important difference between this theory and standard models on firm growth

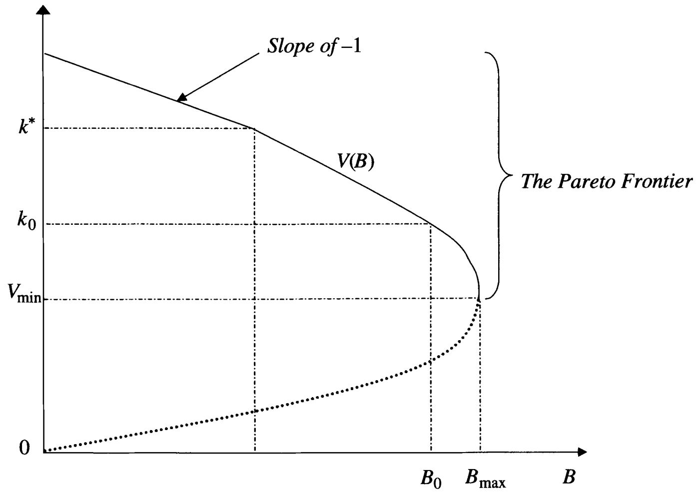

FIGURE 1
The Pareto frontier

with limited borrowing. The latter models consider the firm's dynamic capital accumulation problem subject to some exogenously given borrowing constraint. In our set-up, the forward looking nature of the no-default constraint establishes a link between current debt limits and the structure of future repayments. In this sense, our model considers the optimal design of long-term debt contracts and borrowing constraints.

The growth in the equity value V_t occurs at the same time as the long-term debt decreases. This is no coincidence; as we establish later, the *optimal contract* defines a path along the Pareto frontier defined by the value entitlements of the lender and the firm, B_t, V_t respectively. Figure 1 plots the Pareto frontier, $V(B_t)$. In Figure 1, $B_{\max}$ is the highest level of long-term debt that can be credibly repaid. Any investment project requiring more than $B_{\max}$ cannot be financed. This implicitly defines a level of equity, $V_{\min}$, that is just enough to keep the firm from defaulting were $B_{\max}$ granted. As B decreases, the borrowing constraint is loosened and V increases by more than one-for-one (more on this below). The intuition is that the increase in equity that results from reducing debt, improves entrepreneur's incentives thereby leading to an increase in firm value. As the unconstrained optimum is reached, *i.e.* $V = k^*$, any further decreases in long-term debt result in one-for-one increases in equity, since the total value of the project is not changed.

What are the implications of our contract to the maturity of debt? In our example, short-term capital is constrained by $k_t \leq V_t = V(B_t)$. This defines implicitly a negative relationship between short-term capital and long-term debt. Faced with this short-term borrowing constraint, the firm would choose to repay its long-term debt as fast as possible until this short-term borrowing constraint does not bind.

The limit on total debt $D_t = k_t + B_t \leq V(B_t) + B_t$ is typically not independent of the composition between short and long term. Indeed, as we show later it is generally the case that $V'(B_t) < -1$ implying that total debt can be higher the lower its long term component is: debt structure matters. This is illustrated in the curved portion of the Pareto frontier in Figure 1. Clearly, in the region of the Pareto frontier where $V'(B) = -1$ the maturity structure no longer matters for firm value. This discussion suggests how special are models where firms are confronted with a fixed total borrowing limit. Moreover, it may not be desirable for firms to have the freedom to choose how much to repay each period at will. Indeed, if given such freedom and maintaining the initial loan $I_0 + k_0$, the firm would choose not to repay, keep the revenues, and exit in the following period with an outside value $R(k_0) > (1 + r)k_0 = V_1$. To avoid default, the initial loan would then have to be lower.

In the rest of the paper, our example is generalized in two main dimensions. First, a general class of no-default constraints are considered. Second, we introduce persistent shocks to revenues and consider the possibility of liquidation as part of the lending contract. We derive optimal lending contracts and study their implications for firm growth and survival.

3. THE MODEL

Time is discrete and the time horizon is infinite. At time zero an entrepreneur may start a firm by pursuing a project which requires a fixed initial investment $I_0 \geq 0$. The project gives a random stream of revenues $R(k, s)$ each period, where k is the capital input and $s \in S \subset \Re$ is a revenue shock. The revenue shock s follows a Markov process with conditional cumulative distribution function $F(s', s)$. $F(\cdot)$ is jointly continuous.

The timing of events within a period is as follows. First, the shock s is observed. After observing the revenue shock, the firm can either be liquidated, at a value $L(s)$, or continue in operation. If the firm continues in operation, inputs are purchased, sales take place, and revenues $R(k, s)$ are collected at the end of the period. These revenues are an increasing function of both, k and s. The revenue shock s is publicly known, so there is no asymmetry of information.

The entrepreneur has limited liability. It starts with zero wealth and thus requires a lender to finance the initial investment and the advancements of capital every period.[5] Both, entrepreneur and lender, discount flows using the same discount rate $r > 0$.

Lenders commit to long-term contracts with the firm. However, contracts have limited enforceability as the borrower can choose to default. As in Hart and Moore (1994), only the borrower has the ability to run the firm. If the match is ended either voluntarily or not, the residual value for the borrower is given by a function $O(k, s)$, which is discussed in more detail below.

A long-term contract specifies a contingent liquidation policy e_t ($e_t = 1$ if exit is recommended and $e_t = 0$ otherwise), capital advancements k_t from the lender to the firm that take place at the beginning of each period, and a cash flow distribution consisting of a dividend flow d_t and payments to the lender $R(k_t, s_t) - d_t$ which takes place at the end of the period. Because the firm has no additional funds $d_t \geq 0$. The capital advancement, dividends and liquidation policy at any time t, are contingent on the history $h_t = \{k_{r-1}, d_{r-1}, e_{r-1}, s_r\}_{r=1}^{t}$ of previous transfers and all shocks, including s_t.[6] Let H be the set of all possible histories.

Definition 1. A feasible contract is a mapping $\mathcal{C} : H \to \Re_+^2 \times \{0, 1\}$ such that for all $h_t \in H$ and $(k_t, d_t, e_t) = \mathcal{C}(h_t)$, $d_t \geq 0$, and $e_t = 1$ if $e_{r-1} = 1$, for some $r \leq t$.

The timing of events is as follows. At time zero, a competitive set of lenders offer long-term contracts to the firm. If the firm accepts a contract, the lender pays for the initial investment I_0 and carries the contract as stipulated, provided the agent has not deviated from the corresponding

5. If the firm starts with wealth $w < I_0$, then the project only needs financing of $I_0' = I_0 - w$. If $w > I_0$, then there is no need for external lending.

6. To clarify the notation, we shall use letters without subscripts to denote current period values and with a prime to denote next period's value, except when an explicit reference to longer horizons takes place in which case we shall use subscripts $t, t + 1, \ldots$.

repayment plan or defaulted. If the agent deviates, the contract is terminated and the firm liquidated. Otherwise, the plan defined by the contract continues to be implemented.

3.1. *Contracts with perfect enforceability*

In the absence of enforcement problems, the lender and the firm can commit to the above contract without any additional constraints. Since flows are discounted at the same rate, the optimal contract maximizes total expected discounted profits for the match.

Let $\pi(s) = \max_k R(k, s) - (1+r)k$ denote the profit function. The following assumptions guarantee a solution to this profit maximization problem.

Assumption 1. *The function R has the following properties:*

(1) *$R(k, s)$ is continuous.*
(2) *For each s, $R(k, s) - (1+r)k$ is quasiconcave in k and has a maximum.*
(3) *There exists some $b < \infty$ such that for all s and k, $-b \leq R(k, s) - (1+r)k \leq b$.*

The total surplus of the match $\tilde{W}(s)$ satisfies the following dynamic programming equation:

$$\tilde{W}(s) = \max\left\{L(s), \frac{1}{1+r}\left[\pi(s) + \int \tilde{W}(s')F(ds', s)\right]\right\}. \tag{3}$$

If $\tilde{W}(s) = L(s)$ for all $s \in S$, the firm would not be viable and would be immediately closed. The *survival set*, $\tilde{S} = \{s : \tilde{W}(s) > L(s)\}$, is the set of states at which the firm would continue in the industry.

Assumption 2. *The survival set $\tilde{S}$ is non-empty.*

With perfect enforceability the Modigliani and Miller (1958) theorem applies and the capital structure of the firm is indeterminate. There are two implications from this. First, survival and growth of the firm are independent of its capital structure. Second, there is a multiplicity of optimal debt repayment plans for short- and long-run debt that have the same present values.

3.2. *Contracts with limited enforceability*

As illustrated in our example, when firms have the possibility of default, the long-term contract proposes a unique debt repayment plan. We now give details about the firm's ability to default and the construction of the long-term contract.

3.2.1. Entrepreneur's outside opportunities. If the firm chooses to default it will do so prior to making any payments to the lender. We assume that by defaulting a firm obtains a total value given by a function $O(k, s)$. This function is one of the primitives of the model and summarizes the value of the outside investment opportunities faced by the firm, which is common knowledge to both parties. For example, if the borrower can collect revenues and disappear, without being able to re-establish itself as a new firm, then $O(k, s) = R(k, s)$, as in Thomas and Worral (1994). If the firm can continue operations but is excluded from borrowing, saving and insurance (as in Manuelli (1985), Marcet and Marimon (1992)), then $O(k, s)$ is the value obtained by the firm through optimal self-financing. Alternatively, a firm may be excluded from borrowing but not from saving or purchasing insurance, as in Bulow and Rogoff (1989*b*). $O(k, s)$

will be the value function thus obtained. Another example is obtained if the firm can establish a new contract with a bank, after paying a cost for breach of contract.

If at the beginning of some period the bank decides to liquidate the firm, then the latter obtains a value $O(0, s)$. This value represents an inalienable component of the firm's capital; it is the residual value that cannot be taken away from the entrepreneur such as his opportunity cost. The difference $L(s) - O(0, s)$ thus represents the component of the liquidation value that can be appropriated by the bank. As in Hart and Moore (1994) anything greater than $L(s) - O(0, s)$ could be renegotiated down by the borrower. We assume that $O(0, s) = 0$, for all s. This is not without loss, but it greatly simplifies the exposition of the paper. In Appendix A we show what modifications are required to generalize the results. We make the following assumptions on O.

Assumption 3. *The function O has the following properties:*

(1) $O(0, s) = 0$, *for all* s.
(2) $O(k, s) \geq 0$.
(3) $O(k, s) - k \leq L(s)$.
(4) *O is a continuous function.*
(5) *O is non-decreasing in both arguments.*

Part 2 is in line with our limited liability assumption. Part 3 says that involuntary separations are less efficient than liquidation.

3.2.2. Long-term debt contracts. In this section we formulate the contract in an abstract form; alternative implementations are discussed in Section 5. A contract specifies a liquidation policy, history-dependent contingent advances of capital k_t, and a dividend distribution d_t. This contract implicitly defines an equity value for the firm V_t and the long-term debt level or value to the lender B_t. The equity value for the firm gives the discounted sum of future dividends whereas the long-term debt or value to the lender gives the discounted cash flows to the lender. Thus, the total asset value after history h_t is defined by $W_t \equiv V_t + B_t$.

The total value of debt includes debt originated at possibly different periods of time. However, because there is only one lender, these different vintages of debt are all homogeneous and there are no seniority claims. In spite of this, we label k_t as short-term debt and B_t as long-term debt.

Letting $V_{t+1}(s')$ denote the continuation equity value at the beginning of period $t + 1$ after history $h_{t+1} = (h_t, k_t, d_t, e_t, s')$, the firm will choose not to default in period t provided that the value of outside opportunities is lower than the value of its entitlement by staying in the match:

$$O(k_t, s_t) \leq \frac{1}{1+r}\left(d_t + \int V_{t+1}(s')F(ds', s_t)\right). \tag{4}$$

Since the lender can always include in the contract a recommendation to liquidate the firm and since liquidation is more efficient than default, we require that the participation or enforcement constraint equation (4) be satisfied at all times.

A feasible contract is *enforceable* if, after any history h_t, the triplet $(k_t, d_t, V_{t+1}(s'))$ satisfies equation (4). It is easy to see that after any history, the continuation contract is also an element of the set of enforceable contracts. Letting $\Omega(s) \subset \Re^2$ be the set of values (V, B) such that there exists an enforceable contract with initial values $V_0 = V$ and $B_0 = B$ and initial state s, it follows that $(V_t, B_t) \in \Omega(s_t)$ for all t. The set of optimal contracts gives values that are in the Pareto frontier of $\Omega(s)$. Moreover, as seen below optimal contracts have the property that for all t, (V_t, B_t) are in the Pareto frontier of $\Omega(s_t)$.

The Pareto frontier was depicted in Figure 1 for an example without uncertainty. In that example the Pareto frontier solves the primal problem of maximizing firm value subject to total debt. An alternative to constructing the Pareto frontier is through the dual problem of maximizing the debt value subject to the firm's equity value.[7] In the more general scenario of random productivity, this frontier can be characterized by a function $B(V, s)$ which gives the maximum debt that is enforceable for a given level of equity V and state s and a domain restriction for V. Thus, underlying $B(V, s)$ is the optimal long-term debt contract. In fact, any alternative contract will give a lower frontier, so that for the same level of V and shock s, the firm will be more financially constrained.

3.2.3. Equilibrium contracts. We can now describe the maximum state contingent long-term debt that can be credibly repaid from time 0. The existence of many competitive lenders requires that borrowers receive the highest share value consistent with this initial long-term debt level.

Definition 2. An equilibrium contract $\mathcal{C}(\cdot)$ is feasible, enforceable, and gives the highest possible initial value to the borrower consistent with the lender breaking even: $V_0 = \sup\{V : B(V, s_0) \geq I_0\}$ when the initial shock is s_0.

Referring back to Figure 1, notice that financing of an initial investment I_0 requires that $B_0 \leq B_{\max}$ be raised. Any investment value requiring long-term debt in excess of $B_{\max}$ cannot be financed. This is because, the project does not generate enough cash flows to repay the bondholders and still grant enough future dividends to the firm, that would keep it from defaulting.

If I_0 cannot be financed, no contract is possible unless the firm has funds to contribute to this initial investment. More specifically, the firm must contribute at least $I_0 - B(V_0, s_0)$. For example, a weaker enforcement structure, defined by higher values $O(k, s)$, will reduce the total surplus of the project and thus require a higher initial investment by the firm.

In general, financing of I_0 is feasible only whenever a credible contingent repayment schedule can be agreed upon by both parties. This depends critically on characteristics of the outside-value function. As an example, Bulow and Rogoff (1989*b*) consider the feasibility of long-term debt contracts between a lending and a borrowing country with one-sided commitment. They show that if the borrower cannot be excluded from contingent savings (what they call cash-in-advance contracts), then debt is restricted by the present discounted value of the penalties from breaching the contract. In this case, this present discounted value is given by $B_{\max}$, and when there are no penalties the frontier collapses and there is no feasible lending.

4. THE OPTIMAL CONTRACT

We now construct the dynamic programming problem that solves for the Pareto frontier. The values $V_{t+1}(s')$ provide a summary statistic for the future contract and together with (k_t, d_t, e_t, s_t) are sufficient to verify this non-default or participation constraint. Using V_t as a state variable, following Spear and Srivastava (1987), the contract can be specified in recursive form. Every period, given initial values $V_t = V$ and $s_t = s$ and assuming liquidation is not recommended, the contract specifies a pair (k, d) and continuation values $V(s')$. In turn, the continuation values $V(s')$ will dictate future investment and dividend actions. This formulation

7. This alternative formulation simplifies the analysis considerably.

is consistent with real life bond contracts in which covenants are written establishing restrictions in the firm's investment, dividend and financing policies.

Since the current value for the entrepreneur is the sum of the dividends paid out this period plus the discounted value of the stream of future dividends, the following *equity cash flow* condition must be satisfied if the firm is not liquidated:

$$V = \frac{1}{1+r}\left(d + \int V(s')F(ds', s)\right). \tag{5}$$

The continuation equity values $V\left(s'\right)$ must be supported by an enforceable continuation contract. By Assumption 3(1), any positive continuation value $V(s') \geq 0$ is feasible, since it can be obtained by giving the firm a transfer $t \leq V(s')$ and liquidating the firm (or committing to no future advancements). Any continuation value $V(s') < 0$ is not feasible. Hence, an equity value $V(s')$ can be supported by an enforceable contract if, and only if, $V(s') \geq 0$. This is the domain restriction indicated above.

Using equation (5), the enforcement constraint equation (4) simplifies to

$$O(k, s) \leq V. \tag{6}$$

Finally, the limited liability condition $d \geq 0$ and the equity cash flow equation simplify to[8]

$$\frac{1}{1+r}\int V(s')F(ds', s) \leq V. \tag{7}$$

We now write the problem's Bellman equation. We assume that the lender has access to perfect capital markets; a negative period pay-off results in increased lending. The lender maximizes the debt level $B(V, s)$ by choosing an enforceable contract that gives a current expected value V to the firm when the current revenue shock is s. Recalling that the total surplus of the match is given by $W(V, s) = B(V, s) + V$, and that the lender has full commitment to the contract, it is immediate to see that the optimal debt contract also maximizes $W(V, s)$ given V. The function $W(V, s)$ satisfies the following dynamic programming equation:

$$W(V, s) = \max\Bigg\{L(s),$$
$$\max_{k, V(s')\geq 0} \frac{1}{1+r}\left[R(k, s) - (1+r)k + \int W(V(s'), s')F(ds'; s)\right]\Bigg\} \tag{8}$$

subject to equations (6) and (7). Standard results in dynamic programming imply that there is a unique solution $W(\cdot)$ to this problem.[9]

It is interesting to compare this programme with the one obtained for the case of perfect enforcement, equation (3). Notice that if the no-default constraint equation (6) were never binding, then k would be chosen so that $R(k, s)-(1+r)k = \pi(s)$ and the solution to equation (8) would give $W(V, s) = \tilde{W}(s)$. Also, if there was unlimited liability, V would grow without bound to achieve the efficient level of capital *next* period for all states in $\tilde{S}$.

In the optimal contract, the lender decides to terminate the contract and liquidate the firm whenever its value reaches $B(V, s) = L(s) - O(0, s) = L(s)$. In such a state, the lender gives the lowest feasible value of equity to the firm, $O(0, s) = 0$. Because of limited liability equation (7)

8. Strictly speaking limited liability does not require that d_t is non-negative at any point in time. In principle the entrepreneur could accumulate some form of precautionary saving and invest them at the market interest rate to cover future financial needs. In the optimal contract the lender is implicitly doing these savings for the entrepreneur. See the implementation of the contract discussed in Section 4.5.1 for an example.

9. The possibility of liquidation introduces a non-convexity in the above decision problem. Though not explicitly stated in the above formulation, in our analysis we consider the possibility of randomization on the liquidation decision. This is explicitly addressed in Appendix C.

the lender does not "waste" equity assignments in these states. Since $L(s)$ represents the highest value that can be appropriated by the lender upon liquidation, it can also be thought of as standard collateral.

In the next subsections we will discuss in turn the properties of the debt contract for the design of optimal borrowing constraints, the efficient frontier—a state after which the borrowing constraint will never bind—the growth and survival patterns of firms, and the capital structure policy.

4.1. *Short-run borrowing constraint and equity*

As seen in the example, the level of long-term debt sets a limit on the short-term advancements of working capital. A direct consequence of this, is that a static problem determining short-run financing of k can be separated from the dynamic choice of $V(s')$. In particular, define the indirect profit function

$$\Pi(V, s) = \max_k R(k, s) - (1+r)k \quad \text{subject to } O(k, s) \le V. \tag{9}$$

The solution to this problem is simple. Let $K(s) = \inf\{k : R(k, s) - (1+r)k = \pi(s)\}$, and define $V^u(s) = O(K(s), s)$. This is the smallest continuation value for the firm that, once reached, is compatible with static profit maximization. Thus, if $V \ge V^u(s)$, k is chosen so that $R(k, s) - (1+r)k = \pi(s)$. If $V < V^u(s)$, current profit maximization cannot be enforced and k is chosen so that $O(k, s) = V$. Hence, $R(k, s) - (1+r)k < \pi(s)$, if and only if $V < V^u(s)$. These results follow directly from Assumptions 1 and 3.

The frictions between total debt, equity and financial constraints seen earlier in our example are already apparent here. From the short-run financing constraint equation (6), everything else constant, the higher the debt level B (lower V), the less capital the firm is able to borrow for production. This negative relationship between long-term debt and short-term capital is at the heart of the financing constraint.

We make the following assumptions on the indirect profit function:

Assumption 4. *The function Π has the following properties:*

(1) *Π is twice continuously differentiable, uniformly bounded, increasing in s, strictly increasing in V for $V < V^u(s)$.*
(2) *Π is concave in V, and strictly concave if $V < V^u(s)$.*
(3) *There exists $M < \infty$, such that $V^u(s) \le M$ for every s.*

Lemma 2 in Appendix B gives sufficient conditions for Assumptions 4(1) and 4(2) to hold. Loosely speaking, the assumption that Π is increasing in s, requires that the revenue function increases by more than the outside-value function when shocks increase. Assumption 4(2) requires that the degree of concavity of the revenue function with respect to k be greater than the degree of concavity of the O function. One example that satisfies all of these requirements is $O(k, s) = R(k, s)$ (this is the assumption made in Thomas and Worral (1994)). Another interesting example arises when shocks are project specific, *i.e.* the outside value does not respond to changes in shocks ($O(k, s) = k$).

4.2. *Firm value and the efficient frontier*

The deterministic example we constructed in Section 2 shows that there is a minimum V (maximum level of debt) that is consistent with efficiency. For higher debt levels, the total value of the firm is lower. In this section we extend this result to the general case, where this upper bound on debt is defined for each state. For debt values above that limit, an increase in the equity of the firm raises total firm value, so capital structure matters. Provided debt is below this limit, capital structure is not relevant and the total value of the firm coincides with the perfect enforcement case. Thus, we call this limit—and its corresponding minimum level of equity—the efficient frontier.

From (9) it follows that if the current equity level is sufficiently high, $V \geq V^u(s)$, the period return will be identical to the one obtained in the unconstrained problem, *i.e.* $\Pi(V, s) = \pi(s)$. However, this may not ensure that the current total value $W(V, s) = \tilde{W}(s)$, since the contract must also guarantee that the enforcement constraint will not bind in any future period. For example, if $V^u(s) < V < \frac{1}{1+r} \int V^u(s')F(ds', s)$, then it must be the case that $V(s') < V^u(s')$ with positive probability on a subset of the survival set $\tilde{S}$, and thus next period's unconstrained profit maximum cannot be guaranteed. However, if V is high enough, it may be possible to guarantee that the enforcement constraint will not bind in any future period and thus the unconstrained optimal solution will be attained.

Abusing the notation, let $V^n(s)$ be the minimum level of current initial value for the firm that is needed to guarantee that the enforcement constraint will not bind for at least n periods, including the current one, when the state is s. Then $V^n(s)$, $n \geq 1$, can be defined recursively by

$$V^n(s) = \max\left(V^u(s), \frac{1}{1+r} \int_{\tilde{S}} V^{n-1}(s')F(ds', s)\right) \tag{10}$$

for $s \in \tilde{S}$, with $V^0(s) = 0$. Let

$$\tilde{V}(s) = \lim_{n\to\infty} V^n(s).$$

Since $V^n(s)$ is an increasing sequence, which by Assumption 4(3) is uniformly bounded, this limit exists. Furthermore, using Lebesgue's dominated convergence theorem, it follows that $\tilde{V}(s)$ is a solution to

$$\tilde{V}(s) = \max\left(V^u(s), \frac{1}{1+r} \int_{\tilde{S}} \tilde{V}(s')F(ds', s)\right), \qquad s \in \tilde{S}. \tag{11}$$

This solution is unique, as Blackwell's sufficient conditions can be immediately verified.

The functions $\tilde{V}(s)$ and $\tilde{W}(s)$ define the maximum long-term debt $\tilde{B}(s) = \tilde{W}(s) - \tilde{V}(s)$ consistent with unconstrained financing. We call the function $\tilde{V}(s)$ the efficient frontier, since for equity values above $\tilde{V}(s)$ (*i.e.* debt values below $\tilde{B}(s)$) the firm cannot improve the total surplus by manipulating its debt level. This interpretation is derived from Lemma 1.

Lemma 1. *Let s be in the survival set $\tilde{S}$. Then:*

(1) *$W(V, s)$ is weakly increasing in V.*
(2) *For all $V \geq \tilde{V}(s)$, $\tilde{W}(s) = W(V, s)$.*
(3) *For all $V < \tilde{V}(s)$, $\tilde{W}(s) > W(V, s)$.*

Proof. See Appendix D. ||

An immediate consequence of Lemma 1 is that firm growth and survival are a function of the capital structure below the efficient frontier only. These age and path effects are analysed extensively in what follows.

Lemma 1 helps to identify financially constrained firms in the model. Short-run constrained firms are those that are unable to borrow enough capital to achieve static profit maximization.

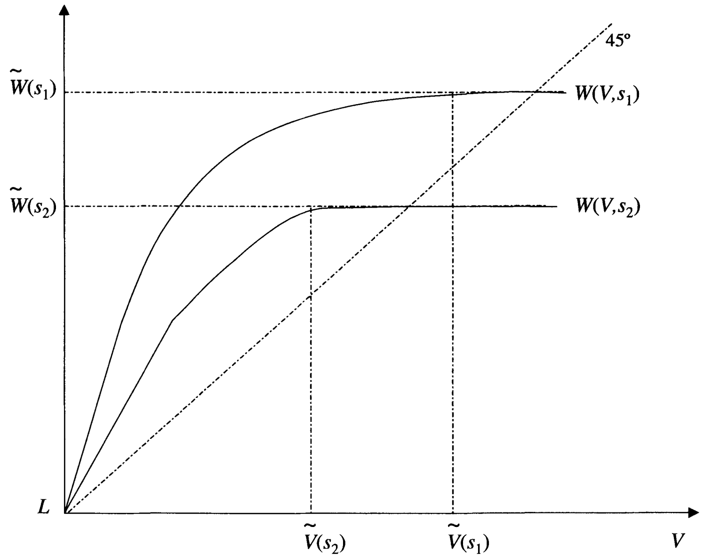

FIGURE 2
The value function for two revenue shocks

These firms have current excess marginal returns. Are these the only constrained firms? No. As discussed above, firms with equity values $V^u(s) < V < \tilde{V}(s)$, face a positive probability of reaching states where the marginal return is positive. All these firms have total surplus $W(V, s) < \tilde{W}(s)$ and future excess marginal returns.

Figure 2 illustrates the function W for two possible revenue shocks $s_1 > s_2$. This figure depicts several properties of the value function. In particular, it strengthens the statement in Lemma 1 that the value function is weakly monotone in V for fixed s, to strictly monotone (see Appendix C for a technical discussion). We leave the discussion of other properties implied by Figure 2 for later.

A direct implication of strict monotonicity of the value function W, is that the optimal contract recommends that no dividends be distributed below the efficient frontier and that all earnings be allocated to the repayment of long-term debt (*i.e.* constraint equation (7) holds as an equality). This is true despite the fact that the borrower and lender are risk neutral and discount flows at the same rate. The reason is that delaying dividend distribution allows for faster equity growth and repayment of long-term debt. More equity reduces the incentives to default and relaxes the short-term borrowing constraint. Thus, total firm value increases as does the maximum long-term debt *at time zero* that can be credibly repaid. Formally:

Proposition 1. *If* $V < \frac{1}{1+r} \int_{\tilde{S}} \tilde{V}(s')F(ds', s)$ *the optimal contract requires that* $V = \frac{1}{1+r} \int V(s')F(ds', s)$, *so no dividends are distributed.*

The relaxation of the borrowing constraint has a simple interpretation for the case of deterministic revenues. The contract gives the firm a certain value, equal to the discounted value of its share of profits. This value, *i.e.* the anticipation of the firm's share of future profits, is precisely what holds the firm from defaulting. Since incentives to default increase with the amount of capital advanced, the higher the equity share, the more capital can thus be advanced. As the outstanding equity grows over time through retained earnings and long-term debt is repaid, the capital advances increase towards the unconstrained level. Clearly, the debt–equity mix is important in determining the borrowing capacity of the firm: the increased equity provides the *bonding* necessary to raise increasing quantities of capital. Once the efficient frontier is reached this role for equity disappears as the long-term contract becomes a function of the current revenue shock only. Also, only after the efficient frontier is reached will dividends be distributed. This feature of the optimal dividend policy is present in other papers as well. In Spence (1979), in order to meet a cash flow constraint, firms distribute dividends only once its capital reaches the long-run size.

4.3. *Firm growth and survival: age effects*

Many studies have documented that firms experience very large growth rates at their early stages of life coupled with dramatic turnover rates. In this section, we discuss the implications of the optimal debt contract for firm growth and survival.

Thomas and Worral (1994) show that capital advancements—and thus firm size—grow monotonically over time. In this section we prove a similar result for our general set-up. We have already established that on average V increases (and thus, B decreases). However, given that revenues are stochastic and debt is contingent, a non-trivial choice of continuation values must be made, trading off borrowing constraints along different future paths. We provide an elementary characterization of the optimal state contingent debt. For any history, contingent debt decreases monotonically over time conditional on the revenue shock.

The results presented in what follows and in the next section use strict concavity of the value function $W(\cdot, s)$ for $L(s) < W(V, s) < \tilde{W}(s)$. Allowing for random liquidation, Appendix C shows this holds under Assumption 4(2).[10]

Take $V_t < \frac{1}{1+r} \int \tilde{V}(s_{t+1}) F(ds_{t+1}, s_t)$, so the constraint in equation (7) binds. It is easy to see that at an interior solution for the optimal contract, $V(s_{t+1})$ will be chosen so that $W_1(V(s_{t+1}), s_t)$ is equalized for all those states were the firm is not liquidated.[11] Notice that this derivative serves as an index for contingent equity and debt. By concavity, a lower derivative corresponds to lower values of contingent debt for all states. By the envelope theorem,[12]

$$W_1(V_t, s_t) = \frac{1}{1+r} \Pi_1(V_t, s_t) + W_1(V(s_{t+1}), s_{t+1}). \tag{12}$$

Given that $\Pi_1 \geq 0$, this implies that the level of contingent debt decreases over time, conditional on the revenue shock. Moreover, it will strictly decrease if and only if $\Pi_1 > 0$, that is when short-run borrowing constraints bind. Notice that this implies that state contingent debt (for all states) will not change after a firm transits through a state where it faces no short-term borrowing constraints and it will strictly decrease otherwise. Interestingly, this implies that even if the firm transits through a period with negative cash flows its contingent debt will not grow and may indeed decrease.

Given the negative relationship between k and long-term debt, the following property for firm size and profits follows:

Proposition 2. *Conditional on the revenue state of the firm s, firm size and profits increase with age.*

10. In particular, our conditions imply that the value function is concave when $O(k, s) = R(k, s)$ and there is no liquidation value, as in Thomas and Worral (1994).

11. Concavity guarantees that the value function is differentiable almost everywhere.

12. Pointwise differentiation of the objective function with respect to $V(s')$ yields the first-order condition $W_1(V(s_{t+1}), s_{t+1}) = \lambda_t$ where λ_t is the Lagrangian multiplier on the limited liability constraint equation (7). The result now follows from using the envelope condition $W_1(V_t, s_t) = \frac{1}{1+r} \Pi_1(V_t, s_t) + \lambda_t$.

Another important corollary of our discussion concerns firm growth. While the firm is in a set where $V < \tilde{V}(s)$, firms will grow with age. Once a region is reached where $V = \tilde{V}(s)$, age will have no further growth effect. This implies that:

Proposition 3. *Younger firms will, on average, grow faster.*

This result does not require mean reversion of productivity shocks or learning, as in models of pure stochastic evolution, like Jovanovic (1982) and Hopenhayn (1992).

The fact that conditional on s, the equity value is an increasing sequence also has an important consequence concerning firm exit. Most industries exhibit very large turnover of firms, predominantly for small and young firms. Empirically, survival rates are lower for smaller and for younger firms. We address this evidence now.

Let $S^*(V)$ be the set of values s for which a firm with equity V remains active. It is immediate to see that the optimal exit rule satisfies the following properties: (i) $V_2 \geq V_1$ implies that $S^*(V_2) \supseteq S^*(V_1)$ (follows from the monotonicity of $W(\cdot)$ with respect to V); (ii) $S^*(V) \subseteq \tilde{S}$; (iii) if $s \in \tilde{S}$, then $s \in S^*(\tilde{V}(s))$. This shows that the exit set is weakly decreasing in V. Intuitively, highly indebted (lower V) firms have lower option value of staying. Thus, the capital structure matters for the exit behaviour of firms in that it induces inefficient liquidation. Zingales (1998) presents evidence for the trucking industry that high debt levels affect the survival of firms by constraining their investment decisions.

Because, V_t is increasing (conditional on s), if a firm does not exit at time t in state s, then it will not exit anytime in the future if it returns to the same state. Thus, according to our theory, limited enforcement contributes to a positive relationship between firm survival and age.

Proposition 4. *Conditional on the revenue state of the firm s, survival probability increases with age.*

These results are a reflection of financial constraints being relaxed with age. Fazzari *et al.* (1988) find that in addition to the firm's q-value cash flows are a significant variable to predict firms' investment. In our model, the q-value of a firm (as measured by the ratio of W to the book value of assets I_0) is a sufficient statistic once the equity value of the firm has reached the efficient frontier. Prior to that point, the size and growth of a firm is determined both by its equity value V and revenue shock s. In this range, a high q-value may reflect either good revenue shocks or lower borrowing constraints. Moreover, since in this region there is a one-to-one map between the pairs (V, s) and (W, Π), cash flows and the firm's q-value jointly determine its decisions and growth. Our model thus suggests that, in addition to the firm's q-value, cash flows are a relevant variable but only for young firms.

4.4. *Firm growth and survival: path effects*

In the previous subsection, we established the general principle of age and growth: conditional on a given shock s, V_t is a non-decreasing process, *i.e.* firm value tends to increase over time. In this subsection, we consider the effect of the history of productivity shocks. The main question we address is: do better histories lead to lower borrowing constraints, larger firm size and higher survival rates? The answer is a qualified yes. This question is particularly important because it implies that age alone does not explain financial constraints: history also matters.

To make our statements precise, consider the following two sequences of shocks:

$$\{s_0, s_1, \ldots, s_{T-1}, s_T\},$$
$$\{s_0, s'_1, \ldots, s'_{T-1}, s_T\}, \tag{13}$$

where both sequences start and end with the same shocks and $s_t \geq s'_t$. Let $\{V_0, V_1, \ldots, V_T\}$ and $\{V_0, V'_1, \ldots, V'_T\}$ denote the corresponding value sequences. As we will see, in general $V'_T \neq V_T$, so history matters. We say that the contract leads to *positive persistence* if $V_t \geq V'_t, t = 1, \ldots, T$. Positive persistence implies that, controlling for the current productivity shock, firm size and value increase with past productivity shocks. Our original question can be restated as: When does the optimal lending contract lead to positive persistence?

To motivate the mechanism for history dependence, consider the case of i.i.d. shocks. Under what conditions will the contract be *path independent*? Take two histories that are identical up to time t and differ only in period t, where $s_t > s'_t$. Let $V_{t+1}(s_{t+1})$ and $V'_{t+1}(s_{t+1})$ denote the corresponding continuation values for the two histories at time $t+1$. Using equation (12) it is easy to see that if $V_{t+1}(s_{t+1}) = V'_{t+1}(s_{t+1})$ then

$$\Pi_1(V_t, s_t) = \Pi_1(V'_t, s'_t).$$

This holds for all s_t and s'_t only if $\Pi_{12} = 0$, *i.e.* when the marginal effect of V is independent of the shock s. This is obviously a very exceptional case. In general, path dependence will be affected by the sign of this cross partial effect. In the case of i.i.d. shocks, positive persistence arises when $\Pi_{12} > 0$ while negative persistence arises in the opposite case. When shocks are not independent, positive correlation of shocks is a key condition for positive persistence, but not sufficient. Given these remarks, for the remaining analysis in this section we make the following two assumptions:

Assumption 5. *For all s', $F(s', s)$ is non-increasing.*

Assumption 6. $\Pi_{12}(V, s) > 0$ *whenever* $V \leq V^u(s)$.

The first assumption requires that the conditional probability distribution increases with the current shock in the first-order stochastic dominance sense. This is a very standard assumption, which is satisfied by many stochastic processes, in particular by first-order auto-regressive processes with absolute persistence less than one. A sufficient condition on the functions R and O for the second assumption (of supermodularity of Π in (V, s)) is given in Appendix B. This condition relates the magnitudes of the cross partials of both R and O. Consider the following extreme cases. If $R_{12}(k, s) > 0$ and $O(k, s) = O(k)$, then $\Pi_{12} > 0$ for $V \leq V^u(s)$. However, if $R(k, s) = R(k)$ and $O_{12}(k, s) > 0$ then $\Pi_{12} < 0$ for $V \leq V^u(s)$ and values of k such that $R_1(k, s) > 1+r$. Thus $\Pi_{12} > 0$ requires that the marginal revenue be more responsive to shocks than the marginal outside option.

The following proposition states an important implication of these assumptions, namely that continuation equity values $V(s_{t+1})$ are increasing in s_{t+1}.

Proposition 5. *Let $W(V, s)$ be strictly concave in V for $V < \tilde{V}(s)$. At an interior solution $W_{12}(V, s) > 0$ so any optimal continuation policy $V(s')$ must be non-decreasing.*

Proof. See Appendix D. ||

In words, Proposition 5 says that revenue states where marginal returns to capital are higher are assigned higher equity values.

Does monotonicity in the equity value assignment imply positive persistence? For the case of i.i.d. shocks, this is easy to prove. Indeed, take $s_t > s'_t$ and let $V_t > V'_t$ be the corresponding equity value assignments. Assume these values are such that at least for V'_t the efficient frontier is not reached in the following period. Now given that

$$EV_{t+1} = V_t(1+r) > V_t'(1+r) = EV_{t+1}'$$

it is straightforward to show that $V_{t+1} > V_{t+1}'$. Thus, even when shocks are i.i.d. the model is able to deliver persistence.

When shocks are serially correlated, positive persistence in equity values need not occur. The difficulty arises because higher shocks in one period increase the likelihood of high shocks in the following period. A higher initial equity value may not be enough to sustain significantly larger expected equity needs for the following period. Consider the following example.

Example 1. There are three states: $s_0 < s_1 < s_2$. Profits are identical under s_0 and s_1, *i.e.* $\Pi(V, s_0) = \Pi(V, s_1)$. However, $\Pi_1(V, s_1) < \Pi_1(V, s_2)$. Moreover, suppose $P(s_2 \mid s_1) = P(s_0 \mid s_0) = 1 - \varepsilon$, $P(s_0, s_1) = P(s_0 \mid s_1) = P(s_2, s_0) = \varepsilon$, for ε small, where $P(s_i, s_j) = P(\tilde{s}_{t+1} = s_i, \tilde{s}_t = s_j)$ and $P(s_i \mid s_j) = P(\tilde{s}_{t+1} = s_i \mid \tilde{s}_t = s_j)$. Now consider the paths

$$\{s_0, s_1, s_2\},$$
$$\{s_0, s_0, s_2\}.$$

From the optimal contract at time $t = 0$, derivatives are equated across states for period 1:

$$W_1(V_0, s_0) = W_1(V_1, s_1).$$

For period 2 we get

$$W_1(V_0, s_0) = \Pi_1(V_0, s_0) + W_1(V_{02}, s_2),$$
$$W_1(V_1, s_1) = \Pi_1(V_1, s_1) + W_1(V_{12}, s_2).$$

The values V_{ij} are the optimal continuation values for a firm starting at s_i and moving into state s_j. Suppose by way of a contradiction that $V_{12} \geq V_{02}$, so that the better history leads to higher entitlement at period 2. Then, by concavity, the optimal contract must similarly assign $V_{10} \geq V_{00}$ (because $W_1(V_{10}, s_0) = W_1(V_{12}, s_2) \leq W_1(V_{02}, s_2) = W_1(V_{00}, s_0)$). Thus

$$\begin{aligned}(1+r)V_1 &= \varepsilon V_{10} + (1-\varepsilon)V_{12} \\ &\geq \varepsilon V_{00} + (1-\varepsilon)V_{02} \\ &= (1+r)V_0 - (1-2\varepsilon)V_{00} - \frac{\varepsilon}{2}V_{01} + \left(1 - \frac{3}{2}\varepsilon\right)V_{02} \\ &> (1+r)V_0,\end{aligned}$$

where the first inequality comes from the previous discussion and the last inequality comes from $V_{02} \geq V_{01} \geq V_{00}$ which follows from supermodularity. But, $V_1 > V_0$, implies that $\Pi_1(V_1, s_1) < \Pi_1(V_0, s_0)$. Together with $W_1(V_{12}, s_2) \leq W_1(V_{02}, s_2)$ we obtain the contradiction that $W_1(V_1, s_1) < W_1(V_0, s_0)$.

It is possible to restrict the nature of persistence in productivity shocks so that positive persistence in equity obtains. For that purpose we make the next assumption.

Assumption 7. *If $\Pi_1(V(s), s)$ is non-decreasing in s, then $\Pi_1\left(\frac{1}{1+r}\int V(s)F(ds, s_0), s_0\right)$ is non-decreasing in s_0.*

Proposition 6. *Suppose that Π and F satisfy Assumption 7. Then the optimal contract displays positive persistence.*

Proof. See Appendix D. ||

Remark 1. This result can be strengthened to strictly positive persistence if the R.H.S. of Assumption 7 is strictly increasing.

The following example shows that Assumption 7 is verified in some usual parametrizations.

Example 2. Let the revenue function be $R(k, s) = sk^\alpha$, with $0 < \alpha < 1$, and the outside opportunity function be $O(k, s) = k$. It follows that

$$\Pi(V, s) = sV^\alpha - (1 + r)V$$
$$\Pi_1(V, s) = \alpha s V^{a-1} - (1 + r),$$

when $V < V^u(s)$. It can be shown (see Appendix D) that if $\Pi_1(V(s), s)$ is non-decreasing in s

$$\frac{d}{ds_0}\Pi_1\left(\frac{1}{1+r}\int V(s)F(ds, s_0), s_0\right) \geq \int V(s)\left(\frac{F_1(s, s_0)s + F_2(s, s_0)s_0}{s}\right)ds.$$

What conditions make the last integral positive? A strong sufficient condition is that the term in curve brackets be positive. An alternative is to assume the term in curve brackets is increasing in s and has positive expectation. Two special cases are of interest. When $s = \rho s_0 + \varepsilon$ and ε is distributed with cdf G, then $F(s, s_0) = G(s - \rho s_0)$. The term in brackets has the same sign as $\frac{\varepsilon}{\rho s_0 + \varepsilon}$. A sufficient condition in this case is that ε has positive support. In the special case where $\ln s = \rho \ln s_0 + \varepsilon$ and ε is distributed with cdf G, then $F(s, s_0) = G(\ln s - \rho \ln s_0)$ and the term in brackets has the same sign as $\frac{1-\rho}{s}$, which is always positive.

Notice that Proposition 6 implies that marginal returns will be higher for higher productivity shocks. Using these marginal returns as a measure of borrowing constraints, it follows that borrowing constraints are tighter for higher shocks.

Do good histories increase the probability of survival under positive persistence? The answer is again a qualified yes. As discussed above, the survival set $S(V)$ increases with V. Consequently, good histories increase the survival set. However, a good history also changes the conditional distribution of shocks. It is conceivable—yet unlikely—that the latter effect could contribute negatively to survival. This could happen if the difference between the value of staying and the value of liquidation is not monotonic in s. The following proposition takes care of this anomaly.

Proposition 7. (i) *$W(V, \cdot)$ is weakly increasing in s.*
(ii) *$L(s)$ is constant.*[13] *Then optimal liquidation is determined by an increasing threshold $s^*(V)$.*

Proof. See Appendix D. ||

Given the assumptions for positive persistence, good histories imply both an increase in V and an increase in the conditional distribution for s. Under the assumptions of part (ii) of the above proposition, survival probabilities also increase.

4.5. *The structure and composition of debt*

In previous sections we focused on the real side, considering the implications of the optimal contract for firm-size dynamics. This section turns to the financial side and considers the financial

13. The condition that $L(s)$ is constant can be replaced by $\frac{1}{1+r}[\Pi(V, s) + \int L(s')F(ds', s)] - L(s)$ is non-decreasing in s. Details are available from the authors upon request.

structure and its dynamics. As indicated before, total firm debt can be decomposed in a long-run component $B_t = W_t - V_t$ and the short-term capital advancement k_t. Most of our discussion is centred on the long-run component.

4.5.1. Debt repayment schedule. Prior to reaching the efficient frontier, no dividends are distributed and all profits are paid to the lender. Since total profits increase with the value of the firm, for a fixed state s debt repayments increase over time. Moreover, if liquidation does not occur:[14]

$$B_t - EB_{t+1} = \Pi_t - \frac{r}{1+r} EB_{t+1},$$

and since Π_t increases and EB_{t+1} decreases over time, it follows that for a given s, debt falls in expectation of an increasing rate. In addition, as the equity of the firm grows, its debt maturity changes: short-term capital advancements increase while long-term debt decreases. These predictions are consistent with the findings in Barclay and Smith (1995). They show that, conditional on the revenue shock, firms with higher market-to-book ratio W/I_0 (or Tobin's Q, as previously defined) display a lower ratio of long-term debt to short-term debt B/k.[15] The intuition for this result in our setting is that conditional on s, a firm with higher market-to-book value of assets W/I_0, is also a firm with higher equity entitlement to the entrepreneur V, and hence weaker short-term borrowing constraints and lower long-term debt. In Barclay and Smith (1995) this result is explained by appealing to Myers (1977). In Myers (1977) the existence of risky fixed claims creates an underinvestment problem according to which stockholders may reject positive net present value investment options. This underinvestment problem is eliminated if these fixed claims mature before any opportunity to exercise the real investment options. Because the value of these options can be measured by the firm's market-to-book ratio (Smith and Watts, 1992), a higher market-to-book ratio should be associated with shorter-maturity debt.

A sharper characterization of the dynamics of long-term debt can be obtained for the deterministic case. Here we provide a generalization of the example in Section 2 by allowing for a general outside-value function $O(k_t)$. Let V_t be the continuation equity value for the firm at time t. The optimal contract implies that $V_{t+1} = \min((1+r)V_t, V^u)$.

Since $B_t = \frac{1}{1+r}[\Pi(V_t) + B_{t+1}]$ decreases over time, it follows that $\Pi(V_t) > rB_t$. In particular at time 0, $\Pi(V_0) > rB_0 = rI_0 \geq 0$, with strict inequality if the lender contributes to an initial investment $I_0 > 0$. Since V_t increases over time, $\Pi(V_t)$ will be positive every period, so the cash flow of the contract will be positive every period. Each period the borrower pays back the capital advanced with interest and contributes the quantity $\Pi(V_t) = R(k_t) - (1+r)k_t$ to the repayment of long-term debt. Once the efficient frontier V^u has been reached, the borrower keeps rV^u as dividends and pays the constant amount $R(k^*) - (1+r)k^* - rV^u$, where k^* is the profit maximizing amount of capital, as interest on the outstanding long-term loan.

In the above implementation of the optimal contract, the cash flow is controlled by the lender. An alternative implementation is possible when $R(k_t) - k_{t+1} \geq 0$. An initial loan $I_0 + k_0$ allows the entrepreneur to self-finance the capital accumulation without the need of short term advancements. At the end of every period t the borrower makes repayments $R(k_t) - k_{t+1}$, keeping the quantity k_{t+1} for production. Once the efficient frontier is reached, the borrower pays every period to the lender $R(k^*) - k^* - rV^u$.[16] Any other long-term debt contract is either not default-proof or leads to a lower initial debt level. This is obvious, because any other repayment schedule

14. When liquidation occurs $B_t = L(s_t)$.

15. Guedes and Opler (1996) and Stohs and Mauer (1996) also present similar evidence. In these papers however, debt maturity is measured as term-to-maturity of new debt issues or weighted average debt maturity of all debt, respectively. Both measures are in years, whereas our measure of long-term debt B is in dollars.

16. It is straightforward to check that this contract gives the same value at time 0 to both lender and borrower as the contract that combines long-term and short-term debt.

results in a lower V_0, and consequently a lower k_0. Thus, the optimal contract is the one that at each point maximizes the total debt that can be sustained.

4.5.2. State contingent debt. Let us now turn to a characterization of the state contingent nature of debt. The question we address in this section is whether the level of long-term debt is monotonic in the productivity shock. This would be the case, for instance, if the lending contract were implemented by an equity-share arrangement. At the efficient frontier, state contingent debt $\tilde{B}(s)$ satisfies the following equation:

$$\tilde{B}(s) = \frac{1}{1+r}\left[\pi(s) - d(s) + \int \tilde{B}(s)F(ds', s)\right].$$

The first term $b(s) = \pi(s) - d(s)$ corresponds to debt payments. Using standard dynamic programming arguments and first-order stochastic dominance of $F(s', s)$, it is easy to show that if $b(s)$ is non-decreasing in s, then $\tilde{B}(s)$ will be a non-decreasing function.

What happens before the efficient frontier is reached? On the one hand, more productive states can sustain more debt. On the other hand, it is advantageous to increase the equity stake in good states as well. Depending on what effect dominates $B(V(s), s)$ may be increasing or decreasing on s. The following proposition presents sufficient conditions that imply that state contingent debt is monotone even outside the efficient frontier.

Proposition 8. *Let $V(s')$ denote the optimal policy function starting from (V, s). Assume that $\Pi(V(s), s) - (1+r)V(s)$ is increasing in s whenever $\Pi_1(V(s), s)$ is constant. Suppose also that the contract displays positive persistence. If $s_2 > s_1$ then $B(V(s_2), s_2) > B(V(s_1), s_1)$.*

Proof. See Appendix D. ||

Let us return to debt payments at the efficient frontier and analyse the conditions under which $b(s)$ is non-decreasing in s. At the efficient frontier, debt payments are obviously increasing in the range where dividends are not distributed, *i.e.* $d(s) = 0$. This occurs in those states where $\tilde{V}(s) > V^u(s)$. In contrast, for states where $\tilde{V}(s) = V^u(s)$, debt payments are

$$b(s) = \pi(s) - (1+r)V^u(s) + \int \tilde{V}(s')F(ds', s).$$

The last term is non-decreasing, by the assumption of stochastic dominance. The remaining term $\pi(s) - (1+r)V^u(s)$, is determined by the revenue and outside-value function:

$$\frac{d(\pi(s) - (1+r)V^u(s))}{ds} = R_2(k(s), s) - (1+r)\left[O_2(k(s), s) - O_1(k(s), s)\frac{R_{12}(k(s), s)}{R_{11}(k(s), s)}\right]$$

If the outside-value function O is not very sensitive to changes in s and O_2 is small, or R_{12} is small, debt payments will be increasing. As an example, if $O(k, s) = \lambda k$ and $R(k, s) = sk^\alpha$, this derivative will be positive if and only if $1 - \alpha > \lambda\alpha$. This provides both an example where $\tilde{B}(s)$ will be increasing as well as a counter-example.[17] Furthermore, the condition that $1 - \alpha > \lambda\alpha$ also implies that $\Pi(V(s), s) - (1+r)V(s)$ is increasing in s whenever $\Pi_1(V(s), s)$ is constant, which, if there is positive persistence, means that $B(V(\cdot), \cdot)$ is increasing in s even outside the efficient frontier.

17. For an example of monotonicity take $1 - \alpha > \lambda\alpha$. For the converse, consider the case where the $1 - \alpha < \lambda\alpha$ and shocks are i.i.d. For another example, take $O = \lambda R$ and $R = sk^\alpha$, where the derivative is positive if and only if $1 - \alpha > (1+r)\lambda$.

4.5.3. A sequence of short-term contracts. It is easy to see that the optimal contingent debt and growth policy can also be implemented by a sequence of short-term contracts. This is accomplished by rolling over the debt from one lender to another while having all lenders coordinate on a short-term borrowing limit. Let $k(B, s)$ be the short-term borrowing limit implicit in the optimal contract and let $\pi(k(B, s), s) = \max_{k' \le k(B,s)}[R(k', s) - (1+r)k']$ be the maximal profit that can be achieved subject to the borrowing limit $k(B, s)$. Then the entrepreneur's optimal borrowing policy can be derived from the following dynamic programming problem:

$$V(B, s) = \max_{\tau, B(s')} \frac{1}{1+r}\left[\pi(k(B, s), s) - \tau + \int V(B(s'), s')F(ds', s)\right]$$

$$\text{subject to:} \quad \tau \le \pi(k(B, s), s)$$

$$\int B(s')F(ds', s) = B(1 + r) - \tau.$$

This is a standard dynamic investment problem for an entrepreneur facing an exogenously given borrowing constraint $k(B, s)$. This borrowing constraint represents the maximal short-term debt the entrepreneur can raise. It is important to notice that implicit in this borrowing limit is the fact that current lenders anticipate that future lenders will also determine short-term financing using this rule. This equilibrium obviously requires coordination on the lenders' side regarding the borrowing limit. This is clearly not the only equilibrium with short-term contracts: if lenders anticipate that the entrepreneur will not be able to raise any short-term capital in the future, all financing breaks down.

4.6. *Some comparative statics*

4.6.1. The role of collateral. One interesting comparative statics exercise regards the role of collateral in the design of the optimal long-term debt contract and the dynamics of the borrowing constraints. Consider two firms with the same initial investment I_0, one with a larger share of this investment being sunk. Otherwise the firms are identical. This can be modelled as a lower liquidation value for all revenue states, which means lower collateral. The firm with lower liquidation value has a larger set of survival states and thus, from equation (11), higher $\tilde{V}(s)$. It also has lower total firm value for given s. Thus, lower collateral (or greater sunk costs) leads to lower debt and leverage for given s.

Lower liquidation also reduces the total value W at any state. Since the initial long-term debt is the same, the starting equity entitlement V_0 must be smaller. Interpreting lower liquidation as lower collateral, the model implies that projects with higher sunk costs will face stronger initial borrowing constraints and will start smaller, taking a longer time to mature.

4.6.2. Firm size, better projects and better enforcement. A project with higher returns, *i.e.* higher $R(k, s)$ values or better enforcement, *i.e.* lower $O(k, s)$, will have a higher indirect profit function $\Pi(V, s)$. This implies that total value $W(V, s)$ will be higher. An immediate consequence is that the project can sustain a larger initial debt and size, exhibit higher survival probability, repayment of long-term debt is faster and borrowing constraints are eliminated sooner. Alternatively, a project can be more attractive if it faces a better initial distribution G or conditional distribution F for revenue shocks. In both cases, it is easy to show[18] that the initial value of the project is higher and consequently initial equity and total debt will increase. Consequently, firms will start larger and face a higher probability of survival. In addition to

18. The proof follows a similar argument as the one used for Proposition 7 and is available from the authors upon request.

these implications and as expected, a better enforcement technology allows firms to have greater leverage.

The prediction that larger firms (with higher $R(k, s)$) have more debt is supported by Titman and Wessels (1988) who present evidence that firms with greater sales also display higher debt to asset ratios. Also, interpreting a low $O(k, s)$ as high credit rating, the model predicts that firms with higher credit rating have more long-term debt. This is supported by evidence presented in Barclay and Smith (1995).

4.6.3. More risky projects. We do not have any general results. However, the following example suggests that a mean preserving increase in the spread of a project's returns can lead to tighter initial borrowing constraints.

Let $R(k, s) = s^{1-\alpha}k^{\alpha}$ for $0 < \alpha < 1$, $O(k, s) = \lambda k$, the liquidation value $L = 0$, and shocks be i.i.d. It follows that

$$\pi(s) = s(1-\alpha)\left(\frac{\alpha}{1+r}\right)^{\frac{\alpha}{1-\alpha}},$$

$$V^u(s) = \lambda s\left(\frac{\alpha}{1+r}\right)^{\frac{1}{1-\alpha}}.$$

Hence a mean preserving increase in spread of $\pi(s)$ results also in a mean preserving increase in spread of $V^u(s)$. Since $\tilde{V}(s) = \max(V^u(s), \frac{1}{1+r}\tilde{V})$ where $\tilde{V}$ is defined implicitly by

$$\tilde{V} = E\max\left(V^u(s), \frac{1}{1+r}\tilde{V}\right),$$

it follows immediately that $\tilde{V}(s)$ increases. Thus, once the project reaches maturity the entitlement to the lender must be smaller on average. Now suppose that under the initial distribution of returns, $\tilde{W}(s_0) - \tilde{V} = I$, where $\tilde{V}(s_0) = \tilde{V}$. This implies that the project starts unconstrained. Yet with a mean preserving increase in the spread of returns, $\tilde{V}$ increases so the project is initially constrained.

5. THE OPTIMAL DEBT CONTRACT AS A CONSTRAINED GROWTH PROBLEM

It is possible to rewrite our model in such a way that it does not explicitly model the accumulation of financial assets (or the repayment of debt). The focus is on constrained capital accumulation. This is helpful to stress the connections between our model and other models of firm growth with exogenous borrowing constraints. Redefine:

$$\Pi(k, s) = \max\{R(x, s) - (1+r)x \mid x \leq k\},$$

and assume that for every $s \in S$ and $V \geq 0$ there exists a unique k such that $O(k, s) = V$. Given s, this establishes a one–one map between equity values V and capital advancements k. This allows us to rewrite problem equation (8) as:

$$W(k, s) = \max\left\{L(s); \max_{k(s')}\frac{1}{1+r}\left[\Pi(k, s) + \int W(k(s'), s')F(ds', s)\right]\right\},$$

where the inner maximization is subject to

$$O(k, s) \geq \frac{1}{1+r}\int O(k(s'), s')F(ds', s). \tag{14}$$

This set-up has the structure of a capital accumulation problem with a constraint on the investment rate. An equilibrium contract chooses k_0 so that $W(k_0, s_0) - O(k_0, s_0) \geq I_0$. This constraint is equivalent to setting the initial long-term debt and equity levels $B(V_0, s_0) \geq I_0$ from before.

An interesting example satisfying our assumptions is the case $O(k, s) = k$, where equation (14) establishes that the expected growth of capital cannot exceed the interest rate. Another special case results when $O(k, s) = R(k, s)$, where the entrepreneur can steal the revenues of the firm. In this case equation (14) represents an upper bound on the growth rate of revenues.

We can use the deterministic case to show that our model specializes to Spence (1979). Interpret k_t as the firm's capital stock and assume there is no depreciation. If $O(k) = R(k)$, then the entrepreneur can keep the revenue, but not the capital. Every period the firm makes an investment decision. Following the notation in Spence, let m_t be the investment at time t, so $k_t = m_t + k_{t-1}$. Specialize to the case in which R is homogeneous, *i.e.* $R(k) = k^{\alpha}$. Then, $k_t = (1+r)^{1/\alpha} k_{t-1}$ and $m_t = ((1+r)^{1/\alpha} - 1)k_{t-1}$, which is the constraint between investment and the firm's capital used in Spence. This coefficient, which in Spence's analysis is taken as an exogenous parameter is not independent of the interest rate.

6. FINAL REMARKS

In this paper we have developed a general model of endogenous borrowing constraints based on the assumptions of limited enforcement and limited liability. In our model, borrowing constraints arise as part of the optimal debt contract. Our model extends previous theories of borrowing and lending, such as Thomas and Worral (1994) and Hart and Moore (1994), allowing for a general specification for firm's shocks, capital accumulation and the possibility of exit. The model has implications for firm growth and survival, implying that younger firms tend to grow faster and have lower survival rates. Both properties are consistent with empirical regularities. The model also suggests that the capital structure is an important determinant of the firm's investment and exit decisions. Firms with higher long-term debt have higher market-to-book asset ratios, greater revenues, better credit ratings and collateral.

We have kept our analysis at a fairly abstract level in order to describe the general properties of a class of models of borrowing and lending based on the idea of limited commitment and limited liability. We certainly believe that our structure is very flexible and could be used to develop more specialized models. For example, though we have identified the project as a new firm, there is no reason why to do so. An established firm may look for a long-term loan to finance a new investment.

Our modelling approach highlights an important dynamic consistency feature of borrowing constraints: equity restricts current access to capital, but is itself limited by future access to credit. Of all dynamically consistent lending plans, the optimal contract is the one that maximizes access to short-term capital and the expected rate of decrease of long-term debt. But there are others, which can be suboptimal and thus lead to tighter borrowing constraints than are justified by the environment. This suggests the importance of our modelling approach. Models of exogenous borrowing constraints always leave open the question of whether there are *better* contracts that could imply weaker constraints.

Some models of exogenous borrowing constraints assume that borrowing is limited by a firm's assets. Our model suggests that the relevant variable is not assets but equity. Kiyotaki and Moore (1997), study the macroeconomic implications of limited borrowing in a model where firms' collateral is subject to endogenous fluctuations. In the presence of sunk costs of investment or when some assets—such as human capital—are not fully appropriable, loans cannot be fully collateralized with assets. In such cases, more lending can be sustained—as in our model—with the threat of depriving the borrower from its equity.

There are several interesting extensions of the theory. One of them is to explore the general equilibrium implications of the type of borrowing constraints considered. In this paper we treat the value of default as exogenous, with fairly general properties to accommodate most existing models. In an equilibrium framework, the value of default will not be exogenous and should in turn be influenced by the optimal contract. There are many alternative ways of *closing* a model of the sort developed here, some of which have been already explored in the literature (see, for instance, Ghosh and Ray, 1996, 2001). As an example, consider a situation where, upon defaulting, borrowers can enter into new long-term contracts with other lenders, at some additional cost. An equilibrium contract would require taking into account this endogeneity. This is obviously a very interesting area of research. Though our paper does not address this equilibrium problem, the general results obtained here could still prove very useful in developing that research programme.

There are obviously alternative ways of modelling borrowing constraints which are worth exploring. In particular, informational asymmetries, absent in this model, are an alternative source of frictions to generate credit constraints. Diamond (1989, 1991*a*,*b*) considers the relationship between the unobservable quality of the borrower's project and the maturity/seniority structure of debt. Green (1987) analyses debt contracts in a repeated environment when agents have unobserved endowments. Atkeson (1991) and Marcet and Marimon (1992) study the effect of moral hazard on growth in the context of international lending. Developing models of firm growth and survival based on such foundations is also an important direction of research in this area. Some recent work include DeMarzo and Fishman (2001) and Clementi and Hopenhayn (2002).

APPENDIX A. GENERALIZING THE OUTSIDE OPPORTUNITY FUNCTION

This appendix shows how to modify the model to allow for $O(0, s) > 0$. Since $O(0, s) > 0$ is the inalienable component of the entrepreneur's share, feasibility of future equity values must be adjusted so that $V(s') \geq O(0, s')$. This means that upon liquidation $V = O(0, s)$, and the lender's value is $B(V, s) = L(s) - O(0, s)$. It also means that in designing the efficient frontier the lender takes into account the payments made to the firm in the liquidation states. That is, the new efficient frontier is given by

$$\tilde{V}(s) = \begin{cases} \max\left(V^u(s), \frac{1}{1+r} \int \tilde{V}(s')F(ds', s)\right), & s \in \tilde{S} \\ O(0, s), & \text{else.} \end{cases}$$

All proofs remain unchanged, except for the construction of the efficient frontier.

APPENDIX B. PROPERTIES OF THE INDIRECT PROFIT FUNCTION

Lemma 2 gives conditions under which the indirect profit function $\Pi(V, s)$ displays the properties listed in the text in Assumption 4.

Lemma 2. (i) *Π is continuous, uniformly bounded, strictly increasing in V for $V < V^u(s)$.*
(ii) *If R and O are continuously differentiable and $O_1(k, s) > 0$, then $\Pi_1(V, s) = \frac{R_1(k,s)-(1+r)}{O_1(k,s)}$.*
(iii) *If R and O are twice continuously differentiable and*

$$R_{11}(k, s)O_1(k, s) - O_{11}(k, s)(R_1(k, s) - (1 + r)) \leq 0(< 0)$$

for $k < K(s)$, then $\Pi_{11}(V, s) \leq 0(< 0)$ for $V < V^u(s)$.
(iv) *If R and O are twice continuously differentiable and*

$$R_{12}(k, s)O_1(k, s) - O_{12}(k, s)(R_1(k, s) - (1 + r)) \geq 0(> O_{12}(k, s))$$

for all $k < K(s)$, then $\Pi_{21}(V, s) \geq 0(> 0)$ for $V < V^u(s)$.

(v) *If R and O are continuously differentiable, $O_1(k,s) > 0$ and*

$$R_2(k,s) > \frac{R_1(k,s) - (1+r)}{O_1(k,s)} O_2(k,s),$$

then Π is increasing in s.

Proof. (i) is an immediate consequence of the maximum theorem and the properties given for R and O in Assumptions 1 and 3, and; (ii) is a direct application of the implicit function theorem; (iii) and (iv) follow from differentiating the expression given in part (ii). Part (v) follows from differentiating the profit function. ||

APPENDIX C. CONCAVITY AND MONOTONICITY OF THE VALUE FUNCTION

The next lemmas discuss concavity and strict monotonicity of the surplus function W. The possibility of exit introduces a potential non-concavity in the total surplus function $W(\cdot, s)$. The first lemma gives conditions under which the function W is concave in V without using randomizations.

Lemma 3. *Suppose $R(k,s) - (1+r)k > L(s)$ is satisfied and $\Pi(V,s)$ is concave in V for each s. Then the total surplus function $W(V,s)$ will be concave in V for all s. Furthermore, if $\Pi(V,s)$ is strictly concave for $V < V^u(s)$, then $W(\cdot, s)$ is strictly concave in V whenever $V < \tilde{V}(s)$.*

Proof. Concavity of $W(\cdot, s)$ follows from the concavity of $\Pi(V,s)$ by Theorem 9.8 in Stokey, Lucas and Prescott (1989). Now suppose that $V < \tilde{V}(s)$. Then there exists an n such that $V < V^n(s)$. We will show by induction on n that this implies that $W(\cdot, s)$ is strictly concave in a neighbourhood of V. For $n = 1$, $V < V^u(s)$. Since in this region the return function Π is strictly concave, then the value function $W(\cdot, s)$ will also be strictly concave. Suppose now that the result is true for all s and $V < V^{n-1}(s)$ and that $V^u(s) \le V < V^n(s)$. Letting $V(s')$ be the optimal continuation values, then $V(s') < V^{n-1}(s')$ on a subset of S with positive measure given s. Now, strict concavity follows by the induction hypothesis. ||

Assuming that $R(k,s) - (1+r)k > L(s)$ for all (k,s) eliminates the possibility of exit, making this assumption too strong for our purposes. An alternative is to allow for randomizations. In particular, randomizations on the exit decision for some values of (V,s) are to be expected. The second lemma shows that these randomizations are also sufficient.

Suppose that at the beginning of the period, after observing a shock s and when the initial value is V, a randomization is used and define the new value function W^r by

$$W^r(V,s) = \max\nolimits_{\lambda \in [0,1], V_2 \ge 0, V_1 \ge 0} \lambda W(V_2, s) + (1-\lambda) L(s)$$
$$\text{subject to } \lambda V_2 + (1-\lambda) V_1 = V,$$

where the function W is defined as in equation (8), but replacing W^r for W under the integral sign.

Lemma 4. *The value function $W^r(\cdot, s)$ is concave for all $s \in S$.*

Proof. The proof follows the lines of the one given for Lemma 3. ||

Finally, it is an immediate consequence of concavity and Proposition 1 that the value function is strictly monotone.

Lemma 5. *Suppose $W(V,s)$ is concave in V for all s. Then if $W(V_1,s) > L(s)$ and $V_1 < \tilde{V}(s)$ it follows that $W(V_2,s) < W(V_1,s)$ for all $V_2 < V_1$.*

Proof. If $W(V_2,s) = W(V_1,s)$ for some $V_2 < V_1 < \tilde{V}(s)$, then by concavity of the value function it follows that $W(V_2,s) = W(\tilde{V}(s),s)$, contradicting Lemma 1. ||

APPENDIX D. OTHER PROOFS

Proof of Lemma 1. The weak monotonicity of $W(\cdot, s)$ follows immediately from the monotonicity of $\Pi(\cdot, s)$ applying standard dynamic programming arguments. We now establish 2. First notice that for $s \notin \tilde{S}$, $W(V,s) = L(s)$, for any V. Now, if $V = \tilde{V}(s)$, then it is possible to choose as continuation values $V(s') = \tilde{V}(s')$. We will show that the dynamic programming equation given by problem equation (8) preserves the property defined by 2. Suppose then that $W(V, s') = \tilde{W}(s')$ for $V \geq \tilde{V}(s')$. Then, letting $V(s') = \tilde{V}(s')$, it follows immediately from equation (8) that

$$\begin{aligned} W(V, s) &= \frac{1}{1+r}\left[\pi(s) + \int W(\tilde{V}(s'), s')F(ds', s)\right] \\ &= \frac{1}{1+r}\left[\pi(s) + \int \tilde{W}(s')F(ds', s)\right] = \tilde{W}(s). \end{aligned}$$

Since the set of functions $W(\cdot)$ such that $W(V, s) = \tilde{W}(s)$ for $s \in \tilde{S}$ is closed in the norm topology, the unique solution to the dynamic programming equation (8) also satisfies this property.

We now establish 3. Note that $\tilde{W}(s) > L(s)$, as $s \in \tilde{S}$. Assume that $W(V, s) > L(s)$, otherwise the result is trivial. If $V < \tilde{V}(s)$, then there exists some n such that $V < V^n(s)$. We now prove by induction that this implies $W(V, s) < \tilde{W}(s)$. The result is immediate for $n = 1$, for in this case $V < V^u(s)$ and letting $V(s')$ be the optimal continuation values starting from (V, s),

$$\begin{aligned} W(V, s) &< \frac{1}{1+r}\left[\pi(s) + \int W(V(s'), s')F(ds', s)\right] \\ &\leq \frac{1}{1+r}\left[\pi(s) + \int \tilde{W}(s')F(ds', s)\right] = \tilde{W}(s). \end{aligned}$$

To continue with the induction argument, suppose that $V < V^{n-1}(s)$ implies that $W(V, s) < \tilde{W}(s)$ for all s. Then if $V < V^n(s)$, either $V < V^u(s)$ or $V < \frac{1}{1+r}\int V^{n-1}(s')F(ds', s)$. In the first case, using the same argument as for the case $n = 1$, it follows immediately that $W(V, s) < \tilde{W}(s)$. In the latter case, $V(s') < V^{n-1}(s')$ for a subset of $\tilde{S}$ with positive measure. Using the induction hypothesis, it follows immediately that $W(V, s) < \frac{1}{1+r}[\pi(s) + \int \tilde{W}(s')F(ds', s)] = \tilde{W}(s)$. ‖

Proof of Proposition 5. We first prove that if W is strictly concave in V for $V < \tilde{V}(s)$ and $W_{12} \geq 0$, then any optimal continuation policy $V(s')$ must be non-decreasing. Consider the problem

$$\begin{aligned} g(V, s) &= \max \int W(V(s'), s')F(ds', s) \\ &\text{subject to } \frac{1}{1+r}\int V(s')F(ds', s) \leq V. \end{aligned} \tag{D.1}$$

Let $s_2 > s_1$. We now show that for any pair (V, s) the solution to equation (D.1) is $V(s_2) \geq V(s_1)$. Since W is concave in V, it is almost everywhere differentiable. The first-order conditions of problem equation (D.1) imply that $W_1(V(s_2), s_2) = W_1(V(s_1), s_1)$ at all points where the solution is interior. Otherwise, by strict concavity and because $W_{12} \geq 0$, it follows immediately that

$$\begin{aligned} W_1(V(s_1), s_1) &< W_1(V(s_2), s_1) \\ &\leq W_1(V(s_2), s_2). \end{aligned}$$

Since $V(s')$ is an increasing function, $\int V(s')F(ds', s_2) \geq \int V(s')F(ds', s_1)$. Letting $V^i(s')$ obtained from initial values (V, s_i), it follows that $V^2(s') \leq V^1(s')$ for some set of positive measure. But then, by the strict concavity of W, this inequality must hold for all s'.

We now prove that if $W(V, s)$ is strictly concave in V for $V < \tilde{V}(s)$ and $\Pi_{12}(V, s) \geq 0$ and concave, then $W_{12}(V, s) \geq 0$. We will show that the Bellman equation maps the set of functions with positive cross partial derivatives into itself. So suppose that we start with a function $W_{12} \geq 0$ and, as assumed, strictly concave for $V < \tilde{V}(s')$. It follows immediately that the function g will be strictly concave in V, for all $V < \tilde{V}(s)$. Using the envelope theorem, it follows that $g_1(V, s_i) = \frac{1}{1+r}W_1(V^i(s'), s')$, where V^i is the optimal solution starting from s_i. Let $s_2 > s_1$ and assume that $V < \tilde{V}(s_1)$. By the result above it follows that $V^1(s') \geq V^2(s')$, so $g_1(V, s_1) \leq g_1(V, s_2)$, and thus $g_{12} \geq 0$. The function W satisfies the following Bellman equation

$$W(V, s) = \max\left\{L(s), \frac{1}{1+r}[\Pi(V, s) + g(V, s)]\right\}$$

where function g is understood to depend on W as defined in equation (D.1). Since $\Pi_{12} \geq 0$ and $g_{12} \geq 0$ as was just proved, it follows that the function $W_{12} \geq 0$ as well. ‖

Proof of Proposition 6. We prove this proposition in a sequence of claims.

Claim 1. *Suppose that* Π *and* F *satisfy Assumption 7. Then*

$$W_1(V(s), s) \text{ constant in } s \Longrightarrow \Pi_1\left(\frac{1}{1+r}\int V(s)F(ds, s_0), s_0\right) \text{ is non-decreasing in } s_0.$$

Corollary 1. *In the optimal contract,* $\Pi_1(V(s), s)$ *is weakly increasing.*

Proof. Suppose, by way of contradiction, that $s_1 > s_0$ and $\Pi_1(V(s_1), s_1) < \Pi_1(V(s_0), s_0)$. Letting c_i be the continuation derivative starting from s_i and since

$$\Pi_1(V(s_1), s_1) + c_1 = \Pi_1(V(s_0), s_0) + c_0,$$

it follows that $c_1 > c_0$. Let $V_1(s)$ denote the continuation values starting at s_1 and $V_0(s)$ the continuation values starting from s_0. By concavity of W, it follows that $V_1(s) < V_0(s)$. Hence

$$\begin{aligned}\Pi_1(V(s_1), s_1) &= \Pi_1\left(\frac{1}{1+r}\int V_1(s)F(ds, s_1), s_1\right)\\ &> \Pi_1\left(\frac{1}{1+r}\int V_0(s)F(ds, s_1), s_1\right).\end{aligned} \tag{D.2}$$

Finally, since $W_1(V_0(s), s)$ is constant in s, by Claim 1 it follows that

$$\begin{aligned}\Pi_1\left(\frac{1}{1+r}\int V_0(s)F(ds, s_1), s_1\right) &\geq \Pi_1\left(\frac{1}{1+r}\int V_0(s)F(ds, s_0), s_0\right)\\ &= \Pi_1(V(s_0), s_0).\end{aligned} \tag{D.3}$$

Combining equations (D.2) and (D.3), it follows that $\Pi_1(V(s_1), s_1) > \Pi_1(V(s_0), s_0)$, thus generating a contradiction. ||

Now we prove Claim 1.

Proof of Claim 1. The proof follows the standard induction in dynamic programming. Suppose that we start with a function W that satisfies this property. Let TW represent the new value function. Then we need to show that TW satisfies this property. Take $V(s)$ such that $TW_1(V(s), s)$ is constant in s. Consider $s_1 > s_0$. Then $TW_1(V(s_1), s_1) = TW_1(V(s_0), s_0)$. Using the argument in the proof of Corollary 1, it follows that $\Pi_1(V(s), s)$ is non-decreasing in s, so using Assumption 7

$$\Pi_1\left(\frac{1}{1+r}\int V(s)F(ds, s_1), s_1\right) \geq \Pi_1\left(\frac{1}{1+r}\int V(s)F(ds, s_0), s_0\right) \qquad \|$$

Claim 2. *Take* (V_1, s_1) *and* (V_0, s_0) *such that* $s_1 > s_0$*, and* $W_1(V_1, s_1) \leq W_1(V_0, s_0)$*. Let* $V_1(s)$ *denote the optimal continuation policy from* (V_1, s_1) *and* $V_0(s)$ *the optimal continuation from* (V_0, s_0)*. Then* $V_1(s) \geq V_0(s)$ *and* $W_1(V_1(s), s) \leq W_1(V_0(s), s)$*.*

Proof. To prove this statement, let $\tilde{V}_1$ satisfy $W_1(\tilde{V}_1, s_1) = W_1(V_0, s_0)$. By concavity of W in V it follows that $\tilde{V}_1 \leq V_1$. Let $\tilde{V}_1(s)$ denote the optimal continuation starting from $(\tilde{V}_1, s_1)$. From the monotonicity of the policy function, it follows that $\tilde{V}_1(s) \leq V_1(s)$. From Claim 1 it follows that $\Pi_1(\tilde{V}_1, s_1) \geq \Pi_1(V_0, s_0)$ and consequently $W_1(\tilde{V}_1(s), s) \leq W_1(V_0(s), s)$. Using the concavity of W in V it follows that $\tilde{V}_1(s) \geq V_0(s)$. Combining the inequalities, it follows that $V_1(s) \geq V_0(s)$ and thus $W_1(V_1(s), s) \leq W_1(V_0(s), s)$. ||

Combining these results we can now prove Proposition 6. Consider two histories $(s_0, s_1, \ldots, s_{T-1}, s_T)$ and $(s_0, \tilde{s}_1, \ldots, \tilde{s}_{T-1}, s_T)$ where $s_t \geq \tilde{s}_t$ with strict inequality for $t = 1$. Let c_t and $\tilde{c}_t$ denote the corresponding derivatives of the value function at t. Obviously $c_0 = \tilde{c}_0$. From Claim 1 it follows that $c_1 \leq \tilde{c}_1$. Applying inductively the claim, it follows that $c_t \leq \tilde{c}_t$ for all $t \geq 1$. Letting $V_T(s)$ and $\tilde{V}_T(s)$ denote the corresponding policies at T following the respective histories, from the concavity of W in V it follows that $V_T(s) \geq \tilde{V}_T(s)$. ||

Proof of Example 2. Since $\Pi_1(V(s), s)$ is non-decreasing in s, it follows that

$$V'(s) \leq \frac{V(s)}{(1-\alpha)s}. \tag{D.4}$$

Now consider

$$\frac{d}{ds_0}\Pi_1\left(\frac{1}{1+r}\int V(s)F(ds,s_0),s_0\right)=\frac{d}{ds_0}\alpha s_0\left(\frac{1}{1+r}\int V(s)F(ds,s_0)\right)^{\alpha-1}$$
$$\stackrel{s}{=}\int V(s)F(ds,s_0)-(1-\alpha)s_0\frac{\partial}{\partial s_0}\int V(s)F(ds,s_0).$$

Integrating by parts the second integral and rearranging we get:

$$\int V(s)F(ds,s_0)-(1-\alpha)s_0\frac{\partial}{\partial s_0}\left(V(\underline{s})+\int V'(s)(1-F(s,s_0))ds\right)$$
$$=\int V(s)F(ds,s_0)+(1-\alpha)s_0\int V'(s)F_2(s,s_0)ds$$
$$\geq\int V(s)F(ds,s_0)+s_0\int V(s)\frac{F_2(s,s_0)}{s}ds$$
$$=\int V(s)\left(\frac{F_1(s,s_0)s+F_2(s,s_0)s_0}{s}\right)ds.$$

The inequality follows from equation (D.4) and $F_2(s,s_0)<0$ (*i.e.* $F(s',\cdot)$ is non-increasing). ||

Proof of Proposition 7. In Lemma 5 we show that W is weakly increasing in V. Here it is shown that $W(V,\cdot)$ is also weakly increasing in s.

Assume first that we start with a weakly increasing function W on the R.H.S. of the Bellman equation. Suppose $s_2\geq s_1$ and let $V^i(s)$ be the optimal continuation values from (V,s_i). Let μ_i, $i=1,2$ denote the probability distribution corresponding to $F(\cdot,s_i)$. By Assumption 5, μ_2 stochastically dominates μ_1, so there exists a transition function $P(s,s')$ with support on the set $\{(s,s')\in S\times S: s'\geq s\}$ such that $\mu_2=P\mu_1$. Let ν_1 be a probability measure on $S\times\Re_+$ with support on the graph of this function $V^1(\cdot)$ such that $\nu_1(ds,V_1(ds))=\mu_1(ds)$. It follows by construction that $\int W(V_1(s),s)F(ds,s_1)=\int W(V,s)\nu_1(dV,ds)$. Define a measure ν_2 on $S\times\Re_+$ by lifting ν_1 with P, *i.e.* for any rectangle set $A\times B$ let $\nu_2(A\times B)=\int P(s,A)\nu_1(ds,B)$. It is easy to verify that since ν_1 has first marginal μ_1 and $\mu_2=P\mu_1$, then ν_2 will have first marginal μ_2. By the Radon–Nikodym theorem, there exists a transition function $Q:S\times\Re_+\to[0,1]$ such that $\nu_2=Q\mu_2$. $Q(s,\cdot)$ is a probability measure on $\Re_+$ for each s, which may be interpreted as a randomized strategy on continuation values V, given that state s. Since W is concave in V, the optimal continuation policy $V^2(s)$, given (s_2,V) must give a pay-off no lower than the one obtained from this randomized strategy, *i.e.*

$$\int W(V^2(s),s)F(ds,s_2)\geq\int W(V,s)Q(s,V)F(ds,s_1)$$
$$=\int W(V,s)\nu_2(dV,ds).\tag{D.5}$$

Finally, comparing ν_2 and ν_1,

$$\int W(V,s)\nu_2(dV,ds)=\int W(V,s)P(s',s)\nu_1(dV,ds')$$
$$\geq\int W(V,s')\nu_1(dV,ds'),\tag{D.6}$$

where the last inequality follows from the fact that $P(s',\cdot)$ has support on values $s\geq s'$ and the induction assumption that W is weakly increasing in s. Combining equations (D.5) and (D.6), it follows that the Bellman equation maps the set of non-decreasing functions into itself, and thus W is non-decreasing.

Now let $e(V)=\sup\{s\mid\frac{1}{1+r}[\Pi(V,s)+g(V,s)]<L\}$, where function g is as defined in the proof of Lemma 5 above, and if this set is empty set it equal to $\inf S$. It is immediate to check that this function is weakly decreasing and gives the exit thresholds. ||

Proof of Proposition 8. Let $s_2>s_1$. Define $V(s')$ as the optimal policy function starting from (V,s), and $V_i(s')$ as the optimal policy starting from $(V(s_i),s_i)$, for $i=1,2$. We want to show that $B(V(s_2),s_2)>B(V(s_1),s_1)$. Using the first-order condition for $V(s')$:

$$B_1(V(s_2),s_2)=\frac{1}{1+r}\Pi_1(V(s_2),s_2)+B_1(V_2(s'),s')$$
$$=\frac{1}{1+r}\Pi_1(V(s_1),s_1)+B_1(V_1(s'),s')$$
$$=B_1(V(s_1),s_1).$$

Positive persistence and concavity imply that $B_1(V_2(s'), s') < B_1(V_1(s'), s')$. Thus, $\Pi_1(V(s_2), s_2) > \Pi_1(V(s_1), s_1)$. By concavity of $\Pi(\cdot, s)$ there exists $\bar{V}_2 > V(s_2)$ such that $\Pi_1(\bar{V}_2, s_2) = \Pi_1(V(s_1), s_1)$. Positive persistence also implies that

$$\int W(V_2(s'), s')F(ds', s_2) > \int W(V_1(s'), s')F(ds', s_1).$$

Hence, $B(V(s), s)$ is increasing in s if

$$\Pi(V(s_2), s_2) - (1+r)V(s_2) \geq \Pi(V(s_1), s_1) - (1+r)V(s_1).$$

The assumption on the proposition implies that

$$\begin{aligned}\Pi(V(s_1), s_1) - (1+r)V(s_1) &\leq \Pi(\bar{V}_2, s_2) - (1+r)\bar{V}_2 \\ &\leq \Pi(V(s_2), s_2) - (1+r)V(s_2),\end{aligned}$$

where the second inequality follows from $\Pi(V, s) - (1+r)V$ being decreasing in V, for fixed s. ||

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