Christiano
Advanced Macro
416, Fall, 2014
Homework 2.

Consider the one-sector stochastic neoclassical growth model discussed in class.

1. Compute the coefficients in the linear, second order perturbation on the policy rule, for several parameterizations of the model. Do the approximation in terms of the capital stock itself, rather than in terms of the log of capital, as in the handout. In all cases,

$$\alpha = 0.36,\ \beta = 0.99,\ \delta = 0.02,\ \rho = 0.95.$$

Solve the model for: $\gamma = 1,\ 5,\ 20$ and for $Var(\varepsilon_t) = 0.01^2$. Also do the case, $\gamma = 5$, $Var(\varepsilon_t) = 1$. Report the coefficients of the approximate policy rules and store them for analysis below. I would like you to do the calculations in a way that is slightly different that what was done in class. First, do the approximations in terms of the actual capital stock, instead of its log. Second, let the state at any point in time be

$$k, a, \varepsilon,$$

where the current period technology shock is $\rho a + \varepsilon$ and a denotes the value of technology in the previous period:

$$c + k' = \exp(\rho a + \varepsilon)\, k^{\alpha} + (1 - \delta)\, k.$$

The state of technology in the next period is $\rho^2 a + \rho\varepsilon + \varepsilon'$. Thus, you are to find the second order approximation to the following policy rule:

$$k' = g(k, a, \varepsilon, \sigma).$$

My guess is that it is not practical to do the computations by hand with a calculator. A practical way to do the calculations is to do them on a computer, in MATLAB, using symbolic algebra. I have attached a handout to the homework that explains how to do that. Note that the sign of $g_{\sigma\sigma}$ switches for $\gamma = 1$ and $\gamma = 5, 20$. Provide a simple economic intuition for this.

2. Consider flow utility, u, in state k, a, ε, for given σ, $u(k, a, \varepsilon, \sigma)$. Consider the first and second order approximation of this object around steady state, $k = k^*, a = \varepsilon = \sigma = 0$. Consider the effect on flow utility of $Var(\varepsilon)$ when the state variables take on their non-stochastic steady state values:

$$u(k^*, 0, 0, 1) - u(k^*, 0, 0, 0).$$

Show that the above expression is zero in the first order approximation. Show that it is non-zero in the second order approximation. How does the sign of this effect change between $\gamma = 1$ and $\gamma = 5, 20$? Provide intuition.

3. One way to quantify the difference between first and second order perturbations, is to investigate their implications for the speed of adjustment. Although shocks potentially play an interesting role in this, let us abstract from the shocks at this point. Thus, the policy rule is:

$$k_{t+1} - k = g_k \times (k_t - k) + \frac{1}{2} g_{kk} \times (k_t - k)^2,$$

where k is the value of the capital stock in non-stochastic steady state (it is not the log of the capital stock, as in lecture). In percent terms:

$$\hat{k}_t = \frac{k_t - k^s}{k^s}.$$

Dividing the second order policy rule by k^s :

$$\hat{k}_{t+1} = g_k \hat{k}_t + \frac{1}{2} g_{kk} \times k \times \left(\hat{k}_t\right)^2. \tag{1}$$

Suppose that at $t = 1$, $k_1 = \lambda \times k$, so that $\hat{k}_1 = \lambda - 1$. Consider first the standard, first order perturbation, in which g_{kk} is set to zero. Then, $\hat{k}_{t+1} = g_k^t (\lambda - 1)$, $t = 0, 1, \ldots$. The time required to close 95 percent of the gap between the capital stock and steady state is the value of t such that

$$\hat{k}_{t+1} = (\lambda - 1) \times 0.05,$$

or,

$$\begin{aligned} g_k^t &= 0.05 \\ t &= \frac{\log(0.05)}{\log g_k}. \end{aligned}$$

Notice that t is independent of the value of λ. Compute the value of t for our model economy.

4. Now, consider the implications of the second-order expansion for speed of adjustment. Set $\hat{k}_1 = \lambda - 1$, and simulate (1) for $t = 1, ..., 100$. Set λ to a very small number, $\lambda = 0.0001$ and put in the large value of γ. Graph the two trajectories, one for the first order approximation and the other for the second order approximation. Are they very similar?

5. (Simulating a rule obtained by second-order perturbation.) Consider the following expression, which we can think of as the second order approximation of some unspecified law of motion:

$$y_t = f\left(y_{t-1}, \varepsilon_t\right) = \rho y_{t-1} + \alpha y_{t-1}^2 + \varepsilon_t,\ 0 < \rho < 1,$$

where ε_t is a mean zero, *iid* process, uncorrelated with past y_t. Note that there are two solutions, $y = f\left(y, 0\right) : y = 0, y = (1-\rho)/\alpha$. The first steady state is locally stable, since the root in its first order expansion is ρ. The second is not. So, if $y_t > (1-\rho)/\alpha$, then y_t is likely to explode. Verify this by simulating 100 observations of y_t with $\varepsilon_t \tilde{} N\left(0, \sigma_\varepsilon\right)$ and $\sigma_\varepsilon = 0.01,\ 0.10$. The problem illustrated here is something that can happen with policy rules that are quadratic expansions: "...they will have extra steady states not present in the original model, and some of these steady states are likely to mark transitions to unstable behavior".[1]

Kim, Kim, Schaumburg and Sims (2005, section 7) argue that the system should be simulated by 'pruning', as follows. Let the first order system be $y_t^{(1)}$:

$$y_t^{(1)} = \rho y_{t-1}^{(1)} + \varepsilon_t.$$

The pruned simulated output is $y_t^{(2)}$, where $y_t^{(2)}$ solves the original system with the exception that wherever y_{t-1} appears in quadratic terms, it is replaced by $y_t^{(1)}$:

$$y_t^{(2)} = \rho y_{t-1}^{(2)} + \alpha \left[y_{t-1}^{(1)}\right]^2 + \varepsilon_t.$$

[1] Taken from page 17, Kim, Jinill, Sunghyun Kim, Ernst Schaumburg, and Christopher Sims, February 3, 2005, 'Calculating and Using Second Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models,' manuscript.

Construct two graphs for the case, $\sigma_\varepsilon = .01$. Display y_t and $y_t^{(2)}$ in one graph. Display $y_t^{(1)}$ and $y_t^{(2)}$ in the other. Note how pruning eliminates the explosive behavior. See Kim, Kim, Schaumburg and Sims for additional discussion.

6. Return to the policy rule in question 1 with $\gamma = 20$ and $Var(\varepsilon_t) = 1$. Simulate 10,000 observations using the linear approximate rule, the quadratic rule (e.g., 'naive simulation'), and the quadratic rule based on pruning (all three should be simulated in response to the same shocks). Display the three sequences of capital stocks in a graph (truncate super large values if you have to). How do mean value of k_t and its variance compare for the three rules? How does the mean value of k_t compare with its non-stochastic steady state value? Repeat this exercise using the more reasonable estimate of the variance, $V_\varepsilon = 0.01^2$. Finally, consider the same experiment with the more 'normal' risk aversion: $\gamma = 2$ (set $V_\varepsilon = 0.01^2$). Note that in the latter case, there is very little difference between the three solutions.