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416, Fall 2014
Homework 1, due Wednesday, October 15.

1. Consider an economy in which household preferences have the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \ \beta = .99, \ \text{and } u(c_t, n_t) = \log(c_t) + \psi \log(1 - n_t).$$

Here, the time endowment has been assumed to be unity. The household budget constraint is:

$$c_t + k_{t+1} - (1 - \delta)k_t = r_t k_t + w_t n_t + \pi_t, \ \delta = .028,$$

where r_t, w_t, π_t denote the rental rate on capital, the wage rate, and profits, respectively. Firms operate the following technology:

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha}, \ \alpha = 0.36.$$

Here,

$$A_t = Y_t^{\gamma},$$

where Y_t denotes the economy-wide level of output, in per capita terms. Firms are competitive, and maximize profits, which are zero in equilibrium. Note that in the special case, $\gamma = 0$, this is the standard neoclassical growth model with endogenous hours worked.

   (a) Derive the household labor and capital Euler equations. Derive the firm Euler equations.

   (b) In the household Euler equations, substitute out the rental rate on capital and the wage rate for labor, using the firm first order conditions (hint: the firm treats A_t as exogenous). You should have a static and dynamic Euler equation. You should focus on symmetric equilibria, in which the economy-wide average stock of capital and level of employment coincide with the firm's stock of capital and employment. After making this identification, the static and dynamic Euler equations should have the following arguments:

$$\begin{aligned} \text{`static'} &: \quad v_h(n_t, k_t, k_{t+1}) = 0 \\ \text{`dynamic'} &: \quad v_k(k_t, k_{t+1}, k_{t+2}, n_t, n_{t+1}) = 0. \end{aligned}$$

Log-linearize these equations about the steady state values of employment and the firm's stock of capital.[1] In computing the steady states, fix steady state employment at 1/3 and choose the value of ψ that rationalizes that. Initially, fix $\gamma = 0.01$.

Hints on the computation of the steady state: I don't think it is possible to compute the steady state in closed form for $\gamma > 0$. A one-dimensional zero-finding algorithm must be used to find the steady state. The algorithm requires a good initial guess. You get a reasonable guess with the steady state associated with $\gamma = 0$, which you can compute in closed form.[2] To compute the steady state for $\gamma > 0$, I suggest you proceed as follows. Set the value of k to the steady state value implied by $\gamma = 0$. Then, compute employment (in closed form) using the steady state intertemporal Euler equation. Then, compute steady state consumption using the steady state resource constraint. Next, evaluate the intratemporal Euler equation for labor. Adjust k until the labor equation is satisfied as a strict equality. The MATLAB routine, fzero, works well for this. You supply it with a function that takes k as input and evaluates v_h above. The routine will quickly adjust k until $v_h = 0$.

(c) Substitute out for labor in the (linearized) dynamic Euler equa-

[1]By 'log-linearize' I mean the following. Suppose the equation is $f(x_t)$. The linear epansion of this equation about $x_t = x^*$ is $f(x^*) + f'(x^*)\,dx$, where $dx \equiv x_t - x^*$. The log-linear expansion is $f(x^*) + f'(x^*)\,x^*\hat{x}_t$, where $\hat{x}_t \equiv (x_t - x^*)/x^*$. The latter is called a log-linear expansion because, approximately, $\hat{x}_t = \log(x_t/x)$.

[2]In the case, $\gamma = 0$, the three equilibrium conditions are the intra-temporal Euler equation (i.e., the one for labor), the inter temporal Euler equation (i.e., the one associated with capital accumulation) and the resource constraint:

$$\begin{aligned}
\text{(intra)}\ \frac{\psi c/n}{(1-n)/n} &= (1-\alpha)\left(\frac{k}{n}\right)^{\alpha} \\
\text{(inter)}\ 1 &= \beta\left[\alpha\left(\frac{k}{n}\right)^{\alpha-1} + 1 - \delta\right] \\
\text{(resource)}\ \left(\frac{k}{n}\right)^{\alpha} &= \frac{c}{n} + [1-(1-\delta)]\,\frac{k}{n}
\end{aligned}$$

The second equation can be solved for k/n. The third can then be solved for c/n. Then, the first equation can be solved for n.

tion, using the static Euler equation. This will give you one dynamic Euler equation in $\hat{k}_t$, $\hat{k}_{t+1}$, $\hat{k}_{t+2}$. Plot the two roots of this equation against a grid of values, $\tilde{\gamma}$, of γ. Let $\tilde{\gamma}$ be composed of two parts: points 0.001 apart from 0 to 0.42, and points 0.001 apart from 0.473 to 0.57. (That is, $\tilde{\gamma} = [[0 : 0.001 : 0.42]\ [0.473 : 0.001 : 0.57]]$.) Note how both roots are less than unity in the second part of $\tilde{\gamma}$. Note also the interesting behavior of the roots: the one that is less than unity for γ small appears to jump discontinuously from nearly unity in the neighborhood of 0.4722 to nearly minus unity, when it rises monotonically back up towards unity. The root that is greater than unity jumps to plus infinity in the neighborhood of 0.47 and returns back from minus infinity. Finally, note that across the different values of $\tilde{\gamma}$, there are many locally stable solutions, $\hat{k}_{t+1} = a\hat{k}_t$, including cases in which a is negative, and a is very close to unity.

2. Set the system up in the first order difference equation format discussed in class. For the case, $\gamma = 0$, display the unique D matrix associated with the unique non-explosive solution. Consider a value γ for which there are two non-explosive eigenvalues. Display two D matrices associated with the MSV solutions.

3. Consider the model economy in question 1, in the case, $\gamma = 0$.

   (a) In questions 1 and 2, you solved the model by first deriving the first order conditions for consumption and hours worked, and then substituting out for consumption in the two equations from the resource constraint. Then, you log linearized the two equations and substituted out for hours worked in the linearized intertemporal equation using the linearized intra temporal equation. Write out the numerical values of the elements in the solution:

$$\begin{aligned} \hat{k}_{t+1} &= \lambda \hat{k}_t \\ \hat{n}_t &= h\hat{k}_t. \end{aligned}$$

   That is, write out λ and h.

(b) Now, use the QZ decomposition to do the same calculations, but without substituting the intratemporal equation into the intertemporal equation. In particular, express the system of two linearized equilibrium conditions as follows:

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0, z_t = \begin{pmatrix} \hat{k}_{t+1} \\ \hat{n}_t \end{pmatrix},$$

where z_{-1} is given at time 0. Define

$$\underset{2n\times 1}{Y_t} = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix},$$

where $n = 2$ and the first order representation of Y_t is

$$aY_{t+1} + bY_t = 0,$$

where

$$\underset{2n\times 2n}{a} = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix}, \underset{2n\times 2n}{b} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix}.$$

(i) Display the numerical value α_0 and note that the rank of α_0 is less than n, so that α_0 is not invertible.

(ii) Compute the QZ decomposition of a and b, i.e., two orthonormal matrices, Q and Z, such that

$$QaZ = H_0, \; QbZ = H_1,$$

where H_0 and H_1 are upper triangular matrices. (That Q and Z are orthonormal implies $QQ' = I$, $ZZ' = I$.) To do this, you need to use the MATLAB function, qz. In addition, you need the separately provided code, qzdiv.m. These are used as follows, given the a, b matrices that you have already computed.

- [H0,H1,q,z,v]=qz(a,b);
- stake = 1e+08; [H0,H1,q,z] = qzdiv(stake,H0,H1,q,z);

The call to qzdiv orders the zeros on the diagonal of $H0$ in the bottom right. The number of these zeros is denoted l. What is the value of l? Verify that the l terms on the bottom right diagonal of H_1 are non-zero.

Consider the following partitioning of H_0 and H_1 :

$$H_0 = \begin{bmatrix} \underset{(2n-l)\times(2n-l)}{G_0} & H_{12}^0 \\ 0 & \underset{l\times l}{H_{22}^0} \end{bmatrix}, \; H_1 = \begin{bmatrix} \underset{(2n-l)\times(2n-l)}{G_1} & H_{12}^1 \\ 0 & \underset{l\times l}{H_{22}^1} \end{bmatrix},$$

where G_0 and H_{22}^0 are upper triangular, the diagonal elements of G_0 are non-zero, while the diagonal elements of H_{22}^0 are all zero. The object, G_1, is upper triangular, and you may assume that the diagonal elements of the upper triangular matrix, H_{22}^1, are nonzero. Define

$$\gamma_t = \begin{pmatrix} \underset{(2n-l)\times 1}{\gamma_{1t}} \\ \underset{l\times 1}{\gamma_{2t}} \end{pmatrix} = Z'Y_t = \begin{pmatrix} \underset{(2n-l)\times 2n}{L_1} \\ \underset{l\times 2n}{L_2} \end{pmatrix} Y_t, \text{ where } Z' = \begin{pmatrix} \underset{(2n-l)\times 2n}{L_1} \\ \underset{l\times 2n}{L_2} \end{pmatrix}.$$

(iv) Compute all the candidate MSV solutions to the system, i.e., $n \times 2n$ matrices, D, such that $DY_0 = 0$. Suppose there are J such matrices, $D^1, ..., D^J$. Partition each of these matrices into two $n \times n$ parts:

$$D^j = \begin{bmatrix} D_1^j & \vdots & D_2^j \end{bmatrix}.$$

How many of these matrices have the property that D_1^j is invertible? Real? In each invertible case, compute

$$A_j = -\left(D_1^j\right)^{-1} D_2^j,$$

and verify that

$$\alpha_0 \left(A_j\right)^2 + \alpha_1 A_j + \alpha_2 = 0,$$

that is, that you have found the zero of a particular matrix polynomial.

(v) How many matrices, A_j, have the property that their eigenvalues are less than unity in absolute value? There should be exactly one. Compare this solution with the one reported in part (a) of this question. They should be the same.