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Advanced Macro
Fall, 2014
Homework #1, due October 1

1. Consider the Runge function discussed in class, $f : [-5, 5] \rightarrow R$. Study the accuracy of two interpolation methods. In the first, construct an $n-$point grid in the domain of f in which there is a fixed interval between the n grid points ('fixed interval method'). In the second, construct an $n-$point grid in the domain of f in which the grid points correspond to the zeros of a Chebyshev polynomial ('Chebyshev method'). For the Chebyshev method, it is useful to have a mapping from any set, $[a, b]$, into $[-1, 1]$. Following is one possibility, with $b > a$:

$$\begin{aligned} \varphi &: [a, b] \rightarrow [-1, 1] \\ \varphi(x) &= 2\frac{x-a}{b-a} - 1. \end{aligned}$$

Obviously, φ is invertible. Interpolate the Runge function using the fixed interval method and using the Chebyshev method setting $n = 10$ and 12. Graph the interpolating functions over the domain of f, $[-5, 5]$. Show that with the fixed interval method, the interpolating function is diverging from the Runge function, while it is getting closer in the case of the Chebyshev method.

2. (Orthogonality property of Chebyshev polynomials). Consider the Chebyshev polynomial:

$$\begin{aligned} T_0(r) &= 1 \\ T_1(r) &= r \\ T_n(r) &= 2rT_{n-1}(r) - T_{n-2}(r), \end{aligned}$$

for $n = 2, 3, 4, ...$ Let $r_1, ..., r_N$ denote the zeros of the N^{th} order Chebyshev polynomial, $T_N(r)$:

$$r_k = \cos\left(\frac{\pi}{2}\frac{2k-1}{N}\right),\ k = 1, ..., N.$$

The Chebyshev polynomial has the following discrete orthogonality property:

$$\sum_{k=1}^{N} T_i(r_k) T_j(r_k) = \begin{cases} 0 & i \neq j \\ N & i = j = 0 \\ \frac{N}{2} & i = j \neq 0 \end{cases}.$$

The way you did $N-$point Chebyshev interpolation in the previous homework question is that you first computed the zeros of the $N-$dimensional Chebyshev polynomial, $r = [r_1, ..., r_N]'$. These map into the $N \times 1$ set of points, x, belonging to the domain of the Runge function via the transformation:

$$x = \frac{(b-a)(r+1)}{2} + a,$$

where $a = -5$, $b = 5$. Then, you constructed a matrix

$$X = \left[\begin{array}{ccc} T_0(r) & \cdots & T_{N-1}(r) \end{array}\right],$$

where $T_j(r)$ is the $N-$dimensional column vector formed by evaluating the j^{th} order Chebyshev polynomial at each of the elements of the column vector, r. Finally, you form the N dimensional column vector, $f(x)$, where f denotes the Runge function evaluated at each of the elements in x. Then the interpolation equation to be solved is given by:

$$X\gamma = f(x).$$

Premultiply by X' :

$$X'X\gamma = X'f(x)$$

Note from the discrete orthogonality property, that $X'X$ is a diagonal matrix with $\frac{N}{2}$ in all but one entry on the diagonal. In the 1,1 element of $X'X$ there appears N. To find γ simply invert the $X'X$ matrix:

$$\gamma = (X'X)^{-1} X'f(x),$$

or

$$\gamma_k = \begin{cases} \frac{T_0(r)'f(x)}{N} & k = 0 \\ 2\frac{T_k(r)'f(x)}{N} & k = 1, ..., N-1 \end{cases}. \quad (1)$$

Notice the relatively trivial nature of this formula for the γ's. For example, it is no problem to make N *very* large. This stands in sharp contrast with the fixed interval interpolation that was done (and which works badly!) in the previous homework question. In that case, for N large you have to invert a matrix, X, with columns of numbers that are of very different orders of magnitude, with negative implications for numerical accuracy.

We conclude that the $N-1^{th}$ order Chebyshev interpolating function for $f(x)$ is

$$f(x) \simeq \sum_{k=0}^{N-1} \gamma_k T_k(\varphi(x)), \quad (2)$$

where φ was defined in the previous homework question. Verify that the γ's you computed in the previous homework question are also the solution to the above formula.

3. (Using the Chebyshev interpolation theorem to approximate an integral). The Chebyshev interpolation theorem tells us that for large enough N, (2) provides an excellent (in the sup norm sense) approximation to virtually *any* function, f (some restrictions, such as continuity, are required for f).

And, as noted above, the Chebyshev orthogonality property implies that making N as large as you want creates no numerical problems. Thus, we have

$$\int_a^b f(x)\,dx \simeq \sum_{k=0}^{N-1} \gamma_k \int_a^b T_k(\varphi(x))\,dx.$$

Since $T_k(r)$ is a polynomial, it is easy to integrate. Show that the formula is valid:

$$\int_{-1}^{1} T_k(r)\,dr = \begin{cases} 2 & k=0 \\ 0 & k=1 \\ \left[\frac{kT_{k+1}(r)}{k^2-1} - \frac{rT_k(r)}{k-1}\right]_{r=-1}^{r=1} & k \geq 2 \end{cases}.$$

Show how to use this latter expression to approximate $\int_a^b f(x)\,dx$ (hint: it is convenient to adopt a change of variables in the integration, from $x \in (a,b)$ to $r = \varphi(x) \in (-1,1)$.) This method for approximating an integral is called Chebyshev Quadrature integration (the Chebyshev quadrature integration formula is usually expressed in a simplified form). Consider the integral,

$$\int_0^1 e^{3x}dx.$$

The exact value of this integral is easy to work out analytically. Approximate it using Chebyshev quadrature, programming everything yourself (there is software all over the web for doing this, but it is most instructive to do it yourself, at least one time). Try it for a small value of N (say, $N = 2$), and determine what value of N you need to go to, to get a decent approximation.