Christiano
FINC 520, Spring 2007
Homework 1, due Monday, April 9.

1. Prove:
$$P\left[Y_1+Y_2|X\right]=P\left[Y_1|X\right]+P\left[Y_2|X\right],$$
where Y_1 and Y_2 are scalar random variables and $X=[X_1,...,X_n]'$, where X_i is a scalar random variable.

2. Suppose x_t is a mean-zero covariance stationary process. Let
$$\varepsilon_t=x_t-P\left[x_t|x_{t-1},x_{t-2},...\right],\ E\varepsilon_t^2=\sigma^2.$$
denote the one-step-ahead forecast error in x_t. Let the indeterministic part of x_t that is the subject of the Wold decomposition theorem be denoted
$$\sum_{j=0}^{\infty}d_j\varepsilon_{t-j},\ d_j=\frac{Ex_t\varepsilon_{t-j}}{\sigma^2},\ j=0,1,...\ .$$
Show:
$$x_t-P\left[x_t|x_{t-3},x_{t-4},...\right]=\varepsilon_t+d_1\varepsilon_{t-1}+d_2\varepsilon_{t-2}.$$

3. Consider a stochastic process with covariance function, $\gamma_j=Ex_tx_{t-j}$, where
$$\begin{aligned}\gamma_j &= \phi^{|j|},\text{ for integer } j,\\ -1 &< \phi<1.\end{aligned}$$
Construct the objects in the Wold decomposition of this stochastic process. That is, find (i) the linear representation,
$$\varepsilon_t=f\left(x_t,x_{t-1},...\right),$$
where
$$\varepsilon_t=x_t-P\left[x_t|x_{t-1},...\right].$$
Also, (ii), express the $\{d_j\}$ identified by the Wold theorem:
$$P\left[x_t|\varepsilon_t,\varepsilon_{t-1},...\right]=\sum_{j=0}^{\infty}d_j\varepsilon_{t-j},$$
explicitly in terms of ϕ. Finally, (iii) display η_t, the purely deterministic (i.e., perfectly forecastable) part of x_t.

4. Consider a stochastic process with covariance function, $\gamma_0 > 0$, $|\gamma_1| < \frac{1}{2}\gamma_0$, $\gamma_j = 0$, $j \geq 2$. Identify two MA(1) representations for x_t :

$$\begin{aligned} x_t &= \nu_t + \theta\nu_{t-1},\ \nu_t\tilde{}\text{white noise with variance } \sigma_\nu^2 \\ x_t &= u_t + \frac{1}{\theta}u_{t-1},\ u_t\tilde{}\text{white noise with variance } \sigma_u^2. \end{aligned}$$

Derive explicit expressions relating θ, σ_ν^2, σ_u^2 to the γ_j's. Construct the objects in (i)-(iii) in question 4. Which white noise, ν_t or u_t, corresponds to the one-step-ahead forecast error in the Wold decomposition theorem?

5. The Markov Chain is a model of a stochastic process (see Hamilton, section 22.2). Consider a random variable,

$$x_t \in \left\{x^1, x^2, ..., x^n\right\}.$$

Consider the matrix

$$P = [p_{ij}],$$

where

$$p_{ij} = prob\left[x_{t+1} = x^i | x_t = x^j\right].$$

(I am following Hamilton's convention here. Ljungqvist and Sargent and others work with P' rather than P.) The elements of the j^{th} column of P is the distribution of x_{t+1} given that $x_t = x^j$. It is easy to confirm that the j^{th} column of P^k is the distribution of x_{t+k} given that $x_t = x^j$. Note that because the columns of P are probability distributions, it must be that

$$\iota' P = \iota',$$

where ι is an $n \times 1$ column vector with unity in each element. Note, $\iota' P^2 = \iota' P = \iota'$, and similarly, $\iota' P^k = \iota'$, as is required by the fact that the columns of P^k are probability distributions. When, a column vector, g, and a scalar, λ, satisfy the property, $g'P = \lambda g'$, we say that g is the left column vector of P associated with the eigenvalue, λ. Thus, we know that P has a unit eigenvalue and that the associated left column vector is ι.

Suppose the eigenvalues of P are all distinct, and that - apart from the unity eigenvalue - they are all less than unity in absolute value. That the eigenvalues are distinct implies that P can be written

$$P = T\Lambda T^{-1},$$

where Λ is a diagonal matrix of eigenvalues, T_i is the i^{th} (right) eigenvector associated with the i^{th} eigenvalue (i.e., the i^{th} diagonal element of Λ), where

$$T = \left[T_1 \vdots \cdots \vdots T_n \right],\ T^{-1} = \left[\begin{array}{c} \tilde{T}_1 \\ \vdots \\ \tilde{T}_n \end{array} \right].$$

Here, $\tilde{T}_i$ is the i^{th} row of T^{-1}, which is the left eigenvector of P associated with the i^{th} eigenvalue. For convenience, we order the eigenvalues so that the top left element of Λ is unity. Thus, $\tilde{T}_1 = \iota'$. Recall that

$$P^k = T\Lambda^k T^{-1},$$

so that, as $k \to \infty$,

$$P^k \to \pi\iota',$$

where $\pi \equiv T_1$. That is, P^k converges to a matrix in which each column is π. Put differently the distribution of x_{t+k} converges to π as $k \to \infty$, regardless of the value of x_t. As a result, the doubly infinite stochastic process, $\{..., x_{t-3}, x_{t-2}, x_{t-1}, x_t, x_{t+1}, x_{t+2}, ...\}$ is strictly stationary, with the distribution of x_t corresponding to π, for each t.

For the remainder of this problem, consider the following two-state Markov Chain:

$$P = \left[\begin{array}{cc} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{array} \right],\ 0 < \alpha < 1,$$

so that the probability of the value of x_t changing is $1-\alpha$ and probability of x_t not changing is α. Suppose that $x^1 = \sigma$, $x^2 = -\sigma$.

(a) Obtain analytic expressions for the eigenvalues of P, the columns of T and the rows of $\tilde{T}$.

(b) Compute

$$\begin{aligned} \gamma_0 &= E x_t^2 \\ \gamma_1 &= E x_t x_{t+1} \end{aligned}$$

(Hint: use the law of iterated mathematical expectations to compute $\gamma_1 : E x_t x_{t+1} = E \left\{ E \left[x_t x_{t+1} | x_t \right] \right\}$. This requires first computing $E \left[x_{t+1} | x_t \right]$ for each possible x_t.)

(c) Show that

$$\gamma_j = (2\alpha - 1)^j \gamma_0, \; j \geq 0,$$

and construct the Wold decomposition, (i)-(iii), in question 4.

(d) Is the two-state Markov Chain ergodic? Explain.