Christiano
FINC 520, Spring 2008
Homework 1, due Monday, April 16.

1. Here are two questions about linear projections. You may use the necessity and sufficiency of the orthogonality property of projections in your answer.

   (a) Prove:
   $$P\left[Y_1 + Y_2|X\right] = P\left[Y_1|X\right] + P\left[Y_2|X\right],$$
   where Y_1 and Y_2 are scalar random variables and $X = [X_1, ..., X_n]'$, where X_i is a scalar random variable. Also, P denotes the linear projection operator.

   (b) Consider three random variables, x, y, z, where $cov\left(y,x\right) = cov\left(x,z\right) = 0$. Prove:
   $$P\left[y|x,z\right] = P\left[y|z\right].$$

   (c) Consider the random variable, $w = \delta z + \psi x$, where δ and ψ are arbitrarily selected non-zero numbers. Prove
   $$P\left[y|w,z\right] = P\left[y|x,z\right],$$
   where the equality holds for each possible realization of x, z and w.

2. Consider a stochastic process with covariance function, $\gamma_0 > 0$, $|\gamma_1| < \frac{1}{2}\gamma_0$, $\gamma_j = 0$, $j \geq 2$. Identify two MA(1) representations for x_t :
   $$x_t = \nu_t + \theta\nu_{t-1},\ \nu_t\tilde{}\text{white noise with variance } \sigma_\nu^2.$$
   That is, identify two sets of values of θ and σ_ν^2 that have the property that the resulting MA(1) is consistent with the given γ_j, $j \geq 0$.

3. Consider the ARMA(2,2) process:
   $$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}.$$

(a) Express the model for y_t as a vector AR(1) (VAR(1)):

$$Y_t = FY_{t-1} + v_t,$$

and display the contents of Y_t, F, v_t.

(b) Use the VAR(1) representation to express y_t as an MA(∞). Explain why the existence of the MA(∞) requires that the eigenvalues of F lie inside the unit circle (i.e., have absolute value less than unity).

(c) Prove (for example, using the expansion by cofactors discussed in class) that whether the eigenvalues of F lie inside the unit circle depends only on whether the roots of the AR polynomial in the ARMA representation lie inside the unit circle, where the 'AR polynomial' means $f(\lambda)$, where

$$f(\lambda) = \lambda^2 - \phi_1\lambda - \phi_2.$$

4. Consider the following parameterization of the ARMA(2,2) process in question 3:

$$\begin{aligned} \phi_1 &= 1.70,\ \phi_2 = -0.7125, \\ \theta_1 &= -0.75,\ \theta_2 = 0.125, \\ \sigma_\varepsilon^2 &= 1. \end{aligned}$$

(a) write out the VAR(1) representation of this ARMA process. Compute ψ_j, $j = 0, 1, ..., 100$, in

$$y_t = \psi_0\varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + ...,$$

and graph ψ_j on the vertical axis and j on the horizontal.

(b) compute the covariance function,

$$\gamma_j = Ey_t y_{t-j},\ j \geq 0,$$

using the VAR(1) representation.
Hint: Note that the covariance function of Y_t, $C_Y(j) \equiv EY_tY_{t-j}'$ is as follows. The covariance solves:

$$C_Y(0) = FC_Y(0)F' + V, \tag{1}$$

where $V = Ev_t v_t'$. You can find $C_Y(0)$ by setting $C_Y^{(0)}(0)$ to an arbitrary positive semidefinite matrix (zero is fine) and computing the sequence, $C_Y^{(r)}(0)$, $r = 1, 2, 3, ...$, using

$$C_Y^{(r)}(0) = FC_Y^{(r-1)}(0)F' + V, \; r \geq 1,$$

which is bound to converge given that the eigenvalues of F lie inside the unit circle. Alternatively, note that (1) is linear in $C_Y(0)$ and can be converted into a standard linear system of equations in an equal number of unknowns using the fact,

$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1) vec(A_2),$$

where $vec(X)$ takes the matrix, X, and converts it into a vector by stacking its columns and $\otimes$ is the Kronecker operator, whose description may be found by typing help kron at the MATLAB command prompt.

With $C_Y(0)$ in hand, note that

$$C_Y(j) = EY_t Y_{t-j}' = FC_Y(j-1), \; j = 1, 2, ... \, .$$

(c) note that the autocovariances of interest, γ_j, lie in the upper left block of $C_Y(j)$. Graph γ_j for $j = 0, 1, ..., 100$.

(d) 'flip' one of the roots in the moving average part of the ARMA model, to obtain an alternative, equivalent ARMA representation. Show that its autocovariance function, γ_j, is in fact the same.