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FINC 520, Spring 2009
Homework 1, due Wednesday, April 8.

1. Consider a stochastic process with covariance function, $\gamma_0 > 0$, $|\gamma_1| < \frac{1}{2}\gamma_0$, $\gamma_j = 0$, $j \geq 2$. Identify two MA(1) representations for x_t :

$$x_t = \nu_t + \theta\nu_{t-1},\ \nu_t\tilde{}\text{white noise with variance } \sigma_\nu^2.$$

That is, identify two sets of values of θ and σ_ν^2 that have the property that the resulting MA(1) is consistent with the given γ_j, $j \geq 0$.

2. Consider the ARMA(2,2) process:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}. \quad (1)$$

Suppose that the zeros of

$$\lambda^2 - \phi_1\lambda - \phi_2 \quad (2)$$

are less than unity in absolute value. There are no restrictions on θ_1 and θ_2.

Express the model for y_t as a vector AR(1) (VAR(1)):

$$Y_t = FY_{t-1} + v_t,$$

where v_t is *iid* and uncorrelated with Y_{t-1}, and display the contents of Y_t, F, v_t. Prove that the eigenvalues of the determinant of $F - \lambda I$ are less than unity in absolute value if, and only if, the roots of (2) (i.e., 'the AR part of the ARMA representation in (1)) are less than unity in absolute value. Here, I denotes the identity matrix. Explain why this means that the square summability of the MA(∞) representation associated with (1) depends only on the roots of its AR part while the moving average terms are irrelevant for square summability.

Hint: one way to do the proof is to use the *expansion by cofactors* result for the determinant of a matrix. To explain this result, let A be a square matrix of order m and let B_{ij} be the matrix obtained by deleting from A its i^{th} row and j^{th} column. The object,

$$A_{ij} = (-1)^{i+j}|B_{ij}|$$

is said to be the *cofactor* of the element, a_{ij}, of A. Here, $|\cdot|$ denotes the determinant operator. The matrix, B_{ij}, is referred to as the $(m-1)-$*order minor* of A. The expansion by cofactors expression for the determinant of A is given by:

$$|A| = \sum_{j=1}^{m} a_{ij} A_{ij}, \quad |A| = \sum_{i=1}^{m} a_{ij} A_{ij}.$$

In the case of the first sum, i is kept fixed at an arbitrary value, $i \in \{1, 2, ..., m\}$, and in the case of the second sum, j is kept fixed at an arbitrary value, $j \in \{1, 2, ..., m\}$. Although this result may look complicated, it greatly simplifies the expression, $|F - \lambda I|$, in our particular case. To see this, work this expression out first for the case, $\theta_2 = 0$. Note that one of the rows of $F - \lambda I$ is composed of all zeros, apart from one entry which is $-\lambda$. This row is ideal for expanding $|F - \lambda I|$ by cofactors. The result in the question is proved for $\theta_2 \neq 0$ recursively by expanding $|F - \lambda I|$ by cofactors and then expanding a cofactor by cofactors. The general result that the square summability of an ARMA(p,q) model depends only on the roots of the AR part can be proved recursively in this way, thought it's a bit of a mess to actually do it.

3. Consider the following parameterization of the ARMA(2,2) process in question 2:

$$\begin{aligned}
\phi_1 &= 1.70, \ \phi_2 = -0.7125, \\
\theta_1 &= -0.75, \ \theta_2 = 0.125, \\
\sigma_\varepsilon^2 &= 1.
\end{aligned}$$

   (a) write out the VAR(1) representation of this ARMA process. Compute ψ_j, $j = 0, 1, ..., 100$, in

   $$y_t = \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + ...,$$

   and graph ψ_j on the vertical axis and j on the horizontal.

   (b) compute the covariance function,

   $$\gamma_j = E y_t y_{t-j}, \ j \geq 0,$$

for $j = 0, 1, ..., 100$ using the VAR(1) representation. Do this by first solving the Riccati equation by the two strategies discussed in class (i.e., the recursive one as well as the equation-solving approach). Use the resulting covariances as initial conditions to solve for the remaining covariances by exploiting the fact that covariances for $j > q$, where q is the order of the MA part satisfy the difference equation formed by the AR part of the ARMA representation. Graph γ_j for $j = 0, 1, ..., 100$.

(c) 'flip' one of the roots in the moving average part of the ARMA model, to obtain an alternative, equivalent ARMA representation. Display the coefficients of the alternative moving average representation, including the moving average coefficients and the variance of the disturbance term. Compute the autocovariance function for this alternative representation and show the γ_j's are in fact the same.