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FINC 520, Spring 2007
Homework 5, due Wednesday, May 10.

The first two questions in this three-part homework have two purposes: to show you the power of the Kalman filter techniques we have talked about and to expose you to some promising recent research that uses these methods. The homework summarizes the recent work by Bernanke, Boivin, Eliasz and Giannone which exploits the fact that in practice there are many different variables which measure the same basic concept (e.g., there are several measures of hours worked, there are several measures of inflation, output, etc.). The homework will also show how to evaluate the term structure implications of equilibrium models, along the lines of recent work by Bekaert, Cho and Moreno and others. All the relevant background papers (not needed to do the homework) are on the course website.

1. Suppose the equilibrium allocations for a particular economy solve the following maximization problem:

$$\begin{aligned} \max \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - \frac{\exp(\tau_t)}{1+\psi} l_t^{1+\psi} \right], \\ C_t + K_{t+1} \leq \; & K_t^{\alpha} \left(\exp(z_t) \, l_t \right)^{(1-\alpha)} = Y_t, \\ & \tau_t, \; z_t : \text{ iid Normal, mean zero random variables, variance } \sigma_\tau^2, \; \sigma_z^2, \end{aligned}$$

   (a) Verify that the capital and employment decisions that solve the intertemporal and intratemporal Euler equations are:

$$\begin{aligned} K &= (\beta\alpha)^{\frac{1}{1-\alpha}} \left(\frac{1-\alpha}{1-\beta\alpha} \right)^{\frac{1}{\psi+1}} \\ Y &= (\beta\alpha)^{\frac{\alpha}{1-\alpha}} \left(\frac{1-\alpha}{1-\beta\alpha} \right)^{\frac{1}{\psi+1}} \end{aligned}$$

$$K_{t+1} = \beta\alpha Y_t, \; l_t = \exp\left[-\frac{\tau_t}{\psi+1} \right] \left(\frac{1-\alpha}{1-\beta\alpha} \right)^{\frac{1}{\psi+1}}.$$

   (b) Verify that after taking logs and reorganizing a little:

$$\begin{pmatrix} k_{t+1} \\ \log(Y_t/l_t) \end{pmatrix} = \begin{pmatrix} \gamma_k \\ \gamma_a \end{pmatrix} + \begin{bmatrix} \alpha & 0 \\ \alpha & 0 \end{bmatrix} \begin{pmatrix} k_t \\ \log(Y_{t-1}/l_{t-1}) \end{pmatrix} + \begin{pmatrix} (1-\alpha)\left(z_t - \frac{1}{1+\psi}\tau_t\right) \\ (1-\alpha) z_t + \frac{\alpha}{1+\psi}\tau_t \end{pmatrix},$$

where

$$\gamma_k = \frac{1-\alpha}{1+\psi}\log\left(\frac{1-\alpha}{1-\beta\alpha}\right)+\log\beta\alpha$$
$$\gamma_a = -\frac{\alpha}{1+\psi}\log\left(\frac{1-\alpha}{1-\beta\alpha}\right).$$

and $k_{t+1} \equiv \log(K_{t+1})$.

(c) Suppose the households in the equilibrium have access to credit markets. Let r_t denote the rate of return, from period t to period $t+1$, of a risk free bond. Show that that rate of return must satisfy:

$$r_t = \frac{1}{\beta\exp\left[(1-\alpha)^2\left(\left(\frac{1}{\psi+1}\right)^2\sigma_\tau^2+\sigma_z^2\right)\right]}\left(\frac{Y}{Y_t}\right)^{1-\alpha},$$

where Y denotes the steady state level of output, when $\sigma_\tau=\sigma_z=0$. (hint: use the relevant intertemporal Euler equation and make use of $E\exp(x)=\exp(\mu+(1/2)\sigma^2)$ when x is normal with mean μ and variance σ^2).

Note that when there is uncertainty, then the one-period interest rate in the loan market is lower. This indicates the presence of a precautionary savings motive. Because everyone is identical in this economy, the loan market must clear at zero borrowing and lending. That a lower interest rate clears the loan market when there is more uncertainty reflects that the supply of saving is increasing with uncertainty.

(d) Let R_t denote the risk-free rate of return 5-year bonds, which translates into a 20 period bond in our quarterly model. Work out a formula for R_t that is analogous to the one for r_t. Calculate the average slope of the term structure in this model, i.e.,

$$E\log\frac{R_t}{r_t}.$$

How does this compare with the corresponding slope in the data used in previous homeworks?

2. Consider the economy of the previous section. Define the state

$$\xi_t = \begin{pmatrix} k_{t+1} \\ \log(Y_t/l_t) \\ \tau_t \\ z_t \\ 1 \end{pmatrix}$$

and define the law of motion for the state:

$$\xi_t = F\xi_{t-1} + v_t,$$

where c is a constant. Suppose the vector of observed variables is

$$Y_t = \begin{pmatrix} \log r_t \\ \log R_t - \log r_t \\ l_t^1 \\ l_t^2 \end{pmatrix},$$

where l_t^1 and l_t^2 are two different empirical measures of per capita hours worked (the first is log per capita hours worked based on a survey of US business establishments and the second is log per capita hours worked recently constructed by Francis and Ramey, "Measures of Hours Per Capita and their Implications for the Technology-Hours Debate"..both series are available on the course website). Write the observer equation as follows:

$$Y_t = H'\xi_t + w_t,$$

where w_t is a 4×1 vector of measurement error with $Ew_t w_t' = R$, a diagonal matrix. The first two elements of w_t reflect different ways in which there might be a mismatch between $log(r_t)$ and $\log(R_t) - \log(r_t)$ in the model and data. For example, the interest rates in the model are assumed to be risk free, while in the data they are nominally risk free, but not risk free in real terms. Also, if there is a substantial and volatile premium in the empirical long rate but not the model, then w_{2t} will have a large variance. The last two elements of w_t reflect that l_t^1 and l_t^2 are two different measures of hours in the data, with different degrees of (orthogonal) measurement error. You may suppose that

$$l_t^1 = \lambda_1 \log(l_t) + w_{3t} \text{ and } l_t^2 = \lambda_2 \log(l_t) + w_{4t},$$

where λ_1 and λ_2 are parameters and l_t is hours worked predicted by the model (see question 1 above). This specification leaves open the possibility that l_t^1 better matches hours worked in the model (by setting $\lambda_1 = 0$ and/or the variance of w_{3t} high) or that l_t^2 is a better match, or some combination of the two.

There are 11 model parameters,

$$\alpha,\ \psi,\ \sigma_\tau^2,\ \sigma_z^2,\ \beta,\ \lambda_1,\ \lambda_2$$

and the four measurement error variances in R. In principle, these could be computed to maximize the Gaussian likelihood. Conceptually, this is a straightforward exercise, given that the model is expressed in the state-space/observation equation format that allows application of the Kalman filter. However, this is not the strategy we will take in this homework.

We will simply assign a priori values to the economic parameters:

$$\alpha = 0.33,\ \psi = 1,\ \sigma_\tau^2 = 0.01,\ \sigma_z^2 = 0.01,\ \beta = 0.99.$$

The value for α is suggested by evidence on the share of income going to capital, the value for ψ implies a Frisch labor supply elasticity of $1/\psi = 1$, a not unreasonable number, the one percent standard deviation numbers are in line with empirical analysis, and the value of β implies a 4 percent annual discount rate.

For now, we will suppose that both time series are equally good indicators of hours worked and we will set $\lambda_1 = \lambda_2 = 1$. There remains, however, the problem of assigning values to the measurement error variances. This is a hard thing about which to develop priors. As a rough-and-ready way to proceed, we simply set

$$R_{ii} = Var\left(Y_{it}\right)/100,\ i = 1, 2, 3, 4.$$

(a) Compute the smoothed estimates of $\log r_t$, $\log R_t - \log r_t$, $\log l_t$ and graph these against their empirical measures. The vertical distances between the smoothed and actual data is the estimated measurement error. How good is the model's prediction of the term structure? Does the failure resemble the shortcomings in the simple term structure hypothesis that you documented in a previous homework (there, you saw a substantial premium not explained by the term structure hypothesis in the long rate in the 1980s and early 1990s).

(b) Compute $\hat{w}_t$, the difference between Y_t and the model's smoothed estimate of $H'\xi_t$. Let

$$\hat{R} = \frac{1}{T}\sum_{t=1}^{T} \hat{w}_t \hat{w}_t'.$$

Redo your calculations in the previous part of this question. Do the results change?

(c) Using $\hat{R}$ from the previous question, compute the variance decomposition of the elements of Y_t into parts due to τ_t, z_t and measurement error. Does this seem like a good model of the elements of Y_t?

3. The idea of this question is to determine how many observations are required before the central limit theorem is relevant. In theory, an infinite number of observations are required. Use the random number generator in MATLAB to draw a sequence of T realizations from a uniform distribution on the interval, $(0, 1)$. Compute the sample average of these observations, and repeat the experiment $N = 5,000$ times. Display the histogram (in MATLAB, the hist command) of the N sample means. Superimpose it on the histogram of a normal distribution. Do this for $T =$ 10, 50, 100, 200. At what point do things start to look Normal?