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FINC 520, Spring 2008
Homework 5, due Friday, May 16.

1. Consider the following time series representation:

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1},\ \theta = 2,\ \sigma_\varepsilon^2 = 1. \tag{1}$$

   (a) Compute the alternative time series representation for y_t, which has the same covariance function as above, in which the moving average root has been 'flipped':

$$y_t = u_t - \mu u_{t-1},\ Eu_t^2 = \sigma_u^2. \tag{2}$$

   Derive and display μ and σ_u^2.

   (b) Verify that, by recursive substitution, one can write (2) as:

$$y_t = \sum_{i=0}^{\infty} \phi_i y_{t-i} + u_t, \tag{3}$$

   where $\{\phi_i\}$ is a square summable sequence. Prove that this expression represents the linear projection of y_t onto its infinite past history. Because u_t is the error in this relation, the representation of y_t is sometimes referred to as its 'fundamental' representation (or, 'Wold representation'). The other representation, (1), is not 'fundamental' (careful, the economic and time series analysis term, 'fundamental', means different things). Motivated by (3), it is said that the shock in the fundamental representation 'lies in the space of past data', while the shock in the alternative representation does not. To see what space ε_t lies in, show by recursive substitution that ε_t can be represented as follows:

$$\varepsilon_t = -\sum_{i=1}^{\infty} \left(\frac{1}{\theta}\right)^i y_{t+i},$$

so that ε_t lies in the space of (*all*) future y_t's. [1]

(c) The reasoning in (a) and (b) suggests that $\hat{\varepsilon}_{t|T}$ will look very different from $\hat{\varepsilon}_{t|t-1}$. Since ε_t does not lie in the (infinite) space of past y_t's the error of the projection, $\hat{\varepsilon}_{t|t-1}$ - denoted $P_{t|t-1}$ - can be expected to be just σ_ε^2. By contrast, the error in the projection, $\hat{\varepsilon}_{t|T}$ - denoted $P_{t|T}$ - should be quite small, except for t close to T. Write the non-fundamental representation, (1), in state space/observer form (hint, let the state be $\xi_t = \begin{pmatrix} y_t & \varepsilon_t \end{pmatrix}'$, and let $H' = \begin{pmatrix} 1 & 0 \end{pmatrix}$). Program the formulas for the forecast error variances in MATLAB and display a graph of $P_{t|t}$, $P_{t|T}$, $P_{t|t-1}$, for $t = 1, ..., T$. Set $T = 10$. Now display the same graph, but for the fundamental representation of y_t. Are the results in the graphs consistent with what intuition suggests?

(d) Consider the state-space/observer representation of the non-fundamental representation of y_t. Iterate on the recursive expression for $P_{t|t-1}$ to obtain:

$$P = \lim_{t\to\infty} P_{t|t-1}.$$

The $1,1$, element of P is the one-step-ahead variance of the error

[1] It is interesting to note that the recursive substitution corresponds to simple manipulation of lag operators:

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1} = (1 - \theta L)\,\varepsilon_t,$$

so

$$\varepsilon_t = \frac{1}{1 - \theta L} y_t.$$

A problem is that the expansion of the above polynomial in positive powers of L has explosive coefficients. However, the polynomial can also be expanded in negative powers of L

$$\frac{1}{1-\theta L} = \frac{-\theta^{-1}L^{-1}}{1-\theta^{-1}L^{-1}},$$

so that

$$\begin{aligned} \varepsilon_t &= \frac{-\theta^{-1}L^{-1}}{1-\theta^{-1}L^{-1}} y_t = \frac{-\theta^{-1}}{1-\theta^{-1}L^{-1}} y_{t+1} \\ &= -\frac{1}{\theta}\left[y_{t+1} + \frac{1}{\theta} y_{t+2} + \left(\frac{1}{\theta}\right)^2 y_{t+3} + ... \right]. \end{aligned}$$

in forecasting y_t given $y_{t-1}, y_{t-2}, \ldots$. Does the quantitative magnitude of this one-step-ahead forecast error variance make sense in light of your results in (b)? Confirm that the $1,1$ element of P is unchanged if you construct the state-space/observer system using the fundamental representation of y_t. What happens to the other elements of P? State your findings in intuitive terms.

2. Denote the real interest rate by r_t, where $r_t \equiv i_t - \pi_t^e$ and i_t denotes a quarterly interest rate and π_t^e represents the (unobserved) quarterly inflation rate. For these calculations, use the data from homework 2. Suppose that the real interest rate evolves according to:

$$r_t = (1 - 0.95)\,\mu + 0.95 r_{t-1} + v_t,$$

where μ is the sample mean of the ex post real rate, $i_t - \pi_t$, so that $\mu = 0.0050$ (i.e., 2 percent, at an annual rate). Let v_t be a white noise with standard deviation, 0.0015. Treat the ex post real rate of interest, $i_t - \pi_t$, as

$$i_t - \pi_t = r_t + w_t,$$

where w_t and v_t satisfy all the properties assumed for the Kalman filter (Hamilton, p. 376). Let the standard deviation of the white noise, w_t, be 0.0034. Compute

$$\hat{r}_{t|T} = \hat{E}\left[r_t | i_1 - \pi_1, \ldots, i_T - \pi_T\right],$$

for $t = 1, \ldots, T$ using the Kalman filter. Also, compute $P_{t|T}$ for $t = 1, \ldots, T$. Place four graphs in one figure: the ex post realized real rate, $i_t - \pi_t$; the estimated ex ante real rate, $\hat{r}_{t|T}$; and $\hat{r}_{t|T}$ plus/minus 2 times $\sqrt{P_{t|T}}$, for $t = 1, \ldots, T$.

3. In class, we motivated the spectral decomposition theorem using the band pass filter. In this question, we motivate the same theorem using a discrete decomposition of data into a set of sinusoidal functions.

   Consider the discrete analog of the spectral representation theorem:

$$\begin{aligned} y_t &= \sum_{j=0}^{(T-1)/2} y_{j,t}, \qquad (4) \\ y_{j,t} &= \alpha_j \cos(\omega_j t) + \delta_j \sin(\omega_j t), \end{aligned}$$

where T is the number of observations on y_t, T is assumed to be odd,

$$\omega_j = \frac{2\pi j}{T},\ j = 0, ..., \frac{T-1}{2},$$

and α_j and δ_j, $j = 0, ..., (T-1)/2$ are a set of parameters to be determined. Although it looks at first like there are $T+1$ parameters here (one more than the number of observations), note that this is in fact not true, since δ_0 multiplies the zero vector and so can be ignored. Recall that $\cos(x) = \cos(x + 2\pi k)$ and $\sin(x) = \sin(x + 2\pi k)$ for any integer k. As a result, the j^{th} component of y_t in (1), $y_{j,t}$, has period $2\pi/\omega_j$. To see this, let t denote an initial date, and let t' denote the next time when $y_{j,t}$ is at the same point in its cycle. Thus, $t'\omega_j = t\omega_j + 2\pi$, so that the period, in units of time, of the cycle in $y_{j,t}$, the j^{th} component of y_t, is

$$t' - t = \Delta t = \frac{2\pi}{\omega_j},\ j > 0.$$

Note that higher values of ω_j (i.e., 'higher frequency components of y_t') are associated with cycles of shorter duration, or period.

The equations in (1) can be written in matrix form like this:

$$y = X\beta,$$

where

$$\beta = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \delta_1 \\ \alpha_2 \\ \delta_2 \\ \vdots \\ \alpha_{\frac{T-1}{2}} \\ \delta_{\frac{T-1}{2}} \end{bmatrix},$$

and X corresponds to the cosine and sine variables in (1) (note how we've ignored δ_0).

(i) Show that

$$\frac{1}{T}\sum_{t=1}^{T} \sin(\omega_j t)\sin(\omega_l t) = \frac{1}{T}\sum_{t=1}^{T} \cos(\omega_j t)\cos(\omega_l t) = \begin{cases} \frac{1}{2} & j = l > 0 \\ 0 & j \neq l \end{cases}$$

$$\frac{1}{T}\sum_{t=1}^{T}\cos(\omega_j t)\sin(\omega_l t) = 0, \text{ all } j,l,$$

so that X is a square matrix with orthogonal columns. Thus, (1) represents an exact decomposition of a time series (*any* time series, not just the covariance stationary and indeterministic series addressed by the spectral decomposition theorem) into orthogonal sinusoidal components, as in the spectral decomposition theorem.

(ii) Show that

$$\widehat{var}(y_t) = \frac{1}{T}\sum_{t=0}^{T}(y_t - \bar{y})^2 = \sum_{j=0}^{\frac{T-1}{2}} \widehat{var}(y_{j,t}),$$

where

$$\widehat{var}(y_{j,t}) = \frac{1}{2}\left[\alpha_j^2 + \delta_j^2\right],\ j = 1, ..., \frac{T-1}{2},$$

and the α_j's and δ_j's are the unique solution to

$$\beta = (X'X)^{-1} X'y.$$

(Note that this reduces to $\beta = X^{-1}y$ since X is square and invertible, but it is somewhat easier to understand the vector β by using the more elaborate formula.)

(iii) Denote the k^{th} order sample covariance of $y_1, ..., y_T$ by

$$\begin{aligned}
\hat{\gamma}_k &= \frac{1}{T}\sum_{t=k+1}^{T}(y_t - \bar{y})(y_{t-k} - \bar{y}) \\
\bar{y} &= \frac{1}{T}\sum_{t=1}^{T} y_t,
\end{aligned}$$

and $\hat{\gamma}_{-k} = \hat{\gamma}_k$, for $k = 0, ..., T-1$. Denote the *sample periodogram* by

$$\hat{S}_y(\omega) = \frac{1}{2\pi}\sum_{k=-T+1}^{T-1}\hat{\gamma}_k e^{-i\omega k}.$$

This is the sample analog of the population spectral density. Show that

$$\hat{S}_y(\omega_j) = \frac{T}{4\pi}\widehat{var}(y_{j,t}),$$

so that the periodogram at frequency ω_j corresponds to the variance of the j^{th} frequency component of the data. This is the analog of a similar result that we derived using an argument based on the band pass filter.

4. Consider the industrial production data made available with this homework. There are two versions of the data. One is seasonally unadjusted and the other is seasonally adjusted. The data cover the period, January 1919 to December 2006. Compute the first difference of the log of the data, to obtain the 1,055 monthly growth rates from February 1919 to December 2006.

   (a) Consider the seasonally unadjusted data first. Compute the sample periodogram for frequencies, $\omega_j = 2\pi j/T$, $j = 0, ..., (T-1)/2$ and graph the results. Note how jagged the curve is (throughout, you should only graph the log of the spectrum). This reflects the result (see Hamilton, p. 164) that $\hat{S}_y(\omega)$ and $\hat{S}_y(\omega')$ are approximately independent (for large T) for $\omega \neq \omega'$. Moreover, although $\hat{S}_y(\omega)$ is an unbiased estimator of the true spectrum, $S_y(\omega)$, its variance does not shrink to zero as $T \to \infty$. The lack of precision in the sample spectrum as an estimator of the spectral density is perhaps not surprising. The function being estimated (i.e., the spectrum as a function of frequency) is a high-dimensional object (there is a continuum of frequencies between 0 and π), and no assumptions are made about the structure of the underlying time series representation. This is an example of 'little input' implies 'little output'.

   (b) Now consider a more parametric way to estimate the spectrum of the seasonally unadjusted data (call this the 'ar estimator of the spectrum'). Use ordinary least squares to fit a 20 lag scalar ar representation to the data, with a constant term. Compute the spectral density of the resulting ar representation over the same range of frequencies used in (a). Graph the two spectral density estimators in the same picture. Note how one appears to be a smooth version of the other.

   (c) Note the local peaks in the spectrum. What period of oscillation do these correspond to?

(d) Apply the ar estimator of the spectrum to the seasonally adjusted data. Graph the spectrum of the adjusted and unadjusted data in the same figure. Are there 'dips' in the spectrum of the seasonally adjusted data, as we were led to expect based on the results for 'optimal seasonal adjustment' in the previous homework? Why or why not?

(e) An alternative strategy that is sometimes used to seasonally adjust monthly data is to regress the data on 12 seasonal dummies (don't include a constant term here, or you'll have perfectly collinear data!)[2] and treat the residual in this regression as the seasonally adjusted data. In the same graph, plot the estimated spectrum of the unadjusted data, the data adjusted by the US government and the data adjusted using the dummy method (in all cases, compute the spectrum using the ar method). Which is the more effective seasonal adjustment procedure, the dummy method or the US government's method?

[2]That is, let the right hand variables be in the T by 12 matrix, X. The i^{th} column of X has zeros everywhere and a unity in the i^{th} entry. A MATLAB routine that will set up X with the right structure is:

```
B=eye(12);
X=[];
for ii = 1:88
X=[X
B];
end
X=X([2:end],:);
```