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FINC 520, Spring 2009
Homework 6, due Thursday, May 21.

1. Denote the real interest rate by r_t, where $r_t \equiv i_t - \pi_t^e$, i_t denotes a quarterly interest rate and π_t^e represents the (unobserved) expected quarterly inflation rate over the period covered by i_t. For these calculations, use the data from homework 2. Suppose that the real interest rate evolves according to:

$$r_t = (1 - 0.95)\,\mu + 0.95 r_{t-1} + v_t,$$

where μ is the sample mean of the ex post real rate, $i_t - \pi_t$. Let v_t be a white noise with standard deviation, 0.0015. Assume the ex post real rate of interest, $i_t - \pi_t$, is related to r_t as follows:

$$i_t - \pi_t = r_t + w_t,$$

where w_t and v_t satisfy all the properties assumed for the Kalman filter (Hamilton, p. 376). Let the standard deviation of the white noise, w_t, be one-fifths of the sample standard deviation of $i_t - \pi_t$. Compute

$$\hat{r}_{t|T} = \hat{E}\left[r_t | i_1 - \pi_1, ..., i_T - \pi_T\right],$$

for $t = 1, ..., T$ using the Kalman filter. Also, compute $P_{t|T}$ for $t = 1, ..., T$. Place four graphs in one figure: the ex post realized real rate, $i_t - \pi_t$; the estimated ex ante real rate, $\hat{r}_{t|T}$; and $\hat{r}_{t|T}$ plus/minus 1.96 times $\sqrt{P_{t|T}}$, for $t = 1, ..., T$.

2. Consider the following time series representation:

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1},\ \theta = 2,\ \sigma_\varepsilon^2 = 1. \tag{1}$$

   (a) Compute the alternative time series representation for y_t, which has the same covariance function as above, in which the moving average root has been 'flipped':

$$y_t = u_t - \mu u_{t-1},\ Eu_t^2 = \sigma_u^2. \tag{2}$$

   Derive and display μ and σ_u^2.

(b) Verify that, by recursive substitution, one can write (2) as:

$$y_t = \sum_{i=0}^{\infty} \phi_i y_{t-i} + u_t, \tag{3}$$

where $\{\phi_i\}$ is a square summable sequence. Prove that this expression represents the linear projection of y_t onto its infinite past history. Because u_t is the error in this relation, the representation of y_t is sometimes referred to as its 'fundamental' representation (or, 'Wold representation'). The other representation, (1), is not 'fundamental' (careful, the economic and time series analysis term, 'fundamental', means different things). Motivated by (3), it is said that the shock in the fundamental representation 'lies in the space of past data', while the shock in the alternative representation does not. To see what space ε_t lies in, show by recursive substitution that ε_t can be represented as follows:

$$\varepsilon_t = -\sum_{i=1}^{\infty} \left(\frac{1}{\theta}\right)^i y_{t+i},$$

so that ε_t lies in the space of (*all*) future y_t's. [1]

[1] It is interesting to note that the recursive substitution corresponds to simple manipulation of lag operators:

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1} = (1 - \theta L)\,\varepsilon_t,$$

so

$$\varepsilon_t = \frac{1}{1-\theta L} y_t.$$

A problem is that the expansion of the above polynomial in positive powers of L has explosive coefficients. However, the polynomial can also be expanded in negative powers of L

$$\frac{1}{1-\theta L} = \frac{-\theta^{-1}L^{-1}}{1-\theta^{-1}L^{-1}},$$

so that

$$\begin{aligned}
\varepsilon_t &= \frac{-\theta^{-1}L^{-1}}{1-\theta^{-1}L^{-1}} y_t = \frac{-\theta^{-1}}{1-\theta^{-1}L^{-1}} y_{t+1} \\
&= -\frac{1}{\theta}\left[y_{t+1} + \frac{1}{\theta} y_{t+2} + \left(\frac{1}{\theta}\right)^2 y_{t+3} + \ldots\right].
\end{aligned}$$

(c) The reasoning in (a) and (b) suggests that $\hat{\varepsilon}_{t|T}$ will look very different from $\hat{\varepsilon}_{t|t-1}$. Since ε_t does not lie in the (infinite) space of past y_t's the error of the projection, $\hat{\varepsilon}_{t|t-1}$ - denoted $P_{t|t-1}$ - can be expected to be just σ_ε^2. By contrast, the error in the projection, $\hat{\varepsilon}_{t|T}$ - denoted $P_{t|T}$ - should be quite small, except for t close to T. Write the non-fundamental representation, (1), in state space/observer form (hint, let the state be $\xi_t = \begin{pmatrix} y_t & \varepsilon_t \end{pmatrix}'$, and let $H' = \begin{pmatrix} 1 & 0 \end{pmatrix}$). Program the formulas for the forecast error variances in MATLAB and display a graph of $P_{t|t}$, $P_{t|T}$, $P_{t|t-1}$, for $t = 1, ..., T$. Set $T = 10$. Now display the same graph, but for the fundamental representation of y_t. Are the results in the graphs consistent with what intuition suggests?

(d) Consider the state-space/observer representation of the non-fundamental representation of y_t. Iterate on the recursive expression for $P_{t|t-1}$ to obtain:

$$P = \lim_{t \to \infty} P_{t|t-1}.$$

The $1,1$, element of P is the one-step-ahead variance of the error in forecasting y_t given $y_{t-1}, y_{t-2}, ...$. Does the quantitative magnitude of this one-step-ahead forecast error variance make sense in light of your results in (b)? Confirm that the $1,1$ element of P is unchanged if you construct the state-space/observer system using the fundamental representation of y_t. What happens to the other elements of P? State your findings in intuitive terms.

3. Consider the following system:

$$\begin{aligned} y_t &= \xi_t + w_t \\ \xi_t &= \rho \xi_{t-1} + \upsilon_t \quad |\rho| < 1. \end{aligned}$$

Consider:

$$\hat{\xi}_{t|t} = P\left[\xi_t | y_1, ..., y_t\right].$$

A class handout establishes, using spectral methods and the Wold decomposition theorem, that

$$\lim_{t \to \infty} \hat{\xi}_{t|t} = \frac{\sigma_\upsilon^2 / \sigma_w^2}{1 - \rho\lambda} \sum_{j=0}^{\infty} \lambda^j y_{t-j}, \tag{4}$$

where σ_w^2 denotes the variance of

$$y_t - P\left[x_t | x_{t-1}, x_{t-2}, ..x_1\right]$$

as $t \to \infty$. Here, the meaning of lim, $t \to \infty$ is that the period, '1' is infinitely far in the past. Also, λ is the moving average representation parameter in the Wold representation for y_t. Construct $\hat{\xi}_{t|t}$ using the Kalman filter and show, driving $t \to \infty$, that the Kalman filter formula coincides with the right side of (4).

4. Generate a time series of 10 observations from

$$y_t = \varepsilon_t + \frac{1}{2}\varepsilon_{t-1}, \ \ \varepsilon_t \tilde{} N(0,1).$$

Use the data to form the Gaussian density function $y_1, ..., y_{10}$. Denote the log Gaussian density by $\mathcal{L}(y; \theta, \sigma_\varepsilon^2)$, where θ denotes the moving average parameter (the true value is 1/2) and σ_ε^2 denotes the variance of ε_t.

   (a) Compute the concentrated log-likelihood,

$$\mathcal{L}^c(y; \theta) = \max_{\sigma_\varepsilon^2} \mathcal{L}\left(y; \theta, \sigma_\varepsilon^2\right).$$

   Graph $\mathcal{L}^c(y; \theta)$ for $\theta \in (-3, 3)$. Note the slope of $\mathcal{L}^c(y; \theta)$ for $\theta = -1, 1$. Note too, the feature, $\mathcal{L}^c(y; \theta) = \mathcal{L}^c(y; 1/\theta)$. What is the value of $\theta \in (-1, 1)$ that maximizes $\mathcal{L}^c(y; \theta)$? Denote this by $\hat{\theta}$.

   (b) Generate 1000 more data sets, each of length 10. Compute $\hat{\theta}$ for each data set. What is the bias of your estimator (i.e., 1/2 minus the average over all 1,000 $\hat{\theta}$'s)? Display the histogram of $\hat{\theta}$'s. Does the sampling distribution of the $\hat{\theta}$'s look normal (i.e., is the distribution symmetric; do 95 percent of the $\hat{\theta}$'s lie within the mean of the $\hat{\theta}$'s plus/minus 1.96 times the standard deviation of the estimator of $\hat{\theta}$?)