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FINC 520, Spring 2007
Midterm Exam.
Here are some trigonometric properties that you may find useful:

$$\begin{aligned} \sin(k\pi) &= 0, \text{ integer } k, \ \sin(\pi/2) = 1 \\ \cos(0) &= 1, \ \cos(\pi/2) = 0, \ \cos(\pi) = -1 \\ \exp(i\omega) &= \cos(\omega) + i \sin(\omega). \end{aligned}$$

There are 100 points possible on this exam. The number of points allocated to each question are indicated, so you can allocate your time accordingly.

1. (10) Suppose $Y$ and $X$ are random variables and $\Omega$ is a collection of random variables. Show that

$$P[Y|\Omega, X] = P[Y|X] + P[Y - P(Y|\Omega)|X - P(Y|\Omega)],$$

where $P[w|z]$ is the linear projection of $w$ onto the set of variables, $z$.

2. (5) Consider the following moving average representation:

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \ \sum_{j=0}^{\infty} \psi_j^2 < \infty,$$

where $\varepsilon_t$ is a finite-variance white noise and

$$\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} = \lim_{q \to \infty} \sum_{j=0}^{q} \psi_j \varepsilon_{t-j}.$$

State what it means for this limit to be well defined, and prove that it is well defined (you may use a theoretical result if you wish, but be sure to state it carefully.)

3. (5) Suppose

$$\varepsilon_t = x_t - P[x_t|x_{t-1}, x_{-2}, ...].$$

Show that $\varepsilon_t$ and $\varepsilon_s$ are uncorrelated for all $t \neq s$.

4. (15) Suppose

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1},\ \ \theta > 1,$$

where $\varepsilon_t$ is a white noise with variance $E\varepsilon_t^2 = \sigma_\varepsilon^2$.

(a) Show that $y_t$ has the following AR($\infty$) representation

$$A(L)\, y_t = u_t,$$

where $u_t$ is a white noise process with variance $Eu_t^2 = \sigma_u^2$ and

$$A(L) = 1 - \phi_1 L - \phi_2 L^2 - ...$$

Display formulas for the $\phi_i$'s and for $\sigma_u^2$ in terms of $\theta$ and $\sigma_\varepsilon^2$.

(b) Consider the Gaussian density function for $y_t$ after the variance of the white noise driving process has been concentrated out. Denote this concentrated density by $L(\mu)$, where $\mu$ is the first order moving average coefficient. Prove that $L'(1) = L'(-1) = 0$.

5. (15) Suppose

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},\ \ \sum_{j=0}^{\infty} \psi_j^2 < \infty,\ \ \psi_0 = 1,$$

where $\varepsilon_t = y_t - P[y_t|y_{t-1}, y_{t-2}, ...]$. Prove that

$$y_t - P[y_t|y_{t-2}, y_{t-3}, ...] = \varepsilon_t + \psi_1\varepsilon_{t-1}.$$

6. (25) Consider a covariance stationary process, $y_t$, with spectral density, $S_y(\omega)$, $\omega \in (0, \pi)$. Consider the time series representation,

$$z_t = (1 - L)\, y_t,$$

obtained by first differencing $y_t$.

(a) Display an expression involving only sines and cosines, which shows what first differencing does to the spectrum of $y_t$ at different frequencies.

(b) Show by how much the first differencing operator multiplies the component of $y_t$ at the highest frequency.

(c) The business cycle is composed of cycles with period 8 quarters or longer. What frequencies does this correspond to? Explain. Provide a formula that could be input into a calculator to answer this question.

(d) What does first differencing do to the component of $y_t$ at business cycle frequencies or below?

7. (25) The Spectral Decomposition Theorem instructs us to think of a covariance stationary process as being the sum of sinusoidal processes at different frequencies, with each process having a different amplitude and phase. Suppose a variable, $y_t$, is obtained by filtering $x_t$ as follows:

$$y_t = \sum_{j=-\infty}^{\infty} h_j x_{t-j}.$$

Application of the filter alters the amplitude and phase of the different frequency components of $x_t$. To understand how this works, it is useful to write the Fourier transform of $h(L)$ in polar form:

$$h\left(e^{-i\omega}\right) = s(\omega)\, e^{i\theta(\omega)}.$$

(a) Explain why $s(\omega)$ captures how the filter modifies the amplitude of the $\omega$ frequency component of $x_t$.

(b) Explain why $\theta(\omega)$ captures how the filter modifies the phase of the $\omega$ frequency component of $x_t$. (Hint: suppose that the component of $x_t$ at frequency $\omega$, $x_t^{\omega}$, is $2\cos(\omega t) = e^{-i\omega t} + e^{i\omega t}$. Work out the transformation from the $\omega$ frequency of $x_t$ to the $\omega$ frequency of $y_t$. You could do this for your answer to part a too.)

(c) Consider two filters, the first difference, $1 - L$, and the centered moving average, $(1 + L)(1 + L^{-1})/2$. Show that the first changes phase at all frequencies, while the second leaves phase unchanged at all frequencies. Provide intuition for this result?