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FINC 520, Spring 2008
Midterm Exam.

There are 100 points possible on this exam. The number of points allocated to each question are indicated, so you can allocate your time accordingly.

1. (12) Consider the stochastic process,

$$y_t^i = \mu^i + \varepsilon_t^i,\ \mu^i \tilde{} N\left(0, \lambda^2\right),$$

where i indexes the realization of the stochastic process, and $t = 0, 1, 2, ...$ denotes time. Also, $\varepsilon_t^i \tilde{} N(0, \sigma^2)$ is an *iid* random variable for each i and t.

   (a) Explain how you would construct the realizations of this stochastic process.

   (b) Compute the mean, variance and τ^{th} lag autocovariance of the stochastic process at time t.

   (c) Compute:

$$\bar{y}^i = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} y_t^i, \qquad \bar{\sigma}_i^2 = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \left[y_t^i - \bar{y}^i\right]^2$$

   for a particular i.

   (d) Is this process ergodic for the mean? Is it ergodic for the variance? Explain.

   (e) Is the process covariance stationary? Explain.

2. (25) Consider the following ARMA(p,q) process:

$$y_t = \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q},$$

where

$$z^p - \phi_1 z^{p-1} - ... - \phi_p = 0,$$

implies $|z| < 1$. Also, ε_t is *iid*, with variance $E\varepsilon_t^2 = \sigma^2$ and $Ey_t\varepsilon_{t-\tau} = 0$, $\tau > 0$.

(a) Show how this stochastic process can be written as a first order vector autoregressive process:

$$Y_t = FY_{t-1} + v_t.$$

In particular, construct F, Y_t and v_t.

(b) Set up a linear system of equations, the solution to which includes the variance-covariance matrix of Y_t (Hint: a linear system of equations is $X\beta = d$, where X is a matrix of known numbers, d is a vector of known numbers and β is to be solved for.)

(c) How can you recover the autocovariances, $Ey_t y_{t-\tau}$, $\tau = 0, ..., p-1$ from the variance covariance matrix of Y_t?

(d) What restrictions does $Ey_t\varepsilon_{t-\tau} = 0$, $\tau > 0$, imply for the variance covariance matrix of Y_t?

(e) Consider the moving average representation of the ARMA(p,q) process:

$$y_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + ...,$$

where ψ_j, $j = 1, 2, ...$ represents a sequence of scalars. Display a formula for the ψ_j's based on the vector representation of y_t. (If you need to make assumptions about the properties of the matrices you work with, explain why and then do so.)

3. (13) Properties of projections.

(a) Suppose Y is a scalar random variable and X is a column vector of random variables. Suppose that

$$E\left[Y - \delta' X\right] X' = 0,$$

for some vector of numbers, δ. Show that

$$\delta' X = P\left[Y|X\right],$$

where P is the linear projection operator.

(b) Prove that $P\left[a + b|X\right] = P\left[a|X\right] + P\left[b|X\right].$

(c) Prove that $P\left[y|x, z\right] = P\left[y|z\right]$, when $y \perp x$, $x \perp z$.

4. (25) Spectral analysis.

   (a) Show that

   $$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega h} d\omega = \begin{cases} 1, & h = 0 \\ 0, & h \neq 0 \end{cases},$$

   where $i^2 \equiv -1$ and e denotes the 'natural' number.

   (b) It can be shown that, for a general (covariance stationary) ARMA process like the one in question 2,

   $$g_y(z) = \frac{\theta(z)\,\theta(z^{-1})\,\sigma_\varepsilon^2}{\phi(z)\,\phi(z^{-1})} = \gamma_0 + \gamma_1\left(z + z^{-1}\right) + \gamma_2\left(z^2 + z^{-2}\right) + ...$$

   Here, $\gamma_\tau = Ey_t y_{t-\tau}$, $\tau \geq 0$. The expression between the two equality signs refers to standard multiplication and division of polynomials in various powers of z. Consider the AR case in which $\theta(z) = 1$ and $\phi(z) = 1 - \phi z$. Do the polynomial division in this case and verify the result by also computing $\gamma_0,\ \gamma_1, ...$ directly (hint: you could first multiply the two polynomials in the denominator and divide the result into the numerator, however an easier way is to divide each polynomial separately into the numerator and multiply the result).

   (c) Display a formula based on (a) for computing the moving average coefficients in 2 (e).

   (d) Display a formula based on (a) for computing γ_τ, $\tau \geq 0$.

5. (25) Following is a general, potentially two-sided, filter:

   $$F(L) = \sum_{\tau=-\infty}^{\infty} F_\tau L^\tau.$$

   (a) Consider the particular filter defined by:

   $$F\left(e^{-i\omega}\right) = \begin{cases} 1 & \omega \in D \\ 0 & \omega \notin D. \end{cases}$$

   Display a formula for F_τ in this case.

(b) The Spectral Decomposition theorem says that we can think about a covariance stationary time series as being the sum of sinusoidal processes at different frequencies, with each process having a different amplitude and phase. A crude example of this is given by the process, y_t :

$$\begin{aligned} y_t &= \frac{\alpha}{2}\left[e^{-i\omega_1 t}+e^{i\omega_1 t}\right]+\frac{\beta}{2}\left[e^{-i\omega_2 t}+e^{i\omega_2 t}\right] \\ &= \alpha\cos\left(\omega_1 t\right)+\beta\cos\left(\omega_2 t\right), \end{aligned}$$

where $\alpha\cos(\omega_1 t)$ is one frequency component of y_t, $\beta\cos(\omega_2 t)$ is another, $\omega_1 \neq \omega_2$, and α, β are two mean-zero, independent random variables, with variance σ_α^2 and σ_β^2, respectively.

i. How many observations on a given realization of y_t would be required before you could predict the rest of the realization of y_t perfectly?

ii. Most covariance stationary processes cannot be predicted perfectly, no matter how much of a given realization has been observed. In general, how does the representation implied by the Spectral Decomposition differ from the simple one given above?

iii. What is Ey_t in the example? What is Ey_t^2? What is $Ey_ty_{t-\tau}$? Explain.

iv. Consider the band pass filter with $\omega_1 \in D$ and $\omega_2 \notin D$. Derive the x_t process, where

$$x_t = F(L)\,y_t,$$

and y_t is the process defined above. Show that

$$x_t = \alpha\cos\left(\omega_1 t\right).$$