Stochastic Simulation of a Nonlinear, Dynamic Stochastic Model

Lawrence J. Christiano^{*}

Martin S. Eichenbaum^{\dagger}

Mathias Trabandt[‡]

September 2, 2014

Abstract

We describe an algorithm for computing the equilibrium response of endogenous variables to a realization of shocks from a stochastic process. We illustrate the algorithm with two numerical experiments. These experiments are of interest in their own right and they illustrate the dynamic properties of the simple New Keynesian model when the zero lower bound on the interest rate is binding. In the first experiment, we show that with an empirically reasonable amount of persistence, a negative technology shock results in a decrease in consumption and employment. In the second set of experiments, we show that the effects of an increase in government spending in the zero lower bound are very similar, whether the increase must be financed by adjustments in distortionary or lump sum taxes. For the sake of simplicity, the second experiment is deterministic.

^{*}Northwestern University, Department of Economics, 2001 Sheridan Road, Evanston, Illinois 60208, USA. Phone: +1-847-491-8231. E-mail: l-christiano@northwestern.edu.

[†]Northwestern University, Department of Economics, 2001 Sheridan Road, Evanston, Illinois 60208, USA. Phone: +1-847-491-8232. E-mail: eich@northwestern.edu.

[‡]Board of Governors of the Federal Reserve System, Division of International Finance, Trade and Financial Studies Section, 20th Street and Constitution Avenue N.W., Washington, D.C. 20551, USA, E-mail: mathias.trabandt@gmail.com.

1. Introduction

The objective of this document is two-fold:

- 1. We describe the nonlinear stochastic simulation method implemented in Christiano-Eichenbaum-Trabant, 'Understanding the Great Recession'. The method corresponds to the extended path method proposed by Fair and Taylor (1989) (see also Gagnon and Taylor, 1990).
- 2. We work two examples with the code. Both use the simple New Keynesian model without capital. The first example is fully stochastic and the second is not. The code for the examples is available online. Following is a brief description of the two examples:
- We investigate the inflation and output effect of a negative technology shock that hits while the economy is in a binding zero lower bound (ZLB). We examine how that impact changes as the degree of persistence in the technology is increased. When technology has the degree of persistence usually assumed in the Real Business Cycle literature, then consumption and employment fall with the negative technology shock.
 - We examine the economic effects of an expansionary government consumption shock while the economy is in the zero lower bound. We find that results for increasing welfare and output are robust to whether shocks to the government budget constraint are financed with distortionary or lump sum taxes.

The first section below describes the computational strategy. The next section describes the model used in the experiments. After that we report on the two numerical examples.

2. Stochastic Simulation of the Model

We denote the vector of exogenous shocks realized at time t by y_t . The $N \times 1$ vector of endogenous variables whose values are determined at time t is denoted by z_t . Time starts at time t = 1, when z_0 is given. We draw a sequence, $y_t, ..., y_T$, from a time series representation, and we compute the responses of the endogenous variables, $z_1, ..., z_T$, subject to the equilibrium conditions, which we specify below. Our method adopts one approximation, that the system satisfies certainty equivalence, so that objects like Ef(x) are replaced with f(Ex).¹

¹See Adjemian and Juillard (2013) for a strategy to improve on the certainty equivalence assumption. The extended path method is implemented in Dynare's command, extended_path. When the parameter, order, is set to zero then the type of calculations described in this manuscript are implemented. When order > 0, then the approach described by Adjemian and Juillard is implemented.

It is convenient to define the *nonstochastic steady state* values of z_t and y_t . We denote these by z and y, respectively. In particular, $y = \lim_{j\to\infty} E_T y_{t+j}$ and z solves:

$$v\left(z, z, z, y, y\right) = 0,$$

where v is an $N \times 1$ vector of functions that contains the steady state version of the equilibrium conditions in non-stochastic steady state.

The system starts at time 0 when z_{-1} is given and z_0 is determined. The time t realized value of $z_t, t \ge 0$, is a function of time t information, Ω_t , where

$$\Omega_t = \{z_0, z_1, ..., z_{t-1}, y_1, ..., y_t\}.$$

Given Ω_t , the $N \times 1$ vector of endogenous variables, z_t , must satisfy

$$E_t v\left(z_{t-1}, z_t, z_{t+1}, y_t, y_{t+1}\right) = 0,$$

where v denotes the equilibrium conditions. According to our certainty equivalence approximation,

$$E_t v(z_{t-1}, z_t, z_{t+1}, y_t, y_{t+1}) \approx v(z_{t-1}, z_t, z_{t+1}^t, y_t, y_{t+1}^t)$$

where x_{t+j}^t denotes the conditional expectation of x_{t+j} given Ω_t . Thus, we require

$$v\left(z_{t-1}, z_t, z_{t+1}^t, y_t, y_{t+1}^t\right) = 0.$$
(2.1)

At time t it must also be that

$$E_t v\left(z_t, z_{t+1}, z_{t+2}, y_{t+1}, y_{t+2}\right) = 0,$$

or, using certainty equivalence,

$$v\left(z_{t}, z_{t+1}^{t}, z_{t+2}^{t}, y_{t+1}^{t}, y_{t+2}^{t}\right) = 0.$$
(2.2)

More generally, at time t is is required that, for all $j \ge 0$,

$$v\left(z_{t+j-1}^{t}, z_{t+j}^{t}, z_{t+j+1}^{t}, y_{t+j}^{t}, y_{t+j+1}^{t}\right) = 0,$$
(2.3)

where it is understood that $z_{t-1}^t \equiv z_{t-1}$, $z_t^t \equiv z_t$. We assume that there exists a $T^* > T$ such that $z_{T^*}^T = z$ and $y_{T^*}^T = y$, where z and y are defined above.

Motivated by the above expressions, we compute $z_1, ..., z_T$ as follows. Suppose that we have computed $z_1, ..., z_{t-1}$ as functions of $\Omega_1, ..., \Omega_{t-1}$, respectively. We compute z_t as a function of Ω_t in two steps. First, we compute the date t forecast of y_{t+j} , for j > 0 using the information in Ω_t and the time series representation for y_t . We denote these forecasts by $y_{t+j}^t, j = 1, 2, ..., y_{T^*}^t$.

In our second step, we use the forecasts for the exogenous shocks and (2.1)-(2.3) to solve for z_t , z_{t+1}^t , z_{t+2}^t ,..., $z_{T^*}^t$, where $T^* > T$. To this end, we define the following stacked system:

$$v\left(z_{t-1}, z_t, z_{t+1}^t, y_t, y_{t+1}^t\right) = 0$$

$$v\left(z_t, z_{t+1}^t, z_{t+2}^t, y_{t+1}^t, y_{t+2}^t\right) = 0$$

$$\vdots$$

$$v\left(z_{T^*-3}^t, z_{T^*-2}^t, z_{T^*-1}^t, y_{T^*-2}^t, y_{T^*-1}^t\right) = 0$$

$$v\left(z_{T^*-2}^t, z_{T^*-1}^t, z, y_{T^*-1}^t, y\right) = 0.$$

This represents T^*-t systems of N equations in the $N(T^*-t)$ unknowns, $z_t, z_{t+1}^t, z_{t+2}^t, ..., z_{T^*-1}^t$. The value of T^* should be chosen so that the above equations actually have a solution with $z_{T^*}^t = z$. The equations can be solved relatively quickly using a numerical routine for solving a system of nonlinear equations with multiple variables. But, it is important to take advantage of the structure of the equations. To see this, express the equations to be solved as follows:

$$V(\gamma) = \begin{pmatrix} v\left(z_{t-1}, z_t, z_{t+1}^t, y_t, y_{t+1}^t\right) \\ \vdots \\ v\left(z_{T^*-2}^t, z_{T^*-1}^t, z, y_{T^*-1}^t, y\right) \end{pmatrix}, \quad \gamma = \begin{pmatrix} z_t \\ z_{t+1}^t \\ \vdots \\ z_{T^*-1}^t \end{pmatrix}, \quad (2.4)$$

where z_{t-1} , z, y_t and y are taken as given. We seek γ^* such that $V(\gamma^*) = 0$. A gradient method for doing so computes a sequence, $\gamma_1, \gamma_2, \dots$ that is guaranteed to converge to γ^* as long as the initial guess, γ_0 , is sufficiently close to γ^* . Thus, suppose $\gamma_0, \dots, \gamma_{r-1}$ are given and we seek the next vector of parameters, γ_r , in the sequence. Let

$$V(\gamma) \simeq \hat{V}_{r}(\gamma) \equiv V_{t}(\gamma_{r-1}) + V'(\gamma_{r-1})(\gamma - \gamma_{r-1}),$$

where

$$V'\left(\gamma_{r-1}\right) = \frac{dV\left(\gamma_{r-1}\right)}{d\gamma'_{r-1}},$$

so that V' is a square, $(T^* - 1) N \times (T^* - 1) N$, matrix with a block-Toeplitz pattern, and composed mostly zeros. The computational time required for the algorithm is reduced substantially by taking into account the structure of $V'(\gamma_{r-1})$ when computing its matrix inverse. The value of γ_r is the value of γ such that $\hat{V} = 0$. That is,

$$\gamma = \gamma_{r-1} - \left[V'\left(\gamma_{r-1}\right) \right]^{-1} V\left(\gamma_{r-1}\right)$$

We have found that the MATLAB routine, follow, works well in the application reported below and in the much larger model studied in 'Understanding the Great Recession'.²

 $^{^{2}}$ The numerical problem might appear to be amenable to solution by 'shooting' methods. We did not have success with these methods.

After completing the calculations for $(z_t, z_{t+1}^t, z_{t+2}^t, ..., z_{T^*}^t)$, we (i) increment the value of t by one; (ii) compute the forecasts, y_{t+j}^t , j = 2, 3, ..., using y_{t+1} ; and (iii) use the t+1 version of (2.4) to solve for $(z_{t+1}, z_{t+2}^{t+1}, z_{t+3}^{t+1}, ..., z_{T^*}^{t+1})$ The computationally intensive step is (iii), where it is valuable to have a good initial guess, γ_0 , for solving (2.4). In practice, one can use the steady state values of the variables to initiate the calculations for t = 1 in (2.4). For values of t > 1 we set γ_0 to γ^* from the t - 1 version of (2.4). We proceed in this way until we have obtained the objects sought, $z_1, ..., z_T$.

3. Model

We first describe the equilibrium conditions associated with the private sector. We then describe monetary and fiscal policy.

3.1. Private Sector Equilibrium Conditions

The period t intertemporal first order condition of the representative household is:

$$u_{c,t} = \frac{1}{1+r_t} E_t u_{c,t+1} \frac{R_t}{1+\pi_{t+1}},$$
(3.1)

where $u_{c,t}$ denotes the marginal utility of consumption, C_t , in period t. Also, r_t denotes the rate at which the household at time t discounts time t+1 utility. The conventional assumption is that $\beta = 1/(1 + r_t)$ is a constant over time, but in the first example below we allow for the possibility that the discount rate is time varying. Also, $1 + \pi_{t+1} = P_{t+1}/P_t$, where P_t denotes the time t money price of the consumption good. The household's intratemporal Euler equation is:

$$\frac{mrs_t}{1-\tau_t} = \frac{W_t}{P_t}$$

where τ_t denotes the labor tax rate and mrs_t denotes the marginal rate of substitution, $-u_{n,t}/u_{c,t}$. The object, mrs_t , is the marginal cost of working, in consumption units. The object on the right of the equality is the real wage earned by the household, i.e., the nominal wage, W_t , divided by the price level.

Final output, Y_t , is produced by a representative, competitive firm using the following production function:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

The firm selects Y_t and $Y_{i,t}$, $i \in (0, 1)$ to maximize profits,

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} dj,$$

subject to given output and input prices. The first order necessary conditions associated with an interior solution to this optimum problem are:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}}\right)^{\varepsilon}, \ i \in (0,1).$$

Substituting the latter into the production function implies:

$$P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{1}{1-\varepsilon}}$$

The i^{th} intermediate good, $Y_{i,t}$, is produced by a monopoly producer using the following production function:

$$Y_{i,t} = \exp\left(a_t\right) N_{i,t},$$

where the time series representation of a_t has been given above and $N_{i,t}$ denotes the amount of labor used by the i^{th} monopolist.

Because of the monopolist's linear technology of production and the assumption that he hires labor in a competitive labor market, the marginal cost of production is

$$s_t = \frac{\left(1 - \nu\right) W_t / P_t}{e^{a_t}}$$

for each *i*. In the above expression, ν is a subsidy paid by the government (and, financed by a lump sum tax on households) in an effort to correct the distortions due to monopoly power and labor taxation in the steady state. Note that in this model wages in the labor market are completely flexible and the cost of working by households is always equated to the corresponding benefit provided by the market. To see the implication for ν , recall that efficiency in steady state requires that the cost of working is equated to the social marginal product of work, namely e^{a_t} . In steady state, s_t is equated by monopoly firms to the inverse of the markup, $(\varepsilon - 1)/\varepsilon$. Thus,

$$s_t = \frac{\frac{W_t}{P_t} (1 - \nu)}{M P_{L,t}} = \frac{\frac{W_t}{P_t} (1 - \nu)}{e^{a_t}} \stackrel{\text{household Euler}}{\stackrel{\frown}{=}} \frac{\frac{1 - \nu}{1 - \tau_t} m r s_t}{e^{a_t}} \stackrel{\text{in steady state}}{\stackrel{\frown}{=}} \frac{\varepsilon - 1}{\varepsilon}.$$
 (3.2)

From this expression it is evident that we require

$$\frac{1-\nu}{1-\tau} = \frac{\varepsilon - 1}{\varepsilon}$$

Since monopolists all face the same marginal cost, s_t , if prices were flexible and τ were constant at its steady state value, then the employment and consumption allocations are first best, conditional on the value of government consumption, G_t . However, we assume that prices are not set flexibly. Instead, the i^{th} intermediate good can only set his price, $P_{i,t}$, with probability θ . In particular, we suppose that the monopolist must satisfy the Calvo price distortions:

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1 - \theta \\ P_{i,t-1} & \text{with probability } \theta \end{cases},$$

where \tilde{P}_t denotes the price that the monopolist sets in case he is able set his price in period t.Under these conditions, the expression that relates total purchases of goods to aggregate employment, technology and price dispersion is as follows (see, e.g., Tack Yun, 1996):³

$$G_t + C_t = p_t^* e^{a_t} N_t \tag{3.3}$$

$$p_t^* = \left[(1-\theta) \left(\frac{1-\theta \left(1+\pi_t\right)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \theta \frac{\left(1+\pi_t\right)^{\varepsilon}}{p_{t-1}^*} \right]^{-1}$$
(3.4)

The conditions associated with optimal price setting must be adjusted slightly to accommodate our assumption that the discount rate of households, r_t , is potentially time varying. As usual, the prices of the $1 - \theta$ intermediate good producers that have the right to optimize their price are set as a function of current and future marginal costs, s_t . In particular,

$$\tilde{p}_t = \frac{K_t}{F_t},$$

where 4

$$K_t = E_t \sum_{j=0}^{\infty} \beta_{t,j} \theta^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}.$$

Here,

$$\beta_{t,j} = \begin{cases} \frac{1}{1+r_t} \frac{1}{1+r_{t+1}} \cdots \frac{1}{1+r_{t+j-1}} & j \ge 1\\ 1 & j = 0 \end{cases}, \ \beta_{t,j} = \frac{1}{1+r_t} \beta_{t+1,j-1} \\ X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+1}} \frac{1}{\bar{\pi}_{t+2}} \cdots \frac{1}{\bar{\pi}_{t+j}} & j \ge 1\\ 1 & j = 0 \end{cases}, \ X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}}. \end{cases}$$

Then, using (3.2) to substitute out for s_t ,

$$K_t = (1-\nu)\frac{\varepsilon}{\varepsilon - 1}\frac{mrs_t}{(1-\tau_t)e^{a_t}} + \frac{1}{1+r_t}\theta E_t \left(1+\pi_{t+1}\right)^{\varepsilon} K_{t+1}.$$
(3.5)

Similarly,

$$F_t = 1 + \frac{1}{1 + r_t} \theta E_t \left(1 + \pi_{t+1} \right)^{\varepsilon - 1} F_{t+1}.$$
(3.6)

Finally,

$$\frac{K_t}{F_t} = \left[\frac{1-\theta\left(1+\pi_t\right)^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}.$$
(3.7)

³For a derivation in the case, $\beta_{t,j} = \beta^j$, see http://faculty.wcas.northwestern.edu/~lchrist/course/Korea_2012/intro_NK.j ⁴For a derivation in the case, $\beta_{t,j} = \beta^j$, see http://faculty.wcas.northwestern.edu/~lchrist/course/Korea_2012/intro_NK.j

3.2. Monetary and Fiscal Policy

The flow government budget constraint is:

$$\nu W_t N_t + P_t G_t + (1 + R_t) B_t^g = T_t^g + \tau_t W_t N_t + B_{t+1}^g,$$

where T_t^g denotes (negative of) lump sum transfers. The household's budget constraint is:

$$P_t C_t + B_{t+1} = (1 - \tau_t) W_t N_t + R_{t-1} B_t + \underbrace{T_t}^{\text{profits} - T_t^g},$$

where

$$T_t = \text{profits} - T_t^g$$
.

We consider two policy regimes. In the 'distortionary tax regime', the labor tax rate is adjusted to enforce the government's intertemporal budget constraint and the lump sum tax is an exogenous stochastic process. In the 'lump sum tax regime' the (real value of the) lump sum tax is adjusted to satisfy the flow budget constraint with a constant real value of the debt, while the distortionary labor tax is an exogenous stochastic process.

In the distortionary tax regime, the rule for adjusting τ_t so that the government's intertemporal budget constraint is satisfied is as follows:

$$\tau_t = \tau + \eta \left(\Omega_t - b^g\right),\tag{3.8}$$

where η controls the response of the tax to the discrepancy between an average of past real value of government debt and a target level of the real debt, b^g . The average of past real debts is given by the following expression:

$$\Omega_t = \left(\frac{B_t^g}{P_{t-1}}\right)^{(1-\omega)} \Omega_{t-1}^{\omega}, \ 0 < \omega \le 1.$$
(3.9)

Divide the government budget constraint by P_t :

$$\nu \frac{W_t}{P_t} N_t + G_t + \frac{R_{t-1}}{\pi_t} b_t^g = t_t^g + \tau_t \frac{W_t}{P_t} N_t + b_{t+1}^g$$

where

$$\pi_t \equiv \frac{P_t}{P_{t-1}}, \ b_{t+1}^g \equiv \frac{B_{t+1}^g}{P_t}, \ t_t^g \equiv \frac{T_t^g}{P_t}.$$

Note that b_t^g is a state variable at time t. We can rewrite the budget constraint as follows:

$$\overbrace{\left(G_{t} - \left[\left(\tau_{t} - \nu\right)\overbrace{\left(\frac{mrs_{t}}{1 - \tau_{t}}\right)}^{\mathbb{Primary government deficit}} N_{t} + t_{t}^{g}\right]\right)}^{\mathbb{Primary government deficit}} + \frac{R_{t-1}}{\pi_{t}} b_{t}^{g} = b_{t+1}^{g}.$$
(3.10)

This expression determines the real value of the government debt, which in turn affects taxes by way of the tax rule.

The final equilibrium condition is provided by the monetary policy rule:

$$R_{t} = \max\left\{1, \ \overbrace{R^{ss}\left(\frac{R_{t-1}}{R^{ss}}\right)^{\rho_{R}}\left[\left(1+\pi_{t}\right)^{\phi_{1}}\left(\frac{C_{t}}{C^{ss}}\right)^{\phi_{2}}\right]^{1-\rho_{R}}}^{Z_{t}}\right\}.$$
(3.11)

We say the zlb is 'strictly binding' in period t if $Z_t < 1$ and is 'marginally binding' if $Z_t = 1$. The zlb is 'nonbinding' if $Z_t > 1$.

3.3. Equilibrium Conditions

In the distortionary tax regime, we have ten equilibrium conditions. The first six are the equilibrium conditions of the standard NK economy with lump sum taxes: (3.1), (3.3), (3.4), (3.5), (3.6), (3.7). There are three equations associated with distortionary taxes: (3.8), (3.9), (3.10). Finally, there is the equation associated with monetary policy, (3.11). The ten endogenous variables are:

$$K_t, F_t, \pi_t, C_t, p_t^*, N_t, R_t, \tau_t, b_{t+1}^g, \Omega_t.$$

In the lump sum tax regime, $b_{t+1}^g = \Omega_t = b^g$ for all t and t_t^g is determined by (3.10). In this case, the government budget constraint and taxes play no role in the equilibrium allocations and can be solved for after the equilibrium allocations have been computed. So, there are effectively 7 endogenous variables in this case,

$$K_t, F_t, \pi_t, C_t, p_t^*, N_t, R_t,$$

with seven equilibrium conditions. These are composed of the six equilibrium conditions associated with the standard NK economy with lump sum taxes plus the monetary policy rule.

4. The Impact of a Technology Shock While in the Zero Lower Bound

We now consider our first computational experiment. We examine the effects of a negative technology shock when the economy is in the zero lower bound. Such a shock generates two effects. First, by raising marginal costs it results in higher inflation because of sticky prices. Because the interest rate is at its lower bound, the real interest rate is reduced. Other things the same, this implies that consumption must rise. However, there is also a wealth effect which, if sufficiently large, can be expected to drive consumption down. The wealth effect is obviously larger, the greater is the degree of persistence in the technology shock. We show that with a fairly standard degree of persistence, consumption falls. For this example, we can ignore distortionary taxes and the government budget constraint.

The model has four exogenous shocks, r_t , t_t^g , G_t and a_t . We follow Eggertsson and Woodford in positing that r_t is the realization of a two-state Markov chain in which the larger of the two possible values of r_t , namely r^h , is an absorbing state. The system starts in period 1 with $r_t = r^l < r^h$, and remains in that state with constant probability, q. The forecast of r_{t+j} , j > 0, is computed as follows:

$$E_t \left[r_{t+j} | r_t = r^l \right] = \left[\begin{array}{cc} 0 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 1-q & q \end{array} \right]^j \left(\begin{array}{c} r^h \\ r^l \end{array} \right), \tag{4.1}$$

for j = 1, 2, When $r_t = r^h$, then the forecast of r_{t+j} , j > 0, is simply r^h itself. Also, $\lim_{j\to\infty} E_t r_{t+j} = r^h$ for this process. The time series representation for the log of technology, a_t , is assumed to be:

$$a_t = (\rho_1^a + \rho_2^a) a_{t-1} - \rho_1^a \rho_2^a a_{t-2} + \varepsilon_t^a.$$
(4.2)

Note that the parameters are specified in terms of the roots of the characteristic equation associated with the autoregression. In this case, $\lim_{j\to\infty} E_t a_{t+j} = 0$.

For this experiment we assume that the economy is in the lump-sum tax regime. Let the time t endogenous variables of the system be denoted:

$$z_t = \begin{pmatrix} C_t \\ p_t^* \\ N_t \\ \bar{\pi}_t \\ K_t \\ F_t \\ R_t \end{pmatrix}.$$

In terms of the discussion in the previous section, N = 7. Let the time t exogenous variables be denoted:

$$y_t = \left(\begin{array}{c} r_t \\ a_t \end{array}\right).$$

We adopt the following parameterization of u_c and mrs:

$$u_{c,t} = \frac{1}{C_t}, \ mrs_t = C_t N_t^{\varphi}.$$

We adopt the following model parameterization:

$$\rho_1=0.95,\ \rho_2=0.$$

Also,

$$\begin{array}{lll} \varepsilon & = & 6, \ \beta = 0.99, \ \rho_R = 0, \ \phi_{\pi} = 1.5, \ \phi_c = 0, \ \theta = 0.75, \ \sigma = 1, \ \varphi = 1, \\ q & = & 0.8, \ \nu = 1 - (\varepsilon - 1) \, / \varepsilon \end{array}$$

and

$$r = rac{1}{eta} - 1, \ r^l = -0.01$$

We set $r_t = r^l$ for t = 1, ..., 16 and $r_t = r$ for t > 16. In the case of a_t we set $a_{-1} = a_0 = 0$ and $\varepsilon_t = -0.1$ for t = 1 and $\varepsilon_t = 0$ for $t \neq 1$. In addition, we set $T^* = 116$.⁵ We perform the same simulation with $\varepsilon_1 = 0$, and we display both simultions. In both cases, the realization of r_t is the same.

The results are reported in Figure 1. Consider first the results for the case in which there is not disturbance in technology (solid line). Note that the equilibrium is characterized by sets of constants, as in the sort of linear approximations that Eggertsson and Woodford (2003) use to study the ZLB.⁶ The starred line shows what happens when technology drops by 10 percent in the initial period. Note that output is lower both in the ZLB, as well as afterward.

$$E_t \left[-\hat{C}_{t+1} + (\beta \left[R_t - r_t \right] - \pi_{t+1}) + \hat{C}_t \right] = 0.$$

We consider an equilibrium (as emerged in Figures 1-3) in which deviations from steady state are zero when the economy emerges from the ZLB and output and inflation are constant at $\hat{C}_t = \hat{Y}^l$ and $\pi_t = \pi^l$, respectively. Thus, in the ZLB,

$$\hat{Y}^l = \beta r^l + p\pi^l + p\hat{Y}^l.$$

Log-linearizing (3.5), (3.6) and (3.7) around steady state and rearranging, we obtain the usual Phillips curve:

$$\pi_{t} = \kappa \left(1 + \varphi\right) \hat{C}_{t} + \beta E_{t} \pi_{t+1}, \ \kappa \equiv \frac{\left(1 - \theta\right) \left(1 - \beta \theta\right)}{\theta}$$

In the ZLB, this reduces to:

$$\pi^{l} = \frac{(1-\theta)(1-\beta\theta)}{\theta} (1+\varphi) \hat{Y}^{l} + \beta p \pi^{l}$$

Solving, we obtain:

$$\pi^{l} = \frac{\kappa (1+\varphi) \beta r^{l}}{(1-p) (1-\beta p) - \kappa p (1+\varphi)}$$
$$\hat{Y}^{l} = \frac{\beta r^{l} + p \pi^{l}}{1-p}.$$

Interestingly, the values of π^l and \hat{Y}^l that solve these expressions are different from the values exhibited in Figures 1-3. For example, in our baseline parameterization, $(1-p)(1-\beta p) - \kappa p(1+\varphi) = -0.096$, so that $\pi^l > 0$. We are investigating why the reason for this difference.

⁵In this case, $a_{116}^1 = -0.000027429292657$, which we approximate by zero.

 $^{^{6}}$ It is interesting to compare some of the results in Figures 1-3 with what one obtains using the linearization solution strategy implemented in Eggertsson and Woodford (2003). The log-linear approximation of equation (3.1) around a zero inflation steady state is:

Figure 2 reproduces the same results reported in Figure 1, except that ρ_1 is now set to 0.5, so that there is less persistence in the technology shock. Note that now consumption actually rises in the ZLB. The intuition is that when there is a lot of persistence in a_t , then a negative shock to a_t is associated with a strong negative wealth effect (i.e., $\rho_1 = 0.95$), producing a decline in consumption and, hence, aggravating the drop in output in the zlb. When the wealth effect is more modest (as when $\rho_1 = 0.5$), then the response of consumption is dominated by a rate of return effect, an observation that has been stressed by Eggertsson (2010). In particular, the negative shock to technology raises the marginal cost of production and hence raises expected inflation. This in turn reduces the real interest rate (this is necessarily so because the nominal rate of interest is at its lowest possible level) and so encourages consumption and *reduces* the severity of the output drop in the zlb.

We also considered the parameterization, $\rho_1 = 0.95$ and $\rho_2 = 0.20$ in Figure 3. In this case, the wealth effect is even stronger than what it is in the experiment studied in Figure 1. Note that the fall in consumption is now greater, consistent with the wealth effect intuition sketched here.

5. The Impact of Government Spending In the Presence of Distortionary Taxes

We consider a second computational experiment. This experiment is deterministic and has to do with the economic effects in the zero lower bound when taxes are distortionary. Several papers have shown than when the zero lower bound is binding, then an increase in government spending can drive output up substantially and raise welfare. Much of that literature assumes that we are in the lump sum tax regime. Here, we consider the effects of government spending on output and welfare in the distortionary tax regime. Our results resemble those in Erceg and Linde (JEEA, 2014) who argue that results obtained under the assumption that taxes are lump sum are robust to the assumption that taxes are distortionary and must be adjusted to ensure intertemporal government budget balance.

The first subsection lays out the details - including utility function, parameter values and the nature of the shocks - of the experiment. The second subsection reviews the results of the computational experiments.

5.1. Preliminaries

We assume that in periods 1, 2, ..., 12, $r_t = r^l$ and $r_t = r^h$ for t > 12, where r_t appears in (3.1). We set $r^l = -0.005$ and $r^h = 1/\beta - 1$ for $\beta = 0.99$. We suppose that prior to t the system was in steady state and the period 1 drop in r_t was unanticipated. Although the period 1 drop in r_t was unanticipated, once it occurs the subsequent sequence of r_t 's is deterministic. We consider two scenarios. In one, G_t jumps in period 1 and returns to steady state. In the other, $G_t = G$ for all $t \ge 1$. The law of motion for G_t is:

$$\begin{aligned} G_t &= G^{1-0.8} G_{t-1}^{0.8} \exp\left(\varepsilon_t^g\right), \\ \varepsilon_1^g &= \sigma_{\varepsilon}, \ \varepsilon_t^g = 0, \ t > 1, \ G_0 = G \end{aligned}$$

We consider several different values of σ_{ε} , 0.2, 0.4, 0.6, 0.8. The difference (percent or otherwise) between the equilibrium in which G_t jumps and G_t does not jump defines the 'impulse response function to G_t '. Based on this impulse response function we can compute, for example, the multiplier associated with the jump in government spending.

We adopt the specification of utility used in Christiano, Eichenbaum and Rebelo (JPE, 2011):

$$u(c,N) = \frac{\left[C^{\gamma} \left(1-N\right)^{1-\gamma}\right]^{1-\sigma}}{1-\sigma} + \psi_{g} \frac{G^{1-\sigma}}{1-\sigma}$$

so that

$$u_c = \gamma C^{\gamma(1-\sigma)-1} \left(1-N\right)^{(1-\gamma)(1-\sigma)}, \ mrs = \frac{(1-\gamma)C}{\gamma(1-N)}.$$

We calibrate the ratio of G to output to be 20 percent. We calibrate N to be 0.33, so that the household works one-third of its available time of unity. We calibrate a value for ψ_g by requiring that the marginal utility of consumption be equal to the marginal utility of G in steady state:

$$\gamma C^{\gamma(1-\sigma)-1} \left(1-N\right)^{(1-\gamma)(1-\sigma)} = \psi_g G^{-\sigma}.$$
(5.1)

Implicitly, we assume that in the steady state the marginal social cost of G is one unit of consumption, and tax distortions are zero. If tax distortions were positive in the steady state, then optimality of G would imply that the marginal utility of government consumption would be higher than the marginal utility of private consumption. In this case, the value assigned to ψ_g would be higher. Thus, our calibration of ψ_g is 'conservative' because it depresses the implication of our model for the rise in welfare with a rise in G.

As noted above, we choose the subsidy to intermediate goods producers to be such that the monopoly and labor tax distortions disappear in steady state. Thus, there are three steady state conditions: (5.1), the requirement that the marginal rate of substitution between consumption and leisure (i.e., mrs) equals the marginal product of labor, and the resource constraint. The latter two are given, respectively, by:

$$\frac{(1-\gamma)C}{\gamma(1-N)} = 1, \tag{5.2}$$

$$C + G = N. \tag{5.3}$$

Note the absence of price distortions on the right side of the resource constraint, (5.3). This imposes that, in steady state, monetary policy drives inflation to zero. In addition, we have

imposed that the technology shock, a_t , is zero in steady state. We use equations (5.1)-(5.3) to compute C, γ and ψ_g . Then, the assumption, G/(C+G) = 0.20, allows us to compute G. Thus,

$$\gamma = 0.282655246252677, \ C = 0.264, \ \psi_a = 0.009057159317789, \ G = 0.066.$$

We set $\varepsilon = 6$ and $\tau = 1/3$, so that $\nu = 0.444$. The subsidy must be higher than the labor tax rate in steady state because it must not only undo the effects of that tax rate, but also of monopoly power. This setting for the subsidy rate is conservative from the point of view of our analysis, for two reasons. As we show below, in our calibration of the model an increase in government consumption in the zero lower bound raises welfare. If ν were smaller, say zero, then the monopoly power and labor tax rate distortions would be large in steady state. An unanticipated jump in G_t would ameliorate these distortions by reducing the markup of price over marginal cost.⁷ Thus, by setting the subsidy the way we do, we remove this welfare-based motive for raising G. There is a second reason that our setting of the subsidy rate is 'conservative' in light of our ultimate conclusion that a rise in G_t raises welfare. In particular, when $\nu > 0$ a rise in G_t produces an increase in government transfers to firms and these additional transfers add to the deadweight losses of the tax system.

In our calibration, we set the ratio of government debt to GDP, r^b , to 1/2. That is, $r^b \equiv b^g/(C+G) = 1/2$, so that $b^g = 0.165$. Then, from the steady state government budget constraint,

$$t^{g} = \left(r^{b}R + \left(\frac{\nu - \tau}{1 - \tau} + g\right)\right)N$$
$$= \left(r^{b}\left(\frac{1}{\beta} - 1\right) + \left(\frac{\nu - \tau}{1 - \tau} + g\right)\right)N$$
$$= 0.1227.$$

We set

 $\sigma=2,\ \eta=0.05,\ \beta=0.99,\ \phi_1=1.5,\ \phi_2=0,\ \rho_R=0,\ \theta=0.844956044718208,\ \omega=0.95.$

We follow convention in reporting welfare in consumption equivalents. In particular to evaluate the welfare gain of an increase in government spending we compute the permanent subsidy to consumption that makes the welfare in the equilibrium without government spending equal to the welfare of the equilibrium with government spending. Steady state welfare is:

$$V = \frac{\left[C^{\gamma} \left(1 - N\right)^{1 - \gamma}\right]^{1 - \sigma} + \psi_g \frac{G^{1 - \sigma}}{1 - \sigma}}{(1 - \sigma) \left(1 - \beta\right)},$$

⁷As the economy expands, P_t rises relatively little because of price setting frictions, while nominal marginal cost, W_t , rises relatively more with the expansion in labor demand.

and this is the same across the equilibria with and without the increase in G. Suppose that we solve for T observations, imposing that the system is in steady state after T (we set T = 412). Welfare in period 1 is:

$$V_1 = \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left[C_t^{\gamma} \left(1 - N_t\right)^{1-\gamma}\right]^{1-\sigma} + \psi_g G_t^{1-\sigma}}{1-\sigma} = \sum_{t=1}^{T} \beta^{t-1} \frac{\left[C_t^{\gamma} \left(1 - N_t\right)^{1-\gamma}\right]^{1-\sigma} + \psi_g G_t^{1-\sigma}}{1-\sigma} + \beta^T V.$$

The allocations in the equilibrium where $G_t = G$ are denoted by a star superscript. The present discounted value of utility in that equilibrium is V_1^* :

$$V_1^* = \sum_{t=1}^T \beta^{t-1} \frac{\left[(C_t^*)^{\gamma} \left(1 - N_t^* \right)^{1-\gamma} \right]^{1-\sigma} + \psi_g G^{1-\sigma}}{1-\sigma} + \beta^T V.$$

We now compute welfare in an equilibrium in which there is a permanent consumption subsidy, so that C_t is replaced by $C_t (1 + \lambda)$. The effect of λ on steady state equilibrium is denoted $V(\lambda)$, or,

$$V(\lambda) = \frac{\left[((1+\lambda)C)^{\gamma} (1-N)^{1-\gamma} \right]^{1-\sigma} + \psi_g G^{1-\sigma}}{(1-\sigma)(1-\beta)} \\ = \frac{(1+\lambda)^{\gamma(1-\sigma)} \left[C^{\gamma} (1-N)^{1-\gamma} \right]^{1-\sigma} + \psi_g G^{1-\sigma}}{(1-\sigma)(1-\beta)}.$$

Let $V_1^*(\lambda)$ denote the discounted utility of a permanent subsidy on consumption in the equilibrium in which $G_t = G$ for all t:

$$\begin{split} V_{1}^{*}\left(\lambda\right) &= \sum_{t=1}^{T} \beta^{t-1} \frac{(1+\lambda)^{\gamma(1-\sigma)} \left[(C_{t}^{*})^{\gamma} \left(1-N_{t}^{*}\right)^{1-\gamma} \right]^{1-\sigma} + \psi_{g} G^{1-\sigma}}{1-\sigma} + \beta^{T} V\left(\lambda\right) \\ &= (1+\lambda)^{\gamma(1-\sigma)} \sum_{t=1}^{T} \beta^{t-1} \frac{\left[(C_{t}^{*})^{\gamma} \left(1-N_{t}^{*}\right)^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} + \psi_{g} \frac{G^{1-\sigma}}{(1-\sigma) \left(1-\beta\right)} \\ &+ \beta^{T} \frac{(1+\lambda)^{\gamma(1-\sigma)} \left[C^{\gamma} \left(1-N\right)^{1-\gamma} \right]^{1-\sigma} + \psi_{g} G^{1-\sigma}}{(1-\sigma) \left(1-\beta\right)}. \end{split}$$

We wish to identify the value of λ having the property that

$$V_1^*\left(\lambda\right) = V_1.$$

Solving this for λ gives rise to

$$\begin{split} V_{1} &= (1+\lambda)^{\gamma(1-\sigma)} \sum_{t=1}^{T} \beta^{t-1} \frac{\left[(C_{t}^{*})^{\gamma} \left(1-N_{t}^{*}\right)^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} + \psi_{g} \frac{G^{1-\sigma}}{(1-\sigma) \left(1-\beta\right)} \\ &+ \beta^{T} \frac{(1+\lambda)^{\gamma(1-\sigma)} \left[C^{\gamma} \left(1-N\right)^{1-\gamma} \right]^{1-\sigma} + \psi_{g} G^{1-\sigma}}{(1-\sigma) \left(1-\beta\right)} \\ &= (1+\lambda)^{\gamma(1-\sigma)} \left[\underbrace{\sum_{t=1}^{T} \beta^{t-1} \frac{\left[(C_{t}^{*})^{\gamma} \left(1-N_{t}^{*}\right)^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} + \beta^{T} \frac{\left[C^{\gamma} \left(1-N\right)^{1-\gamma} \right]^{1-\sigma}}{(1-\sigma) \left(1-\beta\right)} } \right. \\ &+ \psi_{g} \frac{(1+\beta^{T}) G^{1-\sigma}}{(1-\sigma) \left(1-\beta\right)} \end{split}$$

or, after rearranging,

$$\begin{split} \lambda &= \left[\frac{V_1 - \psi_g \frac{(1+\beta^T)G^{1-\sigma}}{(1-\sigma)(1-\beta)}}{V_1^* - \psi_g \frac{G^{(1-\sigma)}}{(1-\beta)(1-\sigma)}} \right]^{\frac{1}{\gamma(1-\sigma)}} - 1 \\ &= \left[\frac{V_1 (1-\sigma) (1-\beta) - \psi_g (1+\beta^T) G^{1-\sigma}}{V_1^* (1-\beta) (1-\sigma) - \psi_g G^{1-\sigma}} \right]^{\frac{1}{\gamma(1-\sigma)}} - 1 \end{split}$$

We also compute various measures of the impact of the change in government consumption on output. We compute a static multiplier as the ratio,

$$\frac{Y_t - Y_t^*}{G_t - G}$$
, for $t = 1, ..., 10$.

This multiplier is not well defined for t must larger than 10 because eventually the numerator is zero, while the numerator remains non-zero for a while. Another measure of the multiplier is the long run multiplier stressed by Uhlig and Drautzburg and Uhlig. This is defined as follows:

$$\Psi_k = \frac{\sum_{t=1}^k m_t \left(Y_t - Y_t^* \right)}{\sum_{t=1}^k m_t \left(G_t - G \right)}, \ m_t = \begin{cases} \left(R_1 R_2 \cdots R_{t-1} \right)^{-1} & t > 1 \\ 1 & t = 1 \end{cases},$$
(5.4)

for $k = 1, 2, ..., \infty$. The 'long run multiplier' is Ψ_{∞} .

5.2. Computational Results

Consider Table 1 first. This reports λ in percent terms, the short run multiplier, Ψ_1 , and the long run multiplier, Ψ_{∞} . Results are reported for different size shocks to time t = 1government consumption: 0.20, 0.40 and 0.60. These represent 22, 49 and 82 percent jumps in government consumption in period 1, respectively.⁸ Consider the welfare numbers first. Note first of all that they are all positive. Each increase in government spending, though quite large, leads to an increase in welfare. Second, the results are similar whether adjustments to taxes are to the lump sum or the distortionary component of taxes. For example, the optimal value of σ_{ε} is somewhere in the neighborhood of σ_{ε} for both the lump sum and distortionary case. Third, the welfare increases from greater government spending are (slightly) larger when negative shocks to revenues are made up by adjustments to distortionary taxes. This is what one would expect, but it is perhaps surprising that the effects are so very small.

Now consider the government spending multipliers. First, note that all the short run multipliers are greater than unity. Second, note that in the distortionary tax case, long run multipliers are substantially smaller than the short run multipliers, while the two are roughly the same in the case of lump sum taxes. We may infer from (5.4) that the pattern in Ψ_k reflects that $Y_t - Y_t^*$ is positive for small values of t and turns negative for larger values of t when taxes are distortionary (this will be confirmed below). Thus, there is a trade off as output initially increases in response to the rise in G_t and output subsequently falls when the higher distortionary taxes to finance G_t occur.⁹

In our setting, the 'correct' way to aggregate the dynamic effects of government spending and the associated financing is to use the welfare measure and that measure indicates that the lump sum and distortionary tax cases are very similar. Presumably this is because the addition to welfare from the initial increase in output - which occurs at a time when output is very low - is smaller than the reduction in welfare later on as tax distortions increase. In any case, a naive reading of the results based on the long-run multiplier suggests that government spending is relatively ineffective in the zero lower bound, in the (plausible) case that taxes are distortionary. But, according to the welfare measure this inference is simply wrong. Thus, the long run multiplier is not a very useful measure of the effectiveness of government spending in the zero lower bound.

Table 1: Welfare Gain and Long-Run Multiplier, Shock to G Innovation in First Period			
$G_t = G^{1-0.8} G^{0.8}_{t-1} \exp\left(\varepsilon_t^g\right), \varepsilon_1^g = \sigma_\varepsilon, \varepsilon_t^g = 0, t > 1, G_0 = G$			
financing arrangement	$\sigma_{\varepsilon} = 0.2$	$\sigma_{\varepsilon} = 0.4$	$\sigma_{\varepsilon} = 0.6$
lump sum	$100 \times \lambda = 0.53$	$100 \times \lambda = 0.60$	$100 \times \lambda = 0.59$
	$\Psi_1, \Psi_\infty = 1.36, 1.21$	$\Psi_1, \Psi_\infty = 1.36, 1.22$	$\Psi_1, \Psi_\infty = 1.37, 1.23$
distortionary	$100 \times \lambda = 0.50$	$100 \times \lambda = 0.54$	$100 \times \lambda = 0.52$
	$\Psi_1, \Psi_\infty = 1.29, 0.58$	$\Psi_1, \Psi_\infty = 1.30, 0.58$	$\Psi_1, \Psi_\infty = 1.31, 0.57$

⁸For example, $\exp(0.6)=1.82$, after rounding.

⁹A priori, one might have suspected that the longer term decline in Y_t would drag down Y_t in the initial dates too. This effect, which operates via the effect of future consumption on present consumption in the Euler equation, is apparently quantitatively small.

Figures 4 to 8 investigate the dynamic effects of government consumption in the zero lower bound more closely. Figure 4 displays the economic responses to a drop in the discount rate in the first 12 periods when the government consumption remains unchanged. The interest rate falls to its zero lower bound and lifts off for the first time in period 8. Consumption drops by 20 percent right away and it is back to steady state by period 13. Note the very substantial drop in the Tak Yun distortion, p^* . It falls by nearly 8 percent. This is almost half of the almost 18 percent drop in GDP, with the rest coming from the fall in employment. The drop in p^* presumably reflects the very substantial 10 percent drop in inflation in the first period.

Turning to the government debt, note that it drops. One factor operating on the debt is the big deflation, which pushes the debt up (see how the real interest rate jumps). This is obviously *not* the reason why the debt falls! We can see the various elements in the government budget constraint in the bottom left figure. Note that, as expected, revenues from the labor tax rate drop. Of course, this by itself also raises the debt. The key reason for the drop in the debt is the fall in the tax subsidies to firms as the economy slows down (see the solid line). Presumably, this drop in transfers to firms is counterfactual. Our model does not include the many factors which make actual government transfer payments countercyclical. As discussed above, however, this feature of the model should make it harder to get a big positive effect on output and welfare when government consumption increases and taxes are distortionary. So, from our present point of view, this feature of our model is not necessarily a negative.

Figure 5 displays the response of the economy to an increase in government consumption. The increase in government spending does not change the date of 'lift off' in the interest rate, which still occurs in period 8. Note that now the debt does rise. Not surprisingly, the average value of the debt moves more gradually, by less and peaks later. Since the labor tax rate moves with the latter, it also rises only gradually and by relatively little. Note too that inflation falls a little less with the rise in government consumption, so that - not surprisingly - the Tak Yun distortion, p^* , now falls by less.

Perhaps the most useful way to understand the economic response to the increase in government consumption is to study Figure 6. That figure compares the results when Gincreases with the results when it does not. The top left figure shows the difference, $C_t - C_t^*$, expressed as a percent of C_t^* (recall, starred variables are the variables in Figure 5 and unstarred variables are the variables when government consumption rises). Thus, note how consumption rises initially by 2 percent and then goes negative just as the zero lower bound ceases to bind. Note too that consumption is below C_t^* for an extended period, over 100 quarters. The cumulative multiplier, Ψ_k , for k = 1, 2, 3, ..., is reported in the bottom left figure, and note how rapidly it falls. This is not surprising in view of the negative value of $C_t - C_t^{*,10}$ We see a similar pattern in employment. Presumably, the relative fall in consumption and employment reflects the distortionary effect of the labor tax, which slowly rises until reaching a peak in period 42 and is very slow to return to steady state. The fact that the movement in the tax rate is gradual and long lasting is why at the same time the absolute movements are very small. Note that the maximal response of the labor tax rate is only 0.3 *percentage points*. In terms of the actual tax rate it only moves in the third digit after the decimal.

The bottom right graph in Figure 6 yields insight into the efficiency implications of the increase in government spending. The first best allocations are characterized by equality between the marginal cost of working (i.e., mrs, or the marginal rate of substitution between consumption and leisure) and the marginal benefit, which is unity (the mrs in this case is also referred to as the labor wedge). The bottom right graph displays the marginal cost of working when G increases and when its is constant. When G is constant, the marginal cost of working plunges immediately to 0.76 consumption goods, versus the unchanged unit marginal product of labor. When G increases, the drop in the cost of working is less, at 0.80consumption goods. Thus, in the early period, the rise in G increases the efficiency in the level of employment (from p^* we also see that it improves efficiency in the allocation of employment across sectors¹¹). Interestingly, after the zero lower bound is over, the marginal cost of working is further from its efficient point when government consumption is expansionary than when it is not, because of the higher tax distortions. Thus, there is improved efficiency early on and less efficiency later on, with the increase in G. Another way to see this is to compare period by period utility under the two different government consumption scenarios. The 3,3 graph in Figure 6 displays utility with G positive minus utility when G is held constant. It is clear from this figure that the difference is positive initially. Later in the figure the difference appears to go to zero. In fact, it goes to a small negative number and remains there.

Figures 7, 8 and 9 display results for the lump sum tax case. Consider the summary results in Figure 9. Note that now employment and consumption do not go negative eventually. This is not surprising because there is no rising distortionary tax rate later on. An implication of this is that the multiplier, Ψ_k , does not fall so rapidly as it does in the distortionary case. This is why the long and short run multipliers are not to drastically different as they are in the distortionary tax case. Similarly, the bottom right graph shows that the cost of working is boosted early on when government consumption is high. But, the two curves are roughly the same - both equal to unity - later on. In the case of the 3,3 graph, flow utility in the expansionary G equilibrium minus flow utility in the constant G equilibrium approaches zero

¹⁰However, recall that the multiplier is based on GDP, $C_t + G_t - (C_t^* + G_t^*)$.

¹¹This is accomplished by the smaller drop in inflation, which reduces the amount of price dispersion. The welfare benefit of reduced price dispersion is that the allocation of labor across sectors is closer to its first-best pattern, which is equal employment across sectors.

monotonically from positive numbers and never goes negative.

References

- Adjemian, Stephane and Michel Juillard, 2013, "Stochastic Extended Path Approach", manuscript, March 13.
- [2] Chistiano, Lawrence J., Martin Eichenbaum and Mathias Trabandt, forthcoming, "Understanding the Great Recession," *American Economic Joural: Macroeconomics*.
- [3] Eggertsson, Gauti, 2010, "The Paradox of Toil," Federal Reserve Bank of New York Staff Report number 433, March.
- [4] Eggertsson, Gauti, and Michael Woodford, 2003, "The Zero Bound on Interest Rates and Optimal Monetary Policy," Brookings Papers on Economic Activity 1, 212-219.
- [5] Fair, R.C. and John Taylor, 1983, "Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models," *Econometrica*, vol. 51, no. 4, July, pp. 1169-1185.
- [6] Gagnon, Joseph E., and John B. Taylor, 1990, "Solving stochastic equilibrium models with the extended path method," *Economic Modelling: The International Journal of Theoretical and Applied Papers on Economic Modelling*, July.
- [7] Yun, Tack, 1996, "Nominal price rigidity, money supply endogeneity, and business cycles," Journal of Monetary Economics, Elsevier, vol. 37(2-3), pages 345-370, April.









Figure 2: ZLB Episode With and Without Negative Technology Shock AR(2) roots, 0.95 and 0.2







parameter values: $\varphi_1 = 1.5$, $\varphi_2 = 0$, $\varphi_1 = 0.8$, $\varphi_2 = 0$, $\varkappa = 0.03$, $\beta = 0.99$, $\gamma = 0.28266$, N = 0.33, G/Y = 0.2, bg = 0.165, nu = 0.44444, rh = 0.010101, rl = -0.005, tp = 12, T = 412 sig (utility) = 2, $\eta = 0.05$, s = 6, r^{-b} = 0.5, $\nu = 0.44444$, t^{-g} = 0.12267, $\omega_1 = 0.95$, $\omega_2 = 0$, theta = 0.84496, utility function: KPR, rho(G) = 0.8, rho(G) = 0.8, rho(G) = 0.8, rho(G) = 0.8, rho(G) = 0.9, rho



Figure 6: Response to increase in G in ZLB When Shocks to Budget Constraint Financed by Adjusting Distorting Taxes





parameter values: $\varphi_1 = 1.5$, $\varphi_2 = 0$, $\varrho_R = 0$, $\varrho_1 = 0.8$, $\varrho_2 = 0$, $\varkappa = 0.03$, $\beta = 0.99$, lump = 1, $\gamma = 0.28266$, N = 0.33, G/Y = 0.2, bg = 0.165, nu = 0.44444, rh = 0.010101, rl = -0.005, tp = 12, T = 412, psig = 0.0090572 sig (utility) = 2, $\eta = 0.05$, $\varepsilon = 6$, r^{-b} = 0.5, v = 0.44444, t^{-g} = 0.12267, $\omega_1 = 0.95$, $\omega_2 = 0$, theta = 0.84496, utility function: KPR, rho(G) = 0.8, rho(G)



Value of increased government consumption, measured as a constant percent increase in consumption:0.53009

Figure 9: Response to increase in G in ZLB When Shocks to Budget Constraint Financed by Adjusting Lump Sum Taxes