# Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients

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July, 2001

**Abstract**

I present an undetermined coefficients method for obtaining a linear approximating to the solution of a class of dynamic, rational expectations models. I also show how that solution can be used to compute a model's implications for impulse response functions and for second moments.
JEL Classification: C6, C63, C68

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# 1. Introduction

This paper describes a method for solving a system of linear expectational difference equations. The output of the method is a feedback rule relating the current period endogenous variables to a set of state variables. The method, a matrix version of the undetermined coefficients method described in Christiano (1991, Appendix), has been used extensively in applications where the expectational difference equations correspond to the linearized Euler equations of dynamic rational expectations models.[1] It is a blend of the undetermined coefficients method described in McCallum (1983) and the approach of Blanchard and Kahn (1980) (see also King, Plosser and Rebelo (1987).)[2] Because the method focuses on Euler equations, it is not limited to models whose solution can be expressed as the solution to a planning problem. Thus, it can handle models with tax and other distortions.

A distinguishing characteristic of the method is that it can easily accommodate a class of models in which different time $t$ endogenous variables are based on different information sets. This class of models includes limited participation models of money, models of labor hoarding and models with sticky prices.[3] The range of applications is not limited to these models, however. The method can handle any model which does not have occasionally binding inequality constraints.[4] For example, an exciting recent development in macroeconomics is the study of general equilibrium models with heterogeneity. A significant breakthrough in the

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[1] A selection of papers which apply the method includes Alexopoulos (1997), Chari, Christiano and Eichenbaum (1995), Chari, Kehoe and McGrattan (1996), Christiano (1991), Christiano and Eichenbaum (1992, 1995), Christiano, Eichenbaum and Evans (1997a), Christiano and Fisher (1998), Fisher (1997), Gust (1997) and Schlagenhauf and Wrase (1995).

[2] For another approache, see Uhlig (1997).

[3] The limited participation models I have in mind include those in Chari, Christiano and Eichenbaum (1995), Christiano (1991), Christiano and Eichenbaum (1992), Christiano, Eichenbaum and Evans (1997a), Fuerst (1992) and Schlagenhauf and Wrase (1995), in which a financial decision by households is made prior to the realization of the current period shocks, but other decisions are made afterward. Labor hoarding models include those of Burnside and Eichenbaum (1996) and Burnside, Eichenbaum and Rebelo (1993), in which the number of people working is determined prior to the realization of a current period shock, while the intensity with which they work is decided afterward. Other examples include sticky price models in which at least some price setters set prices before the realization of a monetary shock and other decisions are made afterward (see, for example, Chari, Kehoe and McGrattan (1996), Christiano, Eichenbaum and Evans (1997) and Gust (1997).)

[4] For a discussion of strategies for solving such models, see Christiano and Fisher (1997).

quantitative analysis of these models was achieved by Jeff Campbell (1997), when he showed that a class of these models can be solved by linearization methods. We show below how the method presented here can be applied to solve Jeff Campbell's model. Straightforward modifications of the discussion there can be applied to solve the general equilibrium model with worker heterogeneity analyzed by Monika Merz (1996).

Section 2 of the paper presents the expectational difference equations considered and defines precisely what I mean by a solution. Section 3 discusses the computational strategy for finding the solution. The system of expectational difference equations and the solution is presented in full generality, so that the notation is sometimes complicated. For this reason, section 4 presents a series of examples to illustrate aspects of the method. The first example exhibits the method in the simplest possible setting, the one sector stochastic growth model in which hours worked are constant. It shows how the expectational difference equations that are the focus of the method can arise by linearizing the Euler equations of a nonlinear model. The next three examples illustrate various technical aspects of the method. The fifth example shows how the method can be used to solve Campbell's model. Section 5 briefly discusses methods for using the model solution to compute impulse response functions and second moments. Section 6 presents some concluding remarks.

## 2. A System of Expectational Difference Equations

This section defines a system of expectational difference equations and defines a solution. A solution is a liner feedback rule relating the current period endogenous variables to a set of state variables. It is characterized by two matrices. These two matrices can be understood by dividing up the current period state variables into two sets. The first set, the endogenous state variables, are variables like capital which are predetermined at the beginning of the current period, but were determined by the model in some previous period. The second set, the exogenous state variables, are variables that are generated outside the model by some stochastic process. The first matrix considered in the solution strategy (the 'feedback part') is the one that characterizes the impact of the endogenous state variables on the current period endogenous variables. The second matrix (the 'feedforward part') characterizes the impact of the current period exogenous variables on the current period endogenous variables. The procedure for computing these two matrices is

3

described in detail in the next section.

Let $z_{1t}$ denote an $n_1 \times 1$ vector of endogenous variables that is determined at time $t$. Let the $n \times 1$ vector $z_t$ be defined as follows:

$$z_t = \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}, \tag{2.1}$$

where $z_{2t}$ is a $qn_1 \times 1$ vector of $q$ lagged $z_{1t}$'s and $n \equiv n_1(q+1)$. The number of lags of $z_{1t}$ included in $z_{2t}$ is defined by the condition that $z_t$ contain all endogenous state variables relevant for the determination of $z_{1t+1}$ at time $t + 1$.[5] We can accommodate $q = 0$, in which case $z_t \equiv z_{1t}$. The elements in $z$ are expressed as deviations from their respective nonstochastic steady state values.

Suppose that $z_t$ satisfies the following system of $n_1$ linear expectational difference equations:

$$\mathcal{E}_t[\sum_{i=0}^{r} \alpha_i z_{t+r-1-i} + \sum_{i=0}^{r-1} \beta_i s_{t+r-1-i}] = 0, \tag{2.2}$$

for $t = 0, 1, \ldots$, and for given $z_{-1}$. Here, $s_t$ is an $m \times 1$ vector of exogenous shocks. I explain below how this is constructed from current and lagged values of the shocks, $\theta_t$, to agents' environment. In (2.2), $r > q$, and the $\alpha_i$'s and $\beta_i$'s are given $n_1 \times n$ and $n_1 \times m$ matrices, respectively. Also, when $q > 0$, the right $qn_1$ columns of $\alpha_i$, $i = 0, \ldots, r-1$ are composed of zeros. Assume that the rank of $\alpha_0$ exceeds 0 and that the $n_1 \times (r+1)n$ dimensional matrix, $[\alpha_0, \alpha_1, \ldots, \alpha_r]$ has full row rank.[6]

The symbol $\mathcal{E}_t$ in (2.2) represents an expectation operator. This operator differs from the normal expectation operator, $E_t$, in that it allows the conditioning information set to vary across the equations in (2.2). That is, if $X_t$ is an $n_1 \times 1$ vector of random variables,

$$\mathcal{E}_t[X_t] \equiv \begin{bmatrix} E[X_{1t}|\Omega_{1t}] \\ \vdots \\ E[X_{n_1 t}|\Omega_{n_1 t}] \end{bmatrix}, \tag{2.3}$$

where $E[X|\Omega]$ denotes the mathematical expectation of the random variable $X$ conditional on the information set, $\Omega$. In the *standard case*, all the $\Omega_{it}$'s are

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[5] For a simple example of how $q > 0$ can occur, see example 4 in section 4.

[6] For an illustration of how (2.2) can arise by linearizing the Euler equations of an economic model, see example 1 in section 4 below.

4

identical, with $\Omega_{it} = \{z_{t-1-k}, \theta_{t-k}; k \geq 0\}$ for all $i$. The specification of (2.2) is designed to accommodate models in which different time $t$ endogenous variables are determined based on different information sets.[7] It is convenient to adopt a simple vector characterization of the information structure in (2.2). Let $\tau_i$ denote the column vector with unity in locations corresponding to elements of $\theta_t$ which are contained in $\Omega_{it}$ and zero in locations corresponding to elements of $\theta_t$ which are excluded from $\Omega_{it}$, $i = 1, ..., n_1$. Also, let $\tau = [\tau_1, ..., \tau_{n_1}]$.

The law of motion of $s_t$ has the following form:

$$s_t = P s_{t-1} + \epsilon_t, \tag{2.4}$$

where $\epsilon_t$ has mean zero and is uncorrelated with $\epsilon_{t-l}$, $s_{t-l}$, $l > 0$, and is derived from the time series representation for $\theta_t$. I assume:

$$\theta_t = \rho \theta_{t-1} + e_t, \quad E e_t e_t' = V_e, \tag{2.5}$$

where the zero-mean random variable $e_t$ is uncorrelated with lagged values of itself and of $\theta_t$.[8] In the standard case,

$$s_t = \theta_t, \quad P = \rho, \quad \epsilon_t = e_t.$$

If any one of the information sets, $\Omega_{it}$, $i = 1, ..., n_1$ does not contain the whole of $\theta_t$, then (2.4) is constructed slightly differently:

$$s_t = \begin{pmatrix} \theta_t \\ \theta_{t-1} \end{pmatrix}, \quad P = \begin{bmatrix} \rho & 0 \\ I & 0 \end{bmatrix}, \quad \epsilon_t = \begin{pmatrix} e_t \\ 0 \end{pmatrix}.$$

We seek a $\{z_t\}$ process which is consistent with (2.1) - (2.4) for all $z_{t-1}, s_t$; with the initial condition, $z_{-1}$; and with the condition:

$$\lim_{j \to \infty} E_t z_{t+j} \to 0. \tag{2.6}$$

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[7]This possibility is illustrated in example 3.

[8]This law of motion is quite general and will accommodate an arbitrary ARMA(p,q) representation for the underlying economic shocks. For example, suppose $p = q = 1$ and $x_t = \phi x_{t-1} + \varepsilon_t - \gamma \varepsilon_{t-1}$, where $x_t$ is a shock to the economic environment. Then,

$$\theta_t = \begin{pmatrix} x_t \\ \varepsilon_t \end{pmatrix}, \quad \rho = \begin{bmatrix} \phi & -\gamma \\ 0 & 0 \end{bmatrix}, \quad e_t = \begin{pmatrix} \varepsilon_t \\ \varepsilon_t \end{pmatrix}.$$

The assumption that $\theta_t$ has mean zero is also without loss of generality, since we are free to interpret $\theta_t$ as being expressed in deviation from mean.

5

The latter restriction could be motivated by the underlying economic model. In any case, a solution that is not consistent with this condition is not generally of interest. This is because, in practice, the function whose expectation is being taken in (2.2) is a linear approximation of a non-linear function, and the quality of this approximation can be expected to deteriorate as $z_t$ diverges substantially from 0. For a further discussion, see example 1 in section 4 below.

We limit ourselves to solutions which have the following representation:

$$z_t = A z_{t-1} + B s_t, \tag{2.7}$$

where $A$ is the feedback part of the solution and $B$ is the feedforward part. Here, the $n \times n$ matrix $A$ has the following structure:

$$A = \begin{bmatrix} A_1 \\ \cdots \\ A_2 \end{bmatrix},$$

where $A_1$ is an $n_1 \times n$ matrix of (as yet) undetermined coefficients and $A_2$ is the $(n - n_1) \times n$ matrix:

$$A_2 = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}. \tag{2.8}$$

Here, $A_1 \equiv A$ when $q = 0$. In (2.8), $I$ and $0$ denote the $n_1$-dimensional identity and zero matrices, respectively. The requirement, (2.6), corresponds to the restriction that the eigenvalues of $A$ be strictly less than unity in absolute value.

The $n$ by $m$ matrix $B$ has the following structure:

$$B = \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix}. \tag{2.9}$$

Here, $B_1$ is an $n_1 \times m$ matrix composed primarily of as-yet undetermined coefficients, and $B_2$ is an $(n - n_1) \times m$ matrix of zeros. When $q = 0$, then $B \equiv B_1$. Under the standard assumption about information sets, all elements of $B_1$ are treated as undetermined coefficients. When information sets differ, a number of elements of $B_1$ are set to zero. We assume that the variables in $z_{1t}$ are ordered so

6

that if the $l^{th}$ entry in $\tau_i$ is zero, then the $(i, l)^{th}$ entry of $B_1$ is also restricted to be zero. For example, if the $i^{th}$ equation is the Euler equation associated with some variable, then we assume that that variable occupies the $i^{th}$ position in $z_{1t}$. It is natural then, that the policy rule governing that variable exclude the $l^{th}$ element of $s_t$.[9]

In summary, the following objects define the system of expectational difference equations that is addressed in this paper:

$$\rho, \alpha_0, ..., \alpha_r, \beta_0, ..., \beta_{r-1}. \tag{2.10}$$

To find a solution, no further information is needed if the standard assumption about information sets is made. If there is incomplete information in at least one equation in (2.2), then one needs to also specify $\tau$ and $V_e$. In this case, $V_e$ is required only if there is a non-trivial signal extraction problem to solve. This occurs if (i) in one of the equations with incomplete information, some current period exogenous variables are observed and some are not, and (ii) the $V_e$ matrix is non-diagonal.

## 3. An Undetermined Coefficient Solution Method

I describe a strategy for computing the undetermined coefficients in the feedback part, $A$, and the feedforward part, $B$, of the policy rule. I first derive a set of equations that $A$ and $B$ must satisfy if (2.7) is to be consistent with (2.1) - (2.4)

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[9]Not all model formulations necessarily lead to zero restrictions on $B_1$. A non-zero restriction can arise if a variable has been scaled by an exogenous variable in a particular way. The following example suggests that, in a case like this, an alternative scaling will ensure that the relevant restriction on $B_1$ is a zero restriction after all. Let $m_{t+1} = \log(M_{t+1}/P_t)$, where $M_{t+1}$ is a representative household's end-of-period $t$ holdings of money and $P_t$ is the price level. In equilibrium, $M_{t+1}$ must equal what is supplied by the monetary authority, so that $M_{t+1} = M_t \exp(x_t)$, where $x_t$ is the money growth rate which is assumed, for the purpose of the illustration, to be exogenous. Then, $m_{t+1} = m_t - \log(\pi_t) + x_t$, where $\pi_t = P_t/P_{t-1}$. Suppose further that (as in Chari, Kehoe and McGrattan (1996) and Christiano, Eichenbaum and Evans (1997)) $P_t$ is set by firms *prior* to the realization of the date $t$ random variables and that the $i^{th}$ equation in (2.2) is their associated Euler equation. The information set in that Euler equation excludes the current value of $x_t$. If $m_{t+1}$ is the $i^{th}$ element of $z_t$ and $x_t$ is the $k^{th}$ element of $\theta_t$, then the $(i, k)^{th}$ element of $B_1$ is restricted to be unity, not zero. But, this reflects that $P_t$ is transformed with $M_{t+1}$. If the $i^{th}$ variable in $z_t$ were $log(M_t/P_t)$ instead, then the restriction on $B_1$ would be a zero restriction after all.

and (2.6). We find that the feedback matrix is the zero of a particular matrix polynomial. Conditional on $A$, the feedforward part is the solution to a linear system of equations. After deriving these equations, I describe a strategy for solving them.

Solve (2.7) recursively to express $z_{t+j}$, $j = 0, 1, ..., r-1$, as a function of $z_{t-1}$, $s_t$ and $\epsilon_{t+k}$, $k = 1, ..., r-1$. Substituting the resulting expressions into the left side of the equality in (2.2), and taking into account $\mathcal{E}_t \epsilon_{t+k} = 0$, $k > 0$, we obtain

$$\mathcal{E}_t [\sum_{i=0}^{r} \alpha_i z_{t+r-1-i} + \sum_{i=0}^{r-1} \beta_i s_{t+r-1-i}] = \alpha(A) z_{t-1} + \widetilde{F} s_t \tag{3.1}$$

where $\widetilde{F}$ is defined by

$$\mathcal{E}_t F s_t = \widetilde{F} s_t, \tag{3.2}$$

and $F = \widetilde{F}$ in the standard case. Here,

$$F = \sum_{i=0}^{r-1} [\beta_i + \overline{Q}_i B_1] P^{(r-1-i)} \tag{3.3}$$

and

$$\begin{aligned} Q_0 &\equiv \alpha_0, Q_k \equiv Q_{k-1} A + \alpha_k, k = 1, 2, ..., r \\ \alpha(A) &\equiv \alpha_0 A^r + \alpha_1 A^{r-1} + \ ... \ + \alpha_r \equiv Q_r \end{aligned} . \tag{3.4}$$

In (3.3), $\overline{Q}_i$ is the $n_1 \times n_1$ matrix formed from the first $n_1$ columns of $Q_i$. Thus, in addition to the eigenvalue restriction mentioned above, the $A$ that we seek must satisfy:[10]

$$\alpha(A) = 0_{n_1 \times n}. \tag{3.5}$$

Given $A$, the $n_1 \times m$ matrix $B_1$ must solve the restriction:

$$\widetilde{F} = 0_{n_1 \times m}. \tag{3.6}$$

Note that $\alpha(A)$ in (3.5) is an $r^{th}$ order matrix polynomial in $A$. Such a polynomial will in general have many matrix roots. An important substantive question is whether any of these satisfy the eigenvalue restriction and, if so, how many. Below, I develop easy to evaluate restrictions on the $\alpha_i$'s in (2.2) to answer these

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[10] A MATLAB program, feedback.m, which computes $A$ given $\alpha_0, ..., \alpha_r$ using the procedure described in the next subsection, can be obtained from the author on request.

8

questions. I also show that given an arbitrary matrix $A$, the system of equations defined by (3.6) is linear in the non-zero components of $B_1$. The necessary condition for these equations to have a solution is satisfied, since the number of undetermined elements in $B_1$ is equal to the number of nonzero elements in $\tilde{F}$. The following subsection discusses the computation of $A$ and after that I turn to $B$.

### 3.1. Computing the Feedback Part

Since the conditions in (3.5) do not involve the second moment properties of $s_t$, one can abstract from the latter in computing $A$. In particular, I simplify the analysis in this subsection by setting $s_t \equiv 0$ for all $t$. Then, (2.2) reduces to $\sum_{j=0}^{r} \alpha_j z_{t+r-1-j} = 0$, $t = 0, 1, 2, ....$, or, in first order difference equation form:

$$aY_{t+1} + bY_t = 0, \ t \geq 0. \tag{3.7}$$

where $Y_t$ is an $[(r-1)n_1 + n] \times 1$ vector:

$$Y_t = \begin{pmatrix} z_{1,t+r-2} \\ \vdots \\ z_{1,t} \\ z_{t-1} \end{pmatrix}. \tag{3.8}$$

Also, the $[(r-1)n_1+n] \times [(r-1)n_1+n]$ matrices $a$ and $b$ are defined as follows:

$$a = \begin{bmatrix} \tilde{\alpha}_0 & 0_{n_1 \times [(r-2)n_1+n]} \\ 0_{[(r-2)n_1+n] \times n_1} & I_{(r-2)n_1+n} \end{bmatrix}, b = \begin{bmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 & \cdots & \tilde{\alpha}_{r-1} & \alpha_r \\ -I_{n_1} & 0_{n_1} & \cdots & 0_{n_1} & 0_{n_1 \times n} \\ 0_{n_1} & -I_{n_1} & \cdots & 0_{n_1} & 0_{n_1 \times n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{n \times n_1} & 0_{n \times n_1} & \cdots & b_1 & b_2 \end{bmatrix} \tag{3.9}$$

where $\tilde{\alpha}_i$ is the left $n_1$ columns of $\alpha_i$, $i = 1, \ldots, r-1$. (When $q = 0$, then $\tilde{\alpha}_i = \alpha_i$ for all $i$.) Also, $0_{i \times j}$ and $0_i$ denote the $i \times j$ and $i \times i$ dimensional matrices of zeros, respectively, and $I_i$ denotes the $i$ dimensional identity matrix. Finally, when $q > 0$, the $n \times n_1$ and $n \times n$ dimensional matrices $b_1$ and $b_2$ are defined as follows:

$$b_1 = \begin{bmatrix} -I_{n_1} \\ 0_{qn_1 \times n_1} \end{bmatrix}, b_2 = \begin{bmatrix} 0_{n_1 \times qn_1} & 0_{n1} \\ -I_{qn_1} & 0_{qn_1 \times n_1} \end{bmatrix},$$

9

and when $q = 0$ :
$$b_1 = -I_{n_1}, b_2 = 0_{n_1}.$$

The vector, $Y_0$, is restricted by the $n$ initial conditions, $z_{-1}$. The $(r-1)n_1$ elements, $z_{1,0}, \ldots, z_{1,r-2}$, are free.

A *solution* to (3.7) is a sequence, $\{Y_t; t \geq 0\}$, which satisfies the initial conditions and (3.7) at all dates. The dimension of space of solutions is $n_1(r-1)$, with each solution distinguished by a different choice for the values of the free elements of $Y_0$. An element of the space of solutions is said to be a *minimal state variable solution* if there is an $n_1(r-1) \times nr$ dimensioned matrix $D$ which satisfies a certain rank condition and $DY_t = 0$ for all $t \geq 0$. The rank condition is that the square matrix formed from the first $(r-1)n_1$ columns of $D$ corresponding to the free variables in $Y_0$ be invertible. An element in the space of solutions is said to be *nonexplosive* if $Y_t \to 0$ as $t \to \infty$ for all possible initial conditions. The matrix $A$ that we seek is constructed from the $D$ matrix associated with a minimal state variable, nonexplosive solution.

I consider two cases, one in which the matrix $a$ is invertible and the other in which it is not.

### 3.1.1. The Invertible $a$ Case

In the first case, (3.7) implies that all solutions can be expressed as $Y_t = \left(-a^{-1}b\right)^t Y_0$, or,
$$P^{-1}Y_t = \Lambda^t P^{-1}Y_0, \tag{3.10}$$

where $P\Lambda P^{-1} = -a^{-1}b$ is the eigenvector, eigenvalue decomposition of $-a^{-1}b$. Suppose there are exactly $n_1(r-1)$ elements in $\Lambda$ that exceed unity in absolute value, and let $D$ be composed of the $n_1(r-1)$ rows of $P^{-1}$ associated with the explosive elements of $\Lambda$. If $D$ satisfies the rank condition described above, then the set of nonexplosive solutions contains one element, the one corresponding to the $Y_0$ satisfying $DY_0 = 0$. That $Y_0$ corresponds to the unique nonexplosive solution of (3.10). This solution is a minimal state variable solution because of the easily verified fact that $\tilde{p}Y_0 = 0$ implies $\tilde{p}Y_t = 0$ for all $t \geq 0$ when $\tilde{p}$ is any row of $P^{-1}$.

If the number of eigenvalues larger than one in absolute value exceeds $n_1(1-r)$, then there is no minimal state variable solution satisfying nonexplosiveness (there are not enough degrees of freedom in the first $n_1(r-1)$ elements of $Y_0$ to 'zero out' all the explosive eigenvalues in (3.10).) In this case (particularly when (2.2) is a local approximation to the Euler equation associated with a nonlinear model)

the solution method developed here does not directly apply.[11] If the number of elements of $\Lambda$ exceeding unity in absolute value is less than $n_1(r - 1)$, then there may be more than one minimal state variable, nonexplosive solution (the requirement of 'zeroing out' the explosive roots in $\Lambda$ may not exhaust the $n_1(r-1)$ degrees of freedom in $Y_0$.)[12] When (2.2) represents a local approximation about a nonstochastic steady state for $z_t$, then this is equivalent to the finding that the nonstochastic steady state is indeterminate.

Write $DY_t = D^1 Y_t^1 + D^2 z_{t-1}$, where $D^1$ are the $n_1(r-1)$ first columns of $D$ and $D^2$ are the remaining $n$ columns and $Y_t = [Y_t^{1\prime}, z_{t-1}']'$ is partitioned conformably.

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[11]Stokey and Lucas (1989, p. 157-158) provide an example useful for thinking about this case. Their example is a two-sector growth model in which the exact policy rule diverges in a neighborhood of the nonstochastic steady state, and converges to a two-period cycle. When (2.2) is a linear expansion about steady state, then there are only non-convergent solutions to $\alpha(A) = 0$. One of these closely approximates the divergent exact solution in the neighborhood of steady state. Still, this linear approximation is very inaccurate, since the implied trajectory for $z_t$ explodes without bound in contrast to the exact solution which is bounded. The methods of this paper can still be applied to this example, however, by redefining $z_t$ appropriately. See Stokey and Lucas (1989, exercise 6.7f, page 158).

[12]For example, suppose the dimension of $\pi = -a^{-1}b$ is $3 \times 3$, and $n_1(r - 1) = 2$. Also, let $\tilde{p}_i$ denote the $i^{th}$ row of $P^{-1}$, $i = 1, 2, 3$. Suppose the eigenvalues of $\pi$ are distinct, and real, but that only one, the first one, is explosive. Then, $D$ can be constructed from $\tilde{p}_1$ and either of $\tilde{p}_2$ or $\tilde{p}_3$. In this case, there are two reduced rank, convergent solutions. In general, when the number of explosive eigenvalues is less than $n(r - 1)$, a reduced rank, convergent solution may not exist, or it may be unique. As an illustration of the former possibility, consider

$$\pi = \begin{bmatrix} 3.1 & -0.1 & -1.7 \\ 0.8 & 0.2 & -0.5 \\ 1.9 & 0.04 & -0.9 \end{bmatrix}.$$

The eigenvalues of $\pi$ are 1.92 and $0.24 \pm 0.07i$, after rounding, where $i = \sqrt{-1}$. Evidently, there is only one explosive eigenvalue. So, to assure convergence, $Y_0$ must be orthogonal to $\tilde{p}_1$. But, $Y_0$ cannot also be made orthogonal to any one of $\tilde{p}_2$ or $\tilde{p}_3$. This is because these vectors are complex. That is, $\tilde{p}_2 = \alpha + i\beta$ and $\tilde{p}_3 = \alpha - i\beta$, where $\alpha$ and $\beta$ are real $1 \times 3$ row vectors. Then, $\tilde{p}_2 Y_0 = 0$ corresponds to the two conditions, $\alpha Y_0 = \beta Y_0 = 0$. Given the assumed 2 degrees of freedom in $Y_0$ and the fact that one is used up by the convergence requirement, $\tilde{p}_1 Y_0 = 0$, there are not enough degrees of freedom left to impose $\tilde{p}_i Y_0 = 0$ for $i = 2$ or 3. To see that the number of reduced rank, convergent solutions may be unique, even when the number of eigenvalues is less than $n_1(r - 1)$, consider a version of the $\pi$ matrix above in which the real, explosive eigenvalue is replaced by one that is less than one in absolute value. Then, the unique convergent, reduced rank solution is the $Y_0$ such that $DY_0 = 0$, where the rows of $D$ are composed of $\alpha$ and $\beta$.

Then, $DY_t = 0$ implies

$$Y_t^1 = -(D^1)^{-1}D^2 z_{t-1}, \tag{3.11}$$

where $-(D^1)^{-1}D^2$ is an $n_1(r-1) \times n$ matrix. The matrix $A_1$ that we seek is composed of the bottom $n_1$ rows of $-(D^1)^{-1}D^2$.

### 3.1.2. The Non-Invertible $a$ Case

Now consider the case when $a$ is not invertible.[13] The procedure I use for handling this case is based on the QZ decomposition, as implemented by Chris Sims (1989). For notational simplicity, let $\omega \equiv (r-1)n_1 + n$, so that $a$ and $b$ are $\omega \times \omega$ matrices.

The first step is to find the orthonormal matrices $Q$ and $Z$, and the upper triangular matrices $H_0$ and $H_1$ with the properties:

$$QaZ = H_0, \ QbZ = H_1. \tag{3.12}$$

The matrix $H_0$ is structured so that the $l$ zeros on its diagonal are located in the lower right part of $H_0$. Denote the upper $(\omega - l) \times (\omega - l)$ block of $H_0$ by $G_0$. This matrix must be non-singular. Let the corresponding upper left $(\omega - l) \times (\omega - l)$ block in $H_1$ be denoted $G_1$. I assume that the diagonal terms in the lower right $l \times l$ block of $H_1$ are non-zero. Also, it is useful to partition $Z'$ as follows:

$$Z' = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \tag{3.13}$$

where $L_1$ is $(\omega - l) \times \omega$ and $L_2$ is $l \times \omega$.

Inserting $ZZ' (= I)$ before $Y_{t+1}$ and $Y_t$ in (3.7), defining $\gamma_t \equiv Z'Y_t$, and pre-multiplying (3.7) by $Q$, (3.7) becomes:

$$H_0\gamma_{t+1} + H_1\gamma_t = 0, \ t = 0, 1, \ldots . \tag{3.14}$$

Partition $\gamma_t$ as follows:

$$\gamma_t = \begin{pmatrix} \gamma_t^1 \\ \gamma_t^2 \end{pmatrix}, \tag{3.15}$$

---

[13]See example 2 in the next section, for an illustration of how this case can arise in practice.

where $\gamma_t^1$ is $(\omega - l) \times 1$ and $\gamma_t^2$ is $l \times 1$. It is easy to verify that (3.14) implies $\gamma_t^2 = 0$, $t \geq 0$, i.e.,[14]

$$L_2 Y_t = 0, \ t = 0, 1, \ldots \ . \tag{3.16}$$

With (3.16) imposed, the last $l$ equations in (3.14) are redundant, so (3.14) can be written

$$G_0 \gamma_{t+1}^1 + G_1 \gamma_t^1 = 0, \ t = 0, 1, \ldots \ . \tag{3.17}$$

The set of solutions to this system can be expressed as $\gamma_t^1 = (-G_0^{-1} G_1)^t \gamma_0^1$, $t \geq 0$, or,

$$P^{-1} \gamma_t^1 = \Lambda^t P^{-1} \gamma_0^1, \tag{3.18}$$

where $P \Lambda P^{-1} = -G_0^{-1} G_1$ is the eigenvector, eigenvalue decomposition of $-G_0^{-1} G_1$. The $\gamma_t^1$ that solve (3.18) converge to zero asymptotically if, and only if, $\tilde{p} \gamma_0^1 = 0$, where $\tilde{p}$ is composed of the rows of $P^{-1}$ corresponding to diagonal terms in $\Lambda$ that exceed 1 in absolute value. This condition is:

$$\tilde{p} L_1 Y_0 = 0. \tag{3.19}$$

Recall that the number of free elements in $Y_0$ is $n_1(r - 1)$. Equation (3.16) for $t = 0$ represents $l$ restrictions on $Y_0$, so that to pin $Y_0$ down uniquely, $n_1(r - 1) - l$ more restrictions are needed. Thus, uniqueness requires that there be $n_1(r - 1) - l$ explosive eigenvalues in $\Lambda$, i.e., that $\tilde{p} L_1$ contain $n_1(r - 1) - l$ rows. Then, define

$$D = \begin{bmatrix} \tilde{p} L_1 \\ L_2 \end{bmatrix}. \tag{3.20}$$

The matrix $A$ that we seek is then obtained by manipulating $D$ in exactly the same way that was done before (see the discussion after equation (3.11).)

---

[14]To see this, let us temporarily adopt a simpler notation. Let the lower right $l \times l$ block of $H_0$ be denoted $\Gamma$ and let the corresponding block of $H_1$ be denoted $W$. Write $\Gamma = [\Gamma_{ij}]$ and $W = [W_{ij}]$. The matrices, $\Gamma$ and $W$, are upper triangular, with the former having zeros along its diagonal and the latter having non-zero terms along its diagonal. Also, write $x_t = \gamma_t^2$, with $x_t = [x_{1t}, \ldots, x_{lt}]'$. Then we have $\Gamma x_{t+1} + W x_t = 0$ for $t = 0, 1, 2, \ldots$ . Note that the last row of $\Gamma$ is composed of zeros, so that the last row of this system of equations is $W_{l,l} x_{lt} = 0$ for all $t$. Since $W_{l,l} \neq 0$, this implies $x_{lt} = 0$ for all $t$. Now consider the $l - 1$ equation:

$$\Gamma_{l-1,l} x_{l,t+1} + W_{l-1,l-1} x_{l-1,t} + W_{l-1,l} x_{l,t} = 0,$$

for $t = 0, 1, 2, \ldots$ . But, since $x_{,t} = 0$ for all $t$, this implies $W_{l-1,l-1} x_{l-1,t} = 0$ for all $t$. Since $W_{l-1,l-1} \neq 0$, this in turn implies $x_{l-1,t} = 0$ for $t = 0, 1, 2, \ldots$ .Proceeding in this way, we establish recursively that $x_{j,t} = 0$ for all $t$, for $j = l, l - 1, \ldots, 1$.

## 3.2. Computing the Feedforward Part

With the $A$ matrix in hand, I now find the $B$ matrix which solves (3.6). I consider two cases. In the standard case, $\mathcal{E}_t F s_t = F s_t$, so that I require $F = 0$. The following result is useful:

$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1) vec(A_2) \tag{3.21}$$

where $\otimes$ denotes the Kronecker product and $vec(\cdot)$ denotes the vectorization operator.[15] Also, $vec(A + B) = vec(A) + vec(B)$. Applying these results to (3.3), I get

$$vec(F') = vec[\sum_{j=0}^{r-1}(P')^{r-1-j}\beta_j'] + \{\sum_{j=0}^{r-1}[\overline{Q}_j \otimes (P^{r-1-j})']\}vec(B_1'), \tag{3.22}$$

where $P^0 \equiv I$ and $'$ denotes the matrix transposition operator. Then, $vec(F') = 0$ implies

$$d + q\delta = 0, \tag{3.23}$$

where

$$q = \{\sum_{j=0}^{r-1}[\overline{Q}_j \otimes (P^{r-1-j})']\}, \ d = vec[\sum_{j=0}^{r-1}(P')^{r-1-j}\beta_j'], \text{ and } \delta = vec(B_1'). \tag{3.24}$$

Note here that $q$ and $d$ are determined, given $A$. The solution we seek is

$$\delta = -q^{-1}d. \tag{3.25}$$

The matrix $B_1$ can be recovered in a straightforward way from $\delta$.

Now consider the case in which $\mathcal{E}_t F s_t = \widetilde{F} s_t \neq F s_t$. One difference between $F$ and $\widetilde{F}$ is that the latter contains a zero in its $i, j^{th}$ entry if the $j^{th}$ element of $s_t$ is excluded in $\Omega_{it}$. The matrices $F$ and $\widetilde{F}$ have the following linear relationship:

$$\overline{vec}(\widetilde{F}') = R vec(F'). \tag{3.26}$$

---

[15] If $X = [x_1 \vdots x_2 \vdots \cdots \vdots x_n]$, where $x_i$ denotes the $i^{th}$ column of $X$, then

$$vec(X) \equiv \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}.$$

Here, $\overline{vec}(\widetilde{F}')$ coincides with $vec(\widetilde{F}')$, except that entries corresponding to elements in $\widetilde{F}$ that are assigned a zero value by exclusion restrictions in $\Omega_{it}$ are deleted. For a simple illustration of the construction of $R$, see example 3 in section 4. Appendix 1 describes a general algorithm for constructing $R$ from $\tau$, $\rho$, and $V_e$. The matrix, $V_e$, is needed for this mapping in case some elements of $\theta_t$ are observed contemporaneously and others not (i.e., at least one $\tau_i$ has neither all zeros nor all ones). In this case, the elements that are not observed must be projected onto the ones that are, and this projection formula requires the covariance between the various shock innovations.

Given the simple relationship between $F$ and $\widetilde{F}$ given by (3.26), obtaining $B_1$ involves a straightforward modification on the algorithm leading to (3.25). Define $\tilde{d} = Rd$, let $\tilde{\delta} = \overline{vec}(B_1')$, and note that $B_1$ has zeros in the same entries as $\widetilde{F}$. Also, let $\tilde{q}$ denote $Rq$ after the columns of $Rq$ corresponding to the zero elements in $vec(B_1')$ have been removed. Premultiplying (3.23) and taking the latter into account, we obtain, $\tilde{d} + \tilde{q}\tilde{\delta} = 0$, so that

$$\tilde{\delta} = -\tilde{q}^{-1}\tilde{d}. \tag{3.27}$$

The matrix $B_1$ can be recovered in a straightforward way from $\tilde{\delta}$.


## 4. Some Examples of The Solution Method

In this section I describe five examples which help illustrate aspects of the general algorithm described in the previous section. The first example is a basic real business cycle model in which labor is exogenously held fixed. I use this example to establish some notation, and to introduce, in the simplest possible way, the idea of the undetermined coefficient solution method. In addition, I use the example to illustrate how (2.2) could arise as an approximation to a nonlinear model. The second example endogenizes the labor decision. This change causes the matrix $a$ discussed in section 3.1 to be non-invertible. The example is used to illustrate the strategy outlined in section 3.1.2 above for dealing with this. The third example illustrates how the solution algorithm can accommodate situations in which different decisions are based on different information sets. The example modifies the second example by assuming that the investment decision is made prior to the realization of the technology shock, while the labor decision is made afterward. The fourth example is a simple version of Kydland and Prescott (1983)'s 'Time

to Build' model and illustrates the possibility that $q > 0$ in section 2. This corresponds to the case in which the vector $z_t$ must include lagged values of the current period decision variables. My final example illustrates how the solution method can be used to solve a model with non-trivial cross-sectional heterogeneity.

## 4.1. Example 1: Real Business Cycle Model With Exogenous Labor

Consider a one-sector neoclassical growth model in which there is only a saving/consumption decision. A planner solves the following problem: Maximize expected utility $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$ subject to the resource constraint, $C_t + K_{t+1} - (1 - \delta)K_t = f(K_t, \theta_t)$, where $\beta$ is the discount rate, $C_t$ is consumption, $K_t$ is the capital stock, $\delta$ is the depreciation rate on capital, $f$ is the production function and $\theta_t$ is a shock to technology:

$$\theta_t = \rho\theta_{t-1} + e_t, \ e_t \tilde{} N(0, \sigma_e^2), \tag{4.1}$$

where $\rho$ is a scalar less than unity in absolute value. The first order condition for the capital investment decision with the resource constraint imposed to eliminate consumption is:

$$0 = E[v(K_{t+2}, K_{t+1}, K_t, \theta_{t+1}, \theta_t)|\theta_t], \tag{4.2}$$

for all $t \geq 0$, $\theta_t$, $K_t$, where $f_K$ denotes the partial derivative of $f$ with respect to its first argument, $U_c$ denotes the marginal utility of consumption, and

$$
\begin{aligned}
&v(K_{t+2}, K_{t+1}, K_t, \theta_{t+1}, \theta_t) \\
={} & U_c\left(f(K_t, \theta_t) + (1 - \delta)K_t - K_{t+1}\right) \\
& -\beta U_c\left(f(K_{t+1}, \theta_{t+1}) + (1 - \delta)K_{t+1} - K_{t+2}\right)\left[f_K(K_{t+1}, \theta_{t+1}) + 1 - \delta\right].
\end{aligned}
$$

The exact solution to the problem is a policy rule, $K_{t+1} = g(K_t, \theta_t)$, that satisfies equation (4.2). That is,

$$E\left\{v\left(g(g(K_t, \theta_t), \theta_{t+1}), g(K_t, \theta_t), K_t, \theta_{t+1}, \theta_t\right)|\theta_t\right\} = 0 \tag{4.3}$$

for all $K_t \geq 0$ and for all $\theta_t$. Determining $g(\cdot)$ exactly is not feasible in general. In effect, solving for $g(\cdot)$ corresponds to solving a continuum of equations (one for each possible value of $K_t$ and $\theta_t$ in (4.3)) in a continuum of unknowns (a value for $g(K_t, \theta_t)$ corresponding to each possible $K_t$ and $\theta_t$). In practice, one of at least two alternative strategies are adopted. One approximates $g(\cdot)$ with

16

a function having a finite number of unknown parameters. The values of these parameters are determined by requiring that (4.3) (or linear combinations of it) be satisfied at a finite number of values of $K_t$ and $\theta_t$ (see, for example, Judd (1992) or Christiano and Fisher (1997).) This is a nonlinear undetermined coefficients method. The strategy pursued in this paper exploits the observation that if $v$ were linear, then the exact solution is straightforward to find. This strategy replaces $v$ by its linear first order Taylor series expansion about $K_{t+2} = K_{t+1} = K_t = K_s$ and $\theta_{t+1} = \theta_t = E\theta_t$. Here, $K_s$ denotes the nonstochastic steady state value of $K_t$, i.e., the solution to $v(K_s, K_s, K_s, E\theta_t, E\theta_t) = 0$. The rationale for this approach is that $v$ itself is often not far from linear. Since it is not *exactly* linear, however, some sort of linear approximation must be taken. The approximation is taken in the region of the nonstochastic steady state on the grounds that this is where the model variables are with high probability in the exact solution.

Let $z_{1t} = z_t = [K_{t+1} - K_s]'$, $s_t = [\theta_t - E\theta_t]'$, $P = \rho$, $m = 1$ and $\epsilon_t = e_t$. Using this notation, the linear approximation to (??) may be written in the form of (2.2):

$$E_t \{\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t | s_t\} = 0,$$

for all $z_{t-1}$ and $s_t$. Here, $n_1 = n = 1$, $q = 0$ and $r = 2$. Also,

$$\alpha_0 = \frac{dv(K_{t+2}, K_{t+1}, K_t, \theta_{t+1}, \theta_t)}{dK_{t+2}}, \text{ for } K_{t+2} = K_{t+1} = K_t = K_s \text{ and } \theta_{t+1} = \theta_t = E\theta_t,$$

and $\alpha_1$, $\alpha_2$, $\beta_0$, $\beta_1$ are defined similarly. Values for the scalars $A$ and $B$ in

$$z_t = A z_{t-1} + B s_t = g(z_{t-1}, s_t),$$

say, can be found that satisfy

$$E_t \{\alpha_0 g (g(z_{t-1}, s_t), s_{t+1}) + \alpha_1 g(z_{t-1}, s_t) + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t | s_t\} = 0,$$

for all $z_{t-1}$ and $s_t$. Evaluating this expression, we get the analog of (3.1):

$$\alpha(A)z_{t-1} + F s_t + E_t \beta_0 \varepsilon_{t+1} = \alpha(A)z_{t-1} + F s_t = 0.$$

Here, we have used the fact that we are in the 'standard case', $\mathcal{E}_t [X_t] = E_t [X_t]$, with $F = \tilde{F}$. Also,

$$\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2, \ F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (A\alpha_0 + \alpha_1)B].$$

To solve this, first find $A$ that satisfies $\alpha(A) = 0$, and then solve the linear equation in $B$, $F = 0$, to find $B$. It is easily verified that there are two real values of $A$ that solve $\alpha(A) = 0$: one is positive and less than one and the other exceeds $\beta^{-1}$ (see Stokey and Lucas (1989, p. 155).) The appropriate choice for $A$ is the smaller of these two. This is because it is known that in the version of this model with $\sigma_\varepsilon = 0$, the exact policy function, $K' = g(K, \theta)$, cuts the 45 degree line from above at $K_t = K_s$ when $\theta_t = E\theta_t$ (see Stokey and Lucas (1989, p. 135).) Thus, with this choice of $A$, we can hope that the resulting linear policy rule may be a good approximation to the exact policy rule.

As I have just shown, the approximate linear solution to this model can be found by computing the zeros of a particular second order scalar polynomial and then solving a linear equation in one unknown. Clearly, the matrix algebra approach laid out in the previous section for doing this is unnecessary here. Still, it is useful to indicate how that approach applies in this example as a way of illustrating how it works.

Consider the computation of the feedback part, $A$, discussed in section 3.1. The system without $s_t$ may be written in the first order difference equation form, (3.7), with the following settings for $a$, $b$, and $Y_t$ :

$$
Y_t = \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}, \; a = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix}, \; b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix}.
$$

Since we are in the invertible $a$ case (see section 3.1.1), we may define:

$$
P\Lambda P^{-1} = -a^{-1}b, \; \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},
$$

where $\lambda_i$ are the eigenvalues of $-a^{-1}b$, and the columns of $P$ are the associated eigenvectors. For convenience, the eigenvalues are ordered so that $\lambda_1 > \lambda_2$. It is readily verified by studying the second order polynomial in $\lambda$, $det(-a^{-1}b - \lambda I)$, that the $\lambda_i$'s are just the two solutions to $\alpha(A) = 0$ discussed above. In addition,

$$
P^{-1} = \begin{bmatrix} 1 & -\lambda_2 \\ 1 & -\lambda_1 \end{bmatrix}.
$$

The matrix $D$ referred to in section 3.1.1 is composed of the $n_1(r - 1) = 1$ row of $P^{-1}$ corresponding to the explosive eigenvalue, $\lambda_1$. Thus, $D = (1, -\lambda_2)$, with $D^1 = 1$ and $D^2 = -\lambda_2$, so that the policy rule in (3.11) corresponds here to

18

$k_{t+1} - k_s = \lambda_2(k_t - k_s)$. Since $\lambda_2$ is the smaller of the two eigenvalues, this solution coincides with the one identified above. It is trivial to verify that the procedure for computing the feedforward part, $B$, in section 3.2 above coincides with the single equation procedure described above, and we do not discuss this further here.

## 4.2. Example 2: Real Business Cycle Model with a Labor Decision

This example is used to demonstrate the possibility that the matrix $a$ in section 3.1 is singular, thus motivating the strategy described in section 3.1.2 for dealing with this. In the example, a planner solves the following problem: Maximize expected utility $E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$ subject to the resource constraint $C_t + K_{t+1} - (1 - \delta)K_t = f(K_t, N_t, \theta_t)$, where $\beta$ is the discount factor, $C_t$ is consumption, $K_t$ is the capital stock, $N_t$ is labor, $\delta$ is the depreciation rate on capital, $f$ is the production function and $\theta_t$ is a technology shock which has the same form as in the previous example. The efficiency conditions for the capital investment and labor decisions with the resource condition imposed to eliminate consumption are:

$$0 = E[v_K(K_{t+2}, N_{t+1}, K_{t+1}, N_t, K_t, \theta_{t+1}, \theta_t)|\theta_t] \tag{4.4}$$

and

$$0 = v_N(K_{t+1}, N_t, K_t, \theta_t). \tag{4.5}$$

for all $t \geq 0$, and for all $K_t$, $\theta_t$. Here, $f_K$ denotes the partial derivative of $f$ with respect to its first argument, $f_N$ denotes the partial derivative of $f$ with respect to its second argument, and $U_c$, $U_N$ denote the partial derivatives of utility with respect to consumption and labor. Also,

$$
\begin{aligned}
&v_K(K_{t+2}, N_{t+1}, K_{t+1}, N_t, K_t, \theta_{t+1}, \theta_t)\\
=\ & U_c\left(f(K_t, N_t, \theta_t) + (1-\delta)K_t - K_{t+1}, N_t\right)\\
& -\beta U_c\left(f(K_{t+1}, N_{t+1}, \theta_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, N_{t+1}\right)\left[f_K(K_{t+1}, N_{t+1}, \theta_{t+1}) + 1 - \delta\right]
\end{aligned}
$$

and,

$$
\begin{aligned}
v_N(K_{t+1}, N_t, K_t, \theta_t)\ =\ & U_N\left(f(K_t, N_t, \theta_t) + (1-\delta)K_t - K_{t+1}, N_t\right)\\
& + U_c\left(f(K_t, N_t, \theta_t) + (1-\delta)K_t - K_{t+1}, N_t\right) f_N(K_t, N_t, \theta_t).
\end{aligned}
$$

Let $z_{1,t} = z_t = [K_{t+1} - K_s, N_t - N_s]'$, where $K_s$ and $N_s$ solve

$$v_K(K_s, N_s, K_s, N_s, K_s, E\theta_t, E\theta_t) = 0, \quad v_N(K_s, N_s, K_s, E\theta_t) = 0.$$

19

Also, let $s_t = \theta_t$, $P = \rho$, $n_1 = n = 2$, $q = 0$, $r = 2$, and $\epsilon_t = e_t$. After approximating $v_K$ and $v_N$ by their linear Taylor series expansions about steady state, the resulting linearized system can be expressed in the form of (2.2), with $\mathcal{E}_t$ corresponding to the 'standard case', i.e., $\mathcal{E}_t[X_t] = E_t X_t$. To find $A$ in (2.7) using the method of section 3.1, we begin by setting up (3.7):

$$Y_t = \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}, \quad a = \begin{bmatrix} \alpha_0 & 0_{2 \times 2} \\ 0_{2 \times 2} & I_2 \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I_2 & 0_{2 \times 2} \end{bmatrix},$$

where $0_{2 \times 2}$ is the 2 by 2-dimensional matrix of zeros and $I_2$ is a 2 by 2-dimensional identity matrix. Let the first and second rows of (3.7) correspond to the linearized version of equations (4.4) and (4.5), respectively. That is,

$$\alpha_0 = \begin{bmatrix} v_{K,1} & v_{K,2} \\ 0 & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} v_{K,3} & v_{K,4} \\ v_{N,1} & v_{N,2} \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} v_{K,5} & 0 \\ v_{N,3} & 0 \end{bmatrix}, \qquad (4.6)$$

where $v_{K,i}$ and $v_{N,i}$ are the derivatives of $v_K$ and $v_N$, respectively, with respect to their $i^{th}$ argument, evaluated in nonstochastic steady state. The fact that $a$ is not invertible reflects that the second row of $\alpha_0$ and, hence, of $a$ itself, is composed of zeros.[16] This is why the matrix $A$ must be computed using the method of section 3.1.2 and not of section 3.1.1.

Computation of $B$ is straightforward. For this, we require the $\beta$'s:

$$\beta_0 = \begin{pmatrix} v_{K,6} \\ 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} v_{K,7} \\ v_{N,4} \end{pmatrix}. \qquad (4.7)$$

The rest is a straightforward application of the methods in section 3.2.

---

[16] At first glance, it may appear that one way to deal with the problem of lack of invertibility is to instead define $a$ and $b$ as follows:

$$a = \begin{bmatrix} \alpha_0 & \alpha_1 \\ 0_{2 \times 2} & I_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0_{2 \times 2} & \alpha_2 \\ -I_2 & 0_{2 \times 2} \end{bmatrix}.$$

In fact, this does not help the singularity problem. To see this, note that the second row of $\alpha_1$ can be expressed as a linear combination of the rows of $I_2$. But, given that the second row of $\alpha_0$ is composed of zeros, this implies that the second row of $a$ itself can be expressed as a linear combination of its third and fourth rows.

### 4.3. Example 3: Real Business Cycle Model with Labor Decision and Imperfect Information

In some models, the expectation operator applied to different Euler equations is evaluated relative to different information sets. This can happen when some variables are determined before the realization of some or all the current period shocks, while others are determined afterward. We illustrate this possibility using a modified version of the previous example. In the modification, we assume the date $t$ labor decision is made after, and the date $t$ investment decision is made before, the realization of the current value of $\theta_t$. Thus, the date $t$ investment decision is made based on the information set $\Omega_{1t} = \{z_{t-1}, \theta_{t-1}\}$ and the date $t$ hours decision is made based on the information set $\Omega_{2t} = \{z_{t-1}, \theta_t\}$. With this modification, we cannot define $s_t$ as in example 2. Instead, we set $s_t = (\theta_t, \theta_{t-1})'$, $\epsilon_t = (e_t, 0)'$ and

$$P = \begin{bmatrix} \rho & 0 \\ 1 & 0 \end{bmatrix}.$$

To put this problem in the form of (2.2), the $\alpha_i$'s are constructed using (4.6). The discussion of the computation of the matrix $A$ in the previous example also applies here. The impact of the changed information sets occurs in the construction of the $\beta_i$'s and in the formulas for computing $B$. The $\beta_i$'s are obtained as follows:

$$\beta_0 = \begin{pmatrix} v_{K,6} & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} v_{K,7} & 0 \\ v_{N,4} & 0 \end{pmatrix},$$

where the $v_{K,i}$'s and $v_{N,4}$ are defined in the previous example. To get $B$ using the computed matrix $A$ and these $\beta_i$'s, first compute $F$ in (3.3):

$$\begin{aligned} F &= [(\beta_0 + \alpha_0 B)P + (\beta_1 + \alpha_1 B + \alpha_0 AB)] \\ &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \end{aligned}$$

so that,

$$\mathcal{E}_t F s_t = \mathcal{E}_t \begin{bmatrix} F_{11}\theta_t + F_{12}\theta_{t-1} \\ F_{21}\theta_t + F_{22}\theta_{t-1} \end{bmatrix} = \begin{bmatrix} 0 & F_{12} + \rho F_{11} \\ F_{21} & F_{22} \end{bmatrix} s_t = \tilde{F} s_t.$$

Then, it is readily confirmed that the matrix $R$ in (3.26) is:

$$R = \begin{bmatrix} \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Define:
$$\tilde{d} = Rvec[P'\beta_0' + \beta_1'],$$

and $\tilde{q}$ is the $3 \times 3$ matrix obtained by deleting the first column from the $3 \times 4$ matrix,
$$R\{\alpha_0 \otimes P' + [\alpha_0 A + \alpha_1] \otimes I\}.$$

Finally,
$$\tilde{\delta} = \overline{vec}(B') = \begin{bmatrix} B_{12} \\ B_{21} \\ B_{22} \end{bmatrix}.$$

Then, $\tilde{\delta}$ is computed from $\tilde{\delta} = -\tilde{q}^{-1}\tilde{d}$. The matrix, $B$, is obtained as follows:
$$B = \begin{bmatrix} 0 & \tilde{\delta}_1 \\ \tilde{\delta}_2 & \tilde{\delta}_3 \end{bmatrix},$$

where $\tilde{\delta} = (\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3)'$.

## 4.4. Example 4: Real Business Cycle Model with Time-to-Build Investment Technology

This example illustrates the possibility that $q > 0$ in the construction of $z_t$ in $(2.1)$. Consider a two period time-to-build economy, with exogenously fixed labor. This is a modified version of the economy in example 1. The modification is that now resources must be devoted for two periods, rather than just one, to augment the stock of capital. A planner maximizes $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$, subject to the resource constraint, $C_t + I_t \leq f(K_t, \theta_t)$, where $\theta_t$ has the same distribution as in the previous examples. Also, $I_t$ denotes investment, which is composed of resources devoted to projects initiated in the current period and resources devoted to projects initiated in the previous period. That is:

$$I_t = \phi x_t + (1 - \phi)x_{t-1}. \tag{4.8}$$

The investment technology requires that if $x_t$ units of gross investment is to occur during period $t + 1$, i.e.,

$$K_{t+2} - (1 - \delta)K_{t+1} = x_t, \tag{4.9}$$

then resources in the amount $\phi x_t$ must be applied in period $t$ and $(1 - \phi)x_t$ must be applied in period $t + 1$, where $0 \le \phi \le 1$. In (4.9), $\delta$ is the rate of depreciation on a unit of capital. Once an investment project is initiated, its scale cannot be expanded or contracted. As a result, $x_{t-1}$ (hence, $K_{t+1}$) is a state variable at time $t$, in addition to $K_t$ and $\theta_t$. Since the date $t$ choice variable is $K_{t+2}$, the policy rule is $K_{t+2} = g(K_{t+1}, K_t, \theta_t)$. This implies that $n_1 = 1$, $z_{1t} = K_{t+2} - K_s$, and $q = 1$, so that

$$z_t = \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}, \ z_{2t} = z_{1t-1}.$$

I now describe how to set up the problem in the form of (2.2). Substituting out for $I_t$ in the resource constraint using (4.8) and (4.9), we get:

$$C_t = f(K_t, \theta_t) - (\phi_1 K_{t+2} + \phi_2 K_{t+1} + \phi_3 K_t),$$

where

$$\phi_1 = \phi, \ \phi_2 = (1 - \phi) - \phi(1 - \delta), \ \phi_3 = -(1 - \phi)(1 - \delta).$$

The planner's problem is:

$$\max_{\{K_{t+2}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U \left( f(K_t, \theta_t) - (\phi_1 K_{t+2} + \phi_2 K_{t+1} + \phi_3 K_t) \right),$$

which leads to the following Euler equation:

$$E \left\{ \phi_1 U_{c,t} + \phi_2 \beta U_{c,t+1} - \beta^2 U_{c,t+2} \left[ f_{K,t+2} - \phi_3 \right] | \theta_t \right\} = 0,$$

for all $\theta_t$, $K_t$, where $U_{c,t}$ is the marginal utility of consumption. This Euler equation can be written

$$E \left\{ v(K_{t+4}, K_{t+3}, K_{t+2}, K_{t+1}, K_t, \theta_{t+2}, \theta_{t+1}, \theta_t) | \theta_t \right\} = 0.$$

Solve for the nonstochastic steady state by finding the $K_s$ that satisfies

$$v(K_s, K_s, K_s, K_s, K_s, E\theta_t, E\theta_t, E\theta_t) = 0.$$

Let $r = 3$, $s_t = \theta_t$, $P = \rho$, $\epsilon_t = e_t$, $m = 1$ and the expectation operator in (2.2) corresponds to what it is in the standard case. Finally, let $V_i$ denote the

23

derivative of $v$ with respect to its $i^{th}$ argument, evaluated in nonstochastic steady state, $i = 1, ..., 8$ so that,

$$\begin{aligned}
\alpha_0 &= [V_1, 0], \ \alpha_1 = [V_2, 0], \ \alpha_2 = [V_3, 0], \ \alpha_3 = [V_4, V_5], \\
\beta_0 &= V_6, \ \beta_1 = V_7, \ \beta_2 = V_8.
\end{aligned}$$

With the problem set up in the form of (2.2), the solution strategy of the previous section can be applied to develop an approximation to the policy rule, $g$.

### 4.5. Example 5: Model with Heterogeneous Capital

With one exception, the model described here corresponds to the real business cycle model in example 2. Instead of capital being homogeneous, there exist different types of capital, each embodying different levels of technology. Capital embodying relatively low-level technology - which tends to be older capital - is scrapped as its productivity lags behind that of the 'leading edge' technology. The leading edge technology grows as a random walk with drift. The salvage value of scrapped capital and investment goods are used to produce new capital, which incorporates the leading edge technology. The model and basic solution strategy are taken from Campbell (1997). The allocations in the model's competitive equilibrium solve a social planning problem. To keep the discussion brief, I focus on that problem here. For a detailed discussion of the competitive equilibrium, see Campbell (1997). A closely related model, in which the heterogeneity reflects varying productivity of workers, is presented in Merz (1996).

There is one good in this economy, which can be used for consumption or investment. Labor and capital are needed for production. Capital embodies different levels of technology. The quantity of period $t$ capital with level of technology $\theta$ is denoted $k_t(\theta)$, where $\theta$ is an index of productivity and $\theta \in (-\infty, +\infty)$. The aggregate production function of this model economy is:

$$Y_t = N_t^\alpha \left[ \int_{-\infty}^{\infty} e^\theta k_t(\theta) d\theta \right]^{1-\alpha} \tag{4.10}$$

where $N_t$ denotes labor input and $\alpha \in (0, 1)$.

At the end of the period a fraction, $\delta$, of capital at each technology level depreciates. The remaining capital is either scrapped, or it undergoes a random change in the level of its technology. Capital is scrapped if its level of technology

24

is below some endogenously selected cutoff, $\bar{\theta}_t$. A unit of capital that is scrapped has salvage value $s$. The total amount of salvaged capital in period $t$ is:

$$S_t = (1 - \delta)s \int\limits_{-\infty}^{\bar{\theta}_t} k_t(\theta)d\theta. \tag{4.11}$$

Capital that is not scrapped draws another productivity level according to:

$$\theta_{t+1} \sim N(\theta_t, \sigma^2). \tag{4.12a}$$

A unit of newly constructed capital that operates in period $t + 1$ draws its level of technology from the following distribution:

$$\theta_{t+1} \sim N(\nu_t, \sigma^2), \tag{4.13a}$$

where $\nu_t$ is the leading edge technology which evolves according to:

$$\nu_t = \mu_\nu + \nu_{t-1} + \epsilon_t^\nu \text{ where } \epsilon_t^\nu \sim N(0, \sigma_\nu^2), \ \mu_\nu > 0. \tag{4.14}$$

Each unit of capital draws its $\theta$ independently of the others.

The distribution assumptions, (4.12a) and (4.13a), imply that the law of motion of the distribution of capital by state of technology evolves as follows:

$$k_{t+1}(\theta') = (1 - \delta) \int\limits_{\bar{\theta}_t}^{\infty} \frac{1}{\sigma} \phi \left( \frac{\theta' - \theta}{\sigma} \right) k_t(\theta)d\theta \ + \ \frac{1}{\sigma} \phi \left( \frac{\theta' - \nu_t}{\sigma} \right) I_t. \tag{4.15}$$

Here, $k_{t+1}(\theta')$ denotes the amount of capital in period $t + 1$ having a technology level $\theta'$ and $\phi$ is the standard normal density function. The first term on the right side of equation (4.15) is the amount of period $t + 1$ capital with technology level $\theta'$ which evolved from capital which was in place in period $t$. The second term is the new capital with productivity $\theta'$ created in period $t$. Here, $I_t$ denotes total investment in period $t$, and this must satisfy the resource constraint:

$$C_t + I_t = Y_t + S_t. \tag{4.16}$$

The planner's problem is to maximize the utility of a representative agent given equations (4.10)-(4.16). The maximization problem is, therefore:

$$\max_{\substack{\{C_t\}_{t=0}^\infty, \{N_t\}_{t=0}^\infty, \{I_t\}_{t=0}^\infty, \{Y_t\}_{t=0}^\infty, \\ \{\bar{\theta}_t\}_{t=0}^\infty, \{S_t\}_{t=0}^\infty, \{k_{t+1}(\theta)\}_{t=0}^\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{4.17}$$

subject to (4.10)-(4.16), where:

$$U\left(C_t, N_t\right) = \log(C_t) + \kappa \log(1 - N_t). \qquad (4.18)$$

The strategy for approximating the solution to the model proceeds in three steps. First compute the nonstochastic steady state values of the model variables. By this I mean the state to which the system converges when $\varepsilon_t^\nu \equiv 0$. I do allow idiosyncratic uncertainty in the form of $\sigma > 0$ in (4.13a) and (4.12a). Second, linearize the various equations and first order conditions that characterize the solution about the steady state values of the variables. Third, put the linearized equations in the form of (2.2) and apply the solution procedure described in section 3.

Implementation of this computational strategy requires confronting two basic problems. First, because the model exhibits growth, there is no set of constant steady state values to which the variables tend asymptotically. This problem is solved in the usual way. I exploit the model's balanced growth property by finding a scaling of the variables such that they converge in steady state. I then approximate the policy rules governing the evolution of the scaled variables of the model. At the end, a straightforward mapping takes one from the scaled objects solved for in the computations to the unscaled objects of interest. A second problem for the computational strategy is that one of the 'variables' in the model is actually a distribution. I follow Campbell (1997) by discretizing that distribution, so that $k_t(\theta)$ viewed as a function of $\theta$ is approximated by a vector. In effect, the model corresponds to a modification of the real business cycle model of example 1, in which the capital stock is a vector.

### 4.5.1. The Scaled Planning Problem

I begin with a discussion of the scaling of the variables. To understand the nature of the problem that scaling is designed to address, note that in equilibrium $k_t(\theta)$ shifts weight towards higher values of $\theta$ as both the leading edge state of technology and $\bar{\theta}_t$ increase with time. In addition, $k_t(\theta)$ increases in size over time. The scaling must take into account both of these phenomena. To accommodate the right-shift in $k_t(\theta)$ and $\bar{\theta}_t$, define:

$$\tilde{\theta} \equiv \theta - \nu_{t-1}, \ \tilde{\bar{\theta}}_t \equiv \bar{\theta}_t - \nu_{t-1} \ . \qquad (4.19)$$

Also,
$$
\tilde{k}_{t+1}(\tilde{\theta}) \equiv \frac{k_{t+1}(\theta)}{e^{\nu_t(1-\alpha)/\alpha}}, \ \tilde{C}_t \equiv \frac{C_t}{e^{\nu_t(1-\alpha)/\alpha}},
$$
$$
\tilde{I}_t \equiv \frac{I_t}{e^{\nu_t(1-\alpha)/\alpha}}, \ \tilde{S}_t \equiv \frac{S_t}{e^{\nu_t(1-\alpha)/\alpha}}, \ \tilde{Y}_t \equiv \frac{Y_t}{e^{\nu_t(1-\alpha)/\alpha}} \quad . \tag{4.20}
$$

The scaled planner's problem can be expressed in terms of the scaled variables as:

$$
\max_{\substack{\{\tilde{C}_t\}_{t=0}^\infty, \{N_t\}_{t=0}^\infty, \{\tilde{I}_t\}_{t=0}^\infty, \{\tilde{Y}_t\}_{t=0}^\infty, \\ \{\tilde{\bar{\theta}}_t\}_{t=0}^\infty, \{\tilde{S}_t\}_{t=0}^\infty, \{\tilde{k}_{t+1}(\tilde{\theta}); \tilde{\theta} \in (-\infty,\infty)\}_{t=0}^\infty}} E_0 \sum_{t=0}^\infty \beta^t U\left(\tilde{C}_t, N_t\right) \tag{4.21}
$$

subject to:

$$
\begin{aligned}
\tilde{C}_t + \tilde{I}_t &= \tilde{Y}_t + \tilde{S}_t \\
\tilde{Y}_t &= e^{-\Delta\nu_t(1-\alpha)/\alpha} N_t^\alpha \tilde{K}_t^{1-\alpha} \\
\tilde{S}_t &= (1-\delta)s e^{-\Delta\nu_t(1-\alpha)/\alpha} \int_{-\infty}^{\tilde{\bar{\theta}}_t} \tilde{k}_t(\tilde{\theta}) d\tilde{\theta} \\
\tilde{k}_{t+1}(\tilde{\theta}') &= (1-\delta)e^{-\Delta\nu_t(1-\alpha)/\alpha} \int_{\tilde{\bar{\theta}}_t}^{\infty} \frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}'-\tilde{\theta}+\Delta\nu_t}{\sigma}\right) \tilde{k}_t(\tilde{\theta}) d\tilde{\theta} + \frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}'}{\sigma}\right)\tilde{I}_t,
\end{aligned} \tag{4.22}
$$

where $\Delta\nu_t \equiv \nu_t - \nu_{t-1}$ and the effective stock of capital, $\tilde{K}_t$, is defined as:

$$
\tilde{K}_t \equiv \int_{-\infty}^{\infty} e^{\tilde{\theta}} \tilde{k}_t(\tilde{\theta}) d\tilde{\theta}. \tag{4.23}
$$

In (4.22), $\theta'$ is replaced by $\tilde{\theta}' - \nu_t$, consistent with (4.19). Eliminating $\tilde{C}_t$, $\tilde{Y}_t$, and $\tilde{S}_t$ and expressing the problem in Lagrangian form, we get:

$$
\max_{\{N_t\}_{t=0}^\infty, \{\tilde{I}_t\}_{t=0}^\infty, \{\tilde{\bar{\theta}}_t\}_{t=0}^\infty, \{\tilde{k}_{t+1}(\tilde{\theta}); \tilde{\theta} \in (-\infty,\infty)\}_{t=0}^\infty}
$$
$$
E_0 \sum_{t=0}^\infty \beta^t \left\{ U\left(\tilde{Y}_t + \tilde{S}_t - \tilde{I}_t, N_t\right) + \int_{-\infty}^{\infty} \lambda_t(\tilde{\theta}') m_t(\tilde{\theta}') d\tilde{\theta}' \right\}, \tag{4.24}
$$

where

$$
\tilde{Y}_t + \tilde{S}_t = e^{-\Delta\nu_t(1-\alpha)/\alpha} \left[ N_t^\alpha \tilde{K}_t^{1-\alpha} + (1-\delta)s \int_{-\infty}^{\tilde{\bar{\theta}}_t} \tilde{k}_t(\tilde{\theta}) d\tilde{\theta} \right]
$$

and

$$
\begin{aligned}
m_t(\tilde{\theta}') &= (1-\delta)e^{-\Delta\nu_t(1-\alpha)/\alpha} \int_{\tilde{\bar{\theta}}_t}^{\infty} \frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}'-\tilde{\theta}+\Delta\nu_t}{\sigma}\right) \tilde{k}_t(\tilde{\theta}) d\tilde{\theta} + \frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}'}{\sigma}\right)\tilde{I}_t \\
&\quad -\tilde{k}_{t+1}(\tilde{\theta}')
\end{aligned} \tag{4.25}
$$

Here, $\lambda_t(\tilde{\theta}') \geq 0$ is the Lagrange multiplier associated with each period-$t$ constraint. From the Lagrangian in (4.24) we can obtain the necessary conditions for an optimum with respect to the variables $N_t, \tilde{I}_t, \overline{\tilde{\theta}}_t$, and $\tilde{k}_t(\tilde{\theta})$ for each $\tilde{\theta} \in (-\infty, \infty)$ by direct differentiation.[17]

The necessary conditions with respect to $N_t$, $\tilde{I}_t$ and $\overline{\tilde{\theta}}_t$ are, respectively:

$$\frac{\alpha e^{-\Delta\nu_t(1-\alpha)/\alpha}}{\tilde{C}_t} \left(\frac{\tilde{K}_t}{N_t}\right)^{1-\alpha} = \frac{\kappa}{1 - N_t} \tag{4.26}$$

$$1 = \int_{-\infty}^{\infty} q_t(\tilde{\theta}')\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}'}{\sigma}\right) d\tilde{\theta}' \tag{4.27}$$

$$s = \int_{-\infty}^{\infty} q_t(\tilde{\theta}')\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}' - \overline{\tilde{\theta}}_t + \Delta\nu_t}{\sigma}\right) d\tilde{\theta}' \tag{4.28}$$

where $q_t(\tilde{\theta}') \equiv \tilde{C}_t\lambda_t(\tilde{\theta}')$ is the price of end-of-period capital (after it has drawn its state of technology, $\tilde{\theta}$, for next period) in the scaled competitive equilibrium decentralization of this model (see Campbell (1997)). The first of these conditions is the same as in a standard real business cycle model and needs no elaboration. The second two are particular to this model. The condition, (4.27), equates the consumption cost of one investment good, unity, to the value of the extra new capital of all vintages that it produces. According to condition (4.28), the salvage value, $s$, of a unit of capital that is on the margin of being scrapped must be equal to its value in case it is not scrapped, in which case it draws a new $\theta$.

The necessary condition with respect to $\tilde{k}_{t+1}(\tilde{\theta})$ for a particular value of $\tilde{\theta}$, $\tilde{\theta}_l \in (-\infty, \infty)$, is:

$$\beta E_t \frac{e^{-\Delta\nu_{t+1}(1-\alpha)/\alpha}}{\tilde{C}_{t+1}} \left\{ (1 - \alpha)e^{\tilde{\theta}_l}\left(\frac{\tilde{K}_{t+1}}{N_{t+1}}\right)^{-\alpha} + (1 - \delta)s \cdot 1\{\tilde{\theta}_l < \overline{\tilde{\theta}}_{t+1}\} \right.$$
$$\left. + (1 - \delta) \cdot 1\{\tilde{\theta}_l \geq \overline{\tilde{\theta}}_{t+1}\} \int_{-\infty}^{\infty} q_{t+1}(\tilde{\theta}')\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}' - \tilde{\theta}_l + \Delta\nu_{t+1}}{\sigma}\right) d\tilde{\theta}' \right\} \tag{4.29}$$
$$= \frac{1}{\tilde{C}_t}q_t(\tilde{\theta}_l),$$

where $1\{\cdot\}$ is the indicator function and is defined as follows:

$$1\{x\} = \begin{cases} 1 & \text{if x is true} \\ 0 & \text{otherwise} \end{cases}. \tag{4.30}$$

---

[17] One of these 'variables' is a function. For a discussion of maximization with respect to a function, see, for example, Luenberger (1969, chapter 7).

The first order condition in (4.29) corresponds to the intertemporal Euler equation in the standard real business cycle model. The term on the right of the equality is the value, expressed in utility terms, of a marginal unit of capital having technology level $\tilde{\theta}_l$ at the end of period $t$. This valuation occurs after period $t$ units of capital that survive depreciation and scrappage have drawn their period $t+1$ value of $\theta$. The term on the left of the equality is the payoff, in discounted utility terms, from that marginal unit of capital. The first term is the period $t + 1$ marginal product of that capital, and the second and third terms correspond to the end of period $t+1$ value of that unit of capital. The first of these two corresponds to the case in which $\tilde{\theta}_l$ falls below next period's scrappage cutoff, in which case the value of marginal $\tilde{k}_{t+1}(\tilde{\theta}_l)$ is just its scrap value after depreciation. The second term corresponds to the case in which $\tilde{\theta}_l$ lies above next period's scrappage cutoff. In that case, the undepreciated part of a unit of $\tilde{k}_{t+1}(\tilde{\theta}_l)$ is valued at the new value of $\tilde{\theta}$ that it will draw.

### 4.5.2. Model Steady States

To solve for the model's steady state, I drop time subscripts on the variables in equations (4.25) with $m_t(\tilde{\theta}') = 0$ and (4.26)-(4.29), and approximate the integrals with a quadrature formula. In particular, the integrals are approximated by a weighted sum of the integrand evaluated at the following $M$ abscissas:

$$\tilde{\theta} \in \left\{ \tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_M \right\}.$$

These same abscissas are used for each integral, regardless of the range of integration.[18]

I approximate $\tilde{K}$ in (4.23) as follows:

$$\tilde{K} \;=\; \sum_{j=1}^{M} w_j e^{\tilde{\theta}_j} \tilde{k}_j \tag{4.31}$$

---

[18]Specifically, consider the integral, $\int_{-\infty}^{\infty} g(\tilde{\theta}) d\tilde{\theta}$. I first truncate the range of integration to $\tilde{\theta} \in (\tilde{\theta}^l, \tilde{\theta}^u)$. The abscissas, $\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_M$, are uniformly distributed over the interval, $(\tilde{\theta}^l, \tilde{\theta}^u)$. Thus, with $\tilde{u}_j = (j - M/2)/(M/2), j = 0,...,M$ I set $\tilde{\theta}_j = \tilde{\theta}^l + 0.5(\tilde{\theta}^u - \tilde{\theta}^l)(\tilde{u}_j + 1), j = 0,...,M$. Then, the integral is approximated by the weighted sum, $\sum_{i=1}^{M} w_j g(\tilde{\theta}_i)$, where $w_j = (\tilde{\theta}^u - \tilde{\theta}^l)/M$. In (4.25), the lower bound of the range of integration is truncated. I accommodate this by using a weight, $v_j$, that is the product of $w_j$ and the appropriately constructed indicator function. I accommodate integrals in which the upper bound of the range of integration is truncated in a similar way. I refer to the weights used for this by $u_j$.

where $w_j$ is the quadrature weight and $\tilde{k}_j \equiv \tilde{k}(\tilde{\theta}_j)$. Similarly, the resource constraint and equations (4.26), (4.27) and (4.28) yield in steady state:

$$
\begin{aligned}
\tilde{C} &= e^{-\mu_\nu(1-\alpha)/\alpha}\left[N^\alpha \tilde{K}^{1-\alpha} + (1-\delta)s\sum_{j=1}^{M} u_j\tilde{k}_j\right] - \tilde{I} \\
\frac{\alpha e^{-\mu_\nu(1-\alpha)/\alpha}}{\tilde{C}}\left(\frac{\tilde{K}}{N}\right)^{1-\alpha} &= \begin{cases} \kappa & \text{for Hansen utility} \\ \frac{\kappa}{1-N} & \text{for log-log utility} \end{cases} \\
1 &= \sum_{j=1}^{M} w_j \beta e^{-\mu_\nu(1-\alpha)/\alpha}q_j^0\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_j}{\sigma}\right) \\
s &= \sum_{j=1}^{M} w_j \beta e^{-\mu_\nu(1-\alpha)/\alpha}q_j^0\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_j-\tilde{\tilde{\theta}}+\mu_\nu}{\sigma}\right).
\end{aligned}
\tag{4.32}
$$

Here, $w_j$ and $u_j$ are the quadrature weights and $q_j^0$ is defined as follows:

$$
q_j^0 \equiv \frac{q(\tilde{\theta}_j)}{\beta}e^{\mu_\nu(1-\alpha)/\alpha}
\tag{4.33}
$$

Equation (4.29) in non-stochastic steady state becomes:

$$
\begin{aligned}
q_l^0 &= (1-\alpha)e^{\tilde{\theta}_l}\left(\frac{\tilde{K}}{N}\right)^{-\alpha} + (1-\delta)s\cdot 1\{\tilde{\theta}_l < \tilde{\tilde{\theta}}\} \\
&+ (1-\delta)\cdot 1\{\tilde{\theta}_l \geq \tilde{\tilde{\theta}}\}\sum_{j=1}^{M} w_j q_j^0\beta e^{-\mu_\nu(1-\alpha)/\alpha}\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_j-\tilde{\theta}_l+\mu_\nu}{\sigma}\right)
\end{aligned}
\tag{4.34}
$$

Finally, our approximation to the scaled, steady state law of motion of capital is:

$$
\tilde{k}_l = (1-\delta)e^{-\mu_\nu(1-\alpha)/\alpha}\sum_{j=1}^{M} v_j\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_l-\tilde{\theta}_j+\mu_\nu}{\sigma}\right)\tilde{k}_j + \frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_l}{\sigma}\right)\tilde{I}
\tag{4.35}
$$

where $v_j$ are the quadrature weights. Note that equations (4.34) and (4.35) must hold for every $l \in \{1,...,M\}$ and, therefore, constitute a set of $M$ equations each. Solving for the steady state involves solving for the $2M+6$ unknowns:

$$
\{q_l^0\}_{l=1}^{l=M}, \{\tilde{k}_l\}_{l=1}^{l=M}, \tilde{K}, \tilde{\tilde{\theta}}, \tilde{C}, \tilde{I}, \tilde{Y}, \tilde{S}
$$

and the value of the parameter $\kappa$ in the utility function to match an empirically plausible value of $N$.

The first step in computing our approximation to the steady state finds $\{q_l^0\}_{l=1}^{l=M}$, $\tilde{K}$ and $\tilde{\tilde{\theta}}$ using the $M$ equations in (4.34) and the last two equations in (4.32). A

30

simple procedure to do this exploits the fact that, for a given value of $\tilde{\tilde{\theta}}$, equations (4.34) and the third equation in (4.32) form a linear system of $M+1$ equations in the $M+1$ unknowns, $\{q_l^0\}_{l=1}^{l=M}$ and $\left(\tilde{K}/N\right)^{-\alpha}$. The procedure then assumes a value of $\tilde{\tilde{\theta}}$, solves the linear system for $\{q_l^0\}_{l=1}^{l=M}$ and $\left(\tilde{K}/N\right)^{-\alpha}$ and then checks if the last equation in (4.32) is satisfied. If it is satisfied, then a solution has been found; if not, one tries a different value of $\tilde{\tilde{\theta}}$. In effect, $\tilde{\tilde{\theta}}$ and $\{q_l^0\}_{l=1}^{l=M}$ and $\left(\tilde{K}/N\right)^{-\alpha}$ are found by solving a nonlinear function in $\tilde{\tilde{\theta}}$ alone.

The next step solves for $\{\tilde{k}_l\}_{l=1}^{l=M}$ and $\tilde{I}$ using the pre-assigned value of $N$, and the linear system of equations consisting of the $M$ equations in (4.35) and equation (4.31). I solve for $\tilde{C}$ and for $\kappa$ from (4.32). Finally, one can solve for $\tilde{Y}$ and $\tilde{S}$ from the discrete version of (4.22):

$$
\begin{aligned}
\tilde{Y} &= e^{-\mu_\nu(1-\alpha)/\alpha} N^\alpha \tilde{K}^{1-\alpha} \\
\tilde{S} &= (1-\delta)e^{-\mu_\nu(1-\alpha)/\alpha} s \sum_{j=1}^{M} u_j \tilde{k}_j
\end{aligned}
$$

### 4.5.3. Dynamic Model Solution

To approximate the dynamic solution, I linearize the equations that characterize the model solution about the steady state that was just computed. I adopt the same quadrature approximation for the integrals in these equations as was done in the steady state calculations. In particular, they involve the fixed set of abscissas, $\left\{\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_M\right\}$. Define the following functions:

$$
\tilde{K}_t = \sum_{j=1}^{M} w_j e^{\tilde{\theta}_j} \tilde{k}_t(\tilde{\theta}_j) \equiv \tilde{K}(\tilde{k}_t(\tilde{\theta}_1) \ldots \tilde{k}_t(\tilde{\theta}_M))
$$

$$
\tilde{C}_t = \begin{cases} \frac{\alpha e^{-\Delta\nu_t(1-\alpha)/\alpha}}{\kappa}\left(\frac{\tilde{K}_t}{N_t}\right)^{1-\alpha} & \text{for Hansen utility} \\ \frac{\alpha e^{-\Delta\nu_t(1-\alpha)/\alpha}(1-N_t)}{\kappa}\left(\frac{\tilde{K}_t}{N_t}\right)^{1-\alpha} & \text{for log-log utility} \end{cases} \equiv \tilde{C}(\tilde{k}_t(\tilde{\theta}_1) \ldots \tilde{k}_t(\tilde{\theta}_M), N_t, \Delta\nu_t)
$$

$$
\tilde{I}_t = e^{-\Delta\nu_t(1-\alpha)/\alpha}\left\{ N_t^\alpha \tilde{K}_t^{1-\alpha} + (1-\delta)s \sum_{j=1}^{M} u_j(\tilde{\tilde{\theta}}_t)\tilde{k}_t(\tilde{\theta}_j) \right\} - \tilde{C}_t \equiv \tilde{I}(\tilde{k}_t(\tilde{\theta}_1) \ldots \tilde{k}_t(\tilde{\theta}_M), N_t, \tilde{\tilde{\theta}}_t, \Delta\nu_t)
$$

Here,

$$
u_j(\tilde{\tilde{\theta}}_t) = 1\{\tilde{\theta}_j \le \tilde{\tilde{\theta}}_t\} \times (\tilde{\theta}^u - \tilde{\theta}^l)/M,
$$

and $\tilde{\theta}^u$, $\tilde{\theta}^l$ are upper and lower limits of integration which replace $+\infty$ and $-\infty$. These equations are used to eliminate $\tilde{K}_t$, $\tilde{C}_t$, and $\tilde{I}_t$ in what follows. For example, the approximation to the law of motion of capital in equation (4.22) is:

$$\tilde{k}_{t+1}(\tilde{\theta}_l) = (1-\delta)e^{-\Delta\nu_t(1-\alpha)/\alpha} \sum_{j=1}^{M} v_j(\tilde{\bar{\theta}}_t)\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_l - \tilde{\theta}_j + \Delta\nu_t}{\sigma}\right)\tilde{k}_t(\tilde{\theta}_j) \;+\; \frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_l}{\sigma}\right)\tilde{I}_t,$$

for $l \in \{1, ..., M\}$ or,

$$V_1(\tilde{k}_{t+1}, \tilde{k}_t, N_t, \tilde{\bar{\theta}}_t, \Delta\nu_t) = 0_{M\times 1},$$

where

$$\tilde{k}_{t+1} \equiv \left[\tilde{k}_{t+1}(\tilde{\theta}_1) \ldots \tilde{k}_{t+1}(\tilde{\theta}_M)\right]',$$

and

$$v_j(\tilde{\bar{\theta}}_t) = 1\{\tilde{\theta}_j \geq \tilde{\bar{\theta}}_t\} \times (\tilde{\theta}^u - \tilde{\theta}^l)/M, \; j = 1, ..., M.$$

Similarly, the approximations of equations (4.27) and (4.28) are, respectively:

$$1 - \sum_{j=1}^{M} w_j q_t(\tilde{\theta}_j)\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_j}{\sigma}\right) = 0,$$

or

$$V_2(q_t, \Delta\nu_t) \;=\; 0 \tag{4.36}$$

and,

$$s - \sum_{j=1}^{M} w_j q_t(\tilde{\theta}_j)\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_j - \tilde{\bar{\theta}}_t + \Delta\nu_t}{\sigma}\right) = 0$$

or,

$$V_3(q_t, \tilde{\bar{\theta}}_t, \Delta\nu_t) \;=\; 0 \;, \tag{4.37}$$

where

$$q_t = \left[q_t(\tilde{\theta}_1) \ldots q_t(\tilde{\theta}_M)\right]'.$$

Finally, the discrete version of equation (4.29) is:

$$\frac{1}{\tilde{C}_t}q_t(\tilde{\theta}_l) = E_t\frac{\beta e^{-\Delta\nu_{t+1}(1-\alpha)/\alpha}}{\tilde{C}_{t+1}}\left\{(1-\alpha)e^{\tilde{\theta}_l}\left(\frac{\tilde{K}_{t+1}}{N_{t+1}}\right)^{-\alpha} + (1-\delta)s\cdot 1\{\tilde{\theta}_l < \tilde{\bar{\theta}}_{t+1}\}\right.$$
$$\left. + (1-\delta)\cdot 1\{\tilde{\theta}_l \geq \tilde{\bar{\theta}}_{t+1}\}\sum_{j=1}^{M} w_j q_{t+1}(\tilde{\theta}_j)\frac{1}{\sigma}\phi\left(\frac{\tilde{\theta}_j - \tilde{\theta}_l + \Delta\nu_{t+1}}{\sigma}\right)\right\},$$

$$\tag{4.38}$$

for $l = 1, ..., M$, or,

$$E_t V_4(\tilde{k}_t, \tilde{k}_{t+1}, q_t, q_{t+1}, N_t, N_{t+1}, \tilde{\bar{\theta}}_t, \tilde{\bar{\theta}}_{t+1}, \Delta\nu_t, \Delta\nu_{t+1}) = 0_{M \times 1}. \tag{4.39}$$

To apply the solution method of section 3, it is convenient to define:

$$\hat{z}_t \equiv \begin{bmatrix} \hat{k}_{t+1} \\ \hat{q}_t \\ N_t \\ \tilde{\bar{\theta}}_t \end{bmatrix} \quad \text{where } \hat{k}_t \equiv \tilde{k}_t - k^s \text{ and } \hat{q}_t \equiv q_t - q^s,$$

where $k^s$ and $q^s$ correspond to the steady states computed in the previous subsection. Then, one needs to linearize the $2M + 2$ equations $V_1(\cdot)$, $V_2(\cdot)$, $V_3(\cdot)$, and $V_4(\cdot)$ around the steady state computed above. We can now form a system of $2M + 2$ equations of the form:

$$\mathcal{E}_t \left[ \alpha_0 \hat{z}_{t+1} + \alpha_1 \hat{z}_t + \alpha_2 \hat{z}_{t-1} + \beta_0 \Delta\nu_t + \beta_1 \Delta\nu_{t+1} \right] = 0,$$

where $\mathcal{E}_t[X_t] = [X'_{1t}, E_t X'_{2t}]'$ and $X_{2t}$ corresponds to the last $M$ equations in the previous expression. Setting the parameters $n = n_1 = 2M + 2$, $q = 0$, $r = 2$ and $m = 1$ we can use the solution methods in section 3 to solve the model.

## 5. Second Moment Properties and Impulse Response Functions.

The preceding sections discussed the computation of the solution, (2.7), to a system of expectational difference equations. In practice, this solution is used for two purposes, the computation of impulse response functions and of the second moment properties of a model. I briefly discuss these two issues here.

Suppose the variables of interest in the analysis include not just $z_t$, but also another set, $x_t$. Suppose that, after linearization, the latter are related to the former by the relation,

$$x_t = G_1 z_{t-1} + G_2 s_t. \tag{5.1}$$

The variables of interest are $y_t$, where

$$y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}.$$

33

The impulse response functions represent the response, over time, of the elements of $y_t$ to a pulse in one of the elements in $\epsilon_t$ in (2.4). That is, let one element in $\epsilon_0$ be unity and the rest zero. The impulse response function, $y_0, y_1, y_2, \ldots$ is obtained by solving $s_0 = \epsilon_0$, $s_t = P^t s_0$ for $t > 0$ and then $z_0 = B\epsilon_0$, $z_t = A z_{t-1} + B s_t$, (5.1) for $t > 0$.

There are two ways to compute second moments. One works in the time domain, and another, described in Christiano and Eichenbaum (1990, p. 50), works in the frequency domain. The time domain approach is a modification on the method described above for computing impulse response functions. Draw, from a random number generator, a sequence, $\epsilon_t$ for $t = 0, \ldots, T$. Then, compute $s_t$, for $t = 0, \ldots, T$, using (2.4) and $s_{-1} = 0$. Sequences, $z_t$, $x_t$, $t = 0, \ldots, T$, can be computed using (2.7), $z_{-1} = 0$ and (5.1). It is possible that the second moments of interest pertain to $y_t$ after it has been transformed in some possibly nonlinear way. A transformation which is often used in practice is the Hodrick-Prescott filter, as described in Cooley and Prescott (1995) or Prescott (1986). The second moment properties of interest could be computed from the resulting transformed data. If $T$ is very large, then the resulting second moments correspond to the population second moments implied by the model. One might instead be interested in the small sample second moments and choose a smaller value of $T$, say the value that corresponds to the sample length available in a typical data set. In this case, the second moments described will exhibit Monte Carlo sampling variation. The standard way of dealing with this repeats the calculations many times with different Monte Carlo draws of the $\epsilon_t$'s, and reports the average, across different Monte Carlo draws, of the second moment statistics. In addition, some measure of the variation across Monte Carlo draws is also reported in practice.

The frequency domain approach to computing the population (i.e., $T = \infty$) second moments of a model proceeds as follows. Let the representation of $y_t$ in terms of current and past $\epsilon_t$'s be as follows:

$$y_t = G(L)\epsilon_t,$$

where $L$ denotes the lag operator, $L\epsilon_t \equiv \epsilon_{t-1}$, and[19]

$$G(L) = \left[ \begin{array}{c} (I - AL)^{-1}B \\ G_1 L(I - AL)^{-1}B + G_2 \end{array} \right] (I - PL)^{-1}\epsilon_t.$$

---

[19]To see how this equation is derived, note that (2.4) implies $(I - PL)s_t = \epsilon_t$, or, $s_t = (I - PL)^{-1}\epsilon_t$. Similarly, (2.7) implies $z_t = (I - AL)^{-1}B s_t$.

The spectral density of $y_t$ at frequency $\omega \in (-\pi, \pi)$, $S_y(\omega)$, is:

$$S_y(\omega) = G(e^{-i\omega})V_\epsilon G(e^{i\omega})',$$

where $V_\epsilon$ is the variance-covariance matrix of $\epsilon_t$, and $i$ is the complex number, $\sqrt{-1}$ (see Sargent (1987, ch.11, sec.6)). If the moments of interest are a transformation of $y_t$, then it is the spectrum of this transformed process that is of interest. If in addition the transformation is linear, i.e., has the following representation:

$$\tilde{y}_t = g(L)y_t = \sum_{k=-\infty}^{\infty} g_k y_{t-k},$$

then spectrum of the transformed series is:[20]

$$S_{\tilde{y}}(\omega) = g(e^{-i\omega})g(e^{i\omega})S_y(\omega), \text{ for } \omega \in (-\pi, \pi),$$

with $S_{\tilde{y}}(\omega) \equiv S_y(\omega)$ when it is the untransformed data that are of interest.

Let the population covariance function of $\tilde{y}_t$ be denoted by

$$C(\tau) = E\tilde{y}_t\tilde{y}'_{t-\tau}, \ \tau = 0, \pm 1, \pm 2, \dots .$$

Then,

$$
\begin{aligned}
C(\tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{\tilde{y}}(\omega)e^{i\omega\tau}d\omega \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} S_{\tilde{y}}(\omega_k)e^{i\omega_k\tau},
\end{aligned}
$$

where $\omega_k = \frac{2\pi k}{N}$ for $k = -N/2, ..., N/2$ (see Sargent (1987, ch. 11, equation (20))). The second equality reflects that we interpret the integral after the first equality as a Riemann-Stieltjes sum. In practice, the above sum converges for low values of $N$.[21] The ease with which $S_{\tilde{y}}(\omega_k)$ can be computed and the above sum evaluated explains why the frequency domain approach to computing $C(\tau)$ is often used in practice.

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[20]See King and Rebelo (1993) for a derivation of $g(L)$ when the filter is the Hodrick-Prescott filter. See Sargent (1987, ch. 11) for other linear $g(L)$ functions of interest.

[21]In practice, the evaluation of the finite sum can be simplified by taking into account that $S_{\tilde{y}}(\omega_k) = S_{\tilde{y}}(-\omega_k)'$.

# 6. Conclusion

I have described a general approach to solving linearized rational expectations models. The strategy involves two steps. The first step solves for the feedback part of the solution by finding the zero of a matrix polynomial. The second step solves a linear system of equations to obtain the feedforward part of the solution. A simple strategy for computing impulse response functions and second moment properties was then outlined.

# 7. Appendix 1: Computation of the R Matrix

A feature of the algorithm described here is that information sets can vary across individual equations in the system of expectational difference equations, (2.2). The computational strategy requires the construction of a particular matrix, $R$, for use in (3.26). Example 3 illustrates how this matrix can easily be constructed in a simple example, by working out the relationship between two vectorized matrices. However, constructing $R$ using this informal strategy is tedious in more complicated models. In this appendix I provide a general algorithm for constructing $R$.

A basic input of the algorithm is a specification of which elements in the $m_\theta \times 1$ dimensional column vector of shocks, $\theta_t$, are observed contemporaneously, and which are excluded, in the information set, $\Omega_{it}$, associated with the $i^{th}$ expectational difference equation in (2.2), $i = 1, ..., n_1$. This information is summarized in the $m_\theta \times n_1$ matrix $\tau$, where the $i^{th}$ column vector, $\tau_i$, has a 1 in locations corresponding to elements of $\theta_t$ which are included, and 0's corresponding to elements of $\theta_t$ which are excluded from $\Omega_{it}$. Let the number of 1's in $\tau_i$ be denoted by $m_i \geq 0$. The purpose of this appendix is to provide a matrix representation of the mapping from $\tau$, $\rho$, $V_e$ to the $(n_1 m_\theta + \sum_{i=1}^{n_1} m_i) \times n_1 m$ matrix, $R$.

For the case, $m_i > 0$, let $R_i$ denote the $m_i \times m_\theta$ matrix formed by deleting from the $m_\theta \times m_\theta$ identity matrix the rows corresponding to the entries in $\tau_i$ which contain a zero. Then, the components of $\theta_t$ that are observed contemporaneously in the $i^{th}$ equation of (2.2) are given by $R_i\theta_t$. We now compute the projection of $\theta_t$ onto $R_i\theta_t$ and $\theta_{t-1}$:

$$\underset{(m_\theta \times 1)}{P[\theta_t | R_i\theta_t, \theta_{t-1}]} = a_i R_i\theta_t + a_{i\theta}\theta_{t-1}, \tag{7.1}$$

36

with the understanding that when $m_i = 0$, then $a_i R_i = 0$. We can distinguish three cases:

$$[a_i, \ a_{i\theta}] = \begin{cases} [I_{m_\theta}, \ 0_{m_\theta}] & \text{if } m_i = m_\theta \\ [0_{m_\theta}, \ \rho] & \text{if } m_i = 0 \\ \phi\Phi^{-1} & \text{if } 0 < m_i < m_\theta, \end{cases} \tag{7.2}$$

where $I_{m_\theta}$ and $0_{m_\theta}$ denote the $m_\theta \times m_\theta$ identity and zero matrices. The third case represents a solution to the problem $[a_i, \ a_{i\theta}]\Phi = \phi$, which corresponds to the condition that the projection error, $\theta_t - P[\theta_t|R_i\theta_t, \theta_{t-1}]$, be orthogonal to $\theta_t' R_i'$ and $\theta_{t-1}'$.[22] Here, $\Phi$ is the following $(m_i + m_\theta) \times (m_i + m_\theta)$ matrix:

$$\Phi = \begin{bmatrix} R_i C R_i' & R_i \rho C \\ C' \rho' R_i' & C \end{bmatrix}, \ C = E\theta_t\theta_t' = \sum_{i=0}^{\infty} \rho^i V_e (\rho')^i,$$

and $\rho C$ corresponds to $E\theta_t\theta_{t-1}'$.[23] Also, $\phi$ is the $m_\theta \times (m_i + m_\theta)$ matrix:

$$\phi = [C R_i', \ \rho C].$$

Write $F$ in (3.3) as follows:

$$\underset{(n_1 \times m)}{F} = [\underset{(n_1 \times m_\theta)}{F_1} \ \vdots \ \underset{(n_1 \times m_\theta)}{F_2}], \ F_i = \begin{bmatrix} \underset{(1 \times m_\theta)}{F_{1,i}} \\ \vdots \\ \underset{(1 \times m_\theta)}{F_{n_1,i}} \end{bmatrix}, \ i = 1, 2,$$

so that,

$$\mathcal{E}_t F s_t = \mathcal{E}_t [F_1 \theta_t + F_2 \theta_{t-1}] = \begin{bmatrix} F_{1,1} a_1 R_1 \theta_t + F_{1,1} a_{1\theta} \theta_{t-1} & F_{1,2} \theta_{t-1} \\ \vdots & \vdots \\ F_{n_1,1} a_{n_1} R_{n_1} \theta_t + F_{n_1,1} a_{n_1,\theta} \theta_{t-1} & F_{n_1,2} \theta_{t-1} \end{bmatrix} = \tilde{F} s_t,$$

---

[22] It is possible for there to be exact collinearity in the regressors, $(R_i\theta_t, \theta_t)$. In this case, there are many $[a_i, a_{i\theta}]$'s which solve the problem, $[a_i, a_{i\theta}]\Phi = \phi$. Which of these is used does not matter. In practice, I choose the one that results when $\Phi^{-1}$ is evaluated using the pseudoinverse of $\Phi$ provided by the MATLAB routine pinv.m.

[23] A quick way to compute $C$ exploits (3.21) and the fact, $C = \rho C \rho' + V_e$, so that

$$vec(C) = [I - (\rho \otimes \rho)]^{-1} vec(V_e).$$

37

where

$$
\tilde{F} = \left[ \begin{array}{cc} F_{1,1} a_1 R_1 & F_{1,1} a_{1\theta} + F_{1,2} \\ \vdots & \vdots \\ F_{n_1,1} a_{n_1} R_{n_1} & F_{n_1,2} a_{n_1,\theta} + F_{n_1,2} \end{array} \right] .
$$

Now,

$$
\tilde{F}' = \left[ \begin{array}{cccc} (a_1 R_1)' F'_{1,1} & (a_2 R_2)' F'_{2,1} & \cdots & (a_{n_1} R_{n_1})' F'_{n_1,1} \\ a'_{1\theta} F'_{1,1} + F'_{1,2} & a'_{2\theta} F'_{2,1} + F'_{2,2} & \cdots & a'_{n_1\theta} F'_{n_1,1} + F'_{n_1,2} \end{array} \right] ,
$$

so that

$$
\begin{aligned}
vec(\tilde{F}') &= \left[ \begin{array}{c} (a_1 R_1)' F'_{1,1} \\ a'_{1\theta} F'_{1,1} + F'_{1,2} \\ (a_2 R_2)' F'_{2,1} \\ a'_{2\theta} F'_{2,1} + F'_{2,2} \\ \vdots \\ (a_{n_1} R_{n_1})' F'_{n_1,1} \\ a'_{n_1\theta} F'_{n_1,1} + F'_{n_1,2} \end{array} \right] \\[2mm]
&= \left[ \begin{array}{ccccccc} (a_1 R_1)' & 0 & 0 & 0 & \cdots & 0 & 0 \\ a'_{1\theta} & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (a_2 R_2)' & 0 & \cdots & 0 & 0 \\ 0 & 0 & a'_{2\theta} & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (a_{n_1} R_{n_1})' & 0 \\ 0 & 0 & 0 & 0 & \cdots & a'_{n_1\theta} & I \end{array} \right] vec(F') \\[2mm]
&= \tilde{R} vec(F'),
\end{aligned}
$$

say, where $\tilde{R}$ is $n_1 m \times n_1 m$. Note that the rows of $R'_i$ which correspond to the zero entries of $\tau_i$ are composed entirely of zeros. It follows that the rows of $(a_i R_i)'$ which correspond to the zero entries of $\tau_i$ are zero too. The matrix $R$ is just $\tilde{R}$ with these rows deleted. This matrix satisfies equation (3.26).

# 8. References

1. Alexopoulos, Michelle, 1997, *Monetary Shocks and Shirking*, unpublished manuscript.

2. Blanchard, Olivier, and Charles Kahn, 1980, *The Solution of Linear Difference Models under Rational Expectations*, Econometrica, 48(5), pp. 1305-11.

3. Burnside, Craig and Martin Eichenbaum, 1996, *Factor Hoarding and the Propagation of Business Cycle Shocks*, American Economic Review, December.

4. Burnside, Craig, Martin Eichenbaum, and Sergio Rebelo, 1993, *Labor Hoarding and the Business Cycle*, vol. 101, no. 2, April, pp. 245-273.

5. Campbell, J. R., 1997, *Entry, Exit, Embodied Technology, and Business Cycles*, National Bureau of Economic Research Working paper number 5955.

6. Chari, V.V., Lawrence Christiano and Martin Eichenbaum, 1995, *Inside Money, Outside Money, and Short-Term Interest Rates*, Journal of Money, Credit and Banking, November, vol. 2, no. 4, part 2.

7. Chari, V.V., Kehoe, P., and McGrattan, E., 1996, *Sticky Price Models of the Business Cycle: Can the Contract Multiplier Solve the Persistence Problem?* Federal Reserve Bank of Minneapolis Staff Report #217, September.

8. Christiano, L. J., 1991, *Modeling the Liquidity Effects of a Monetary Shock*, Federal Reserve Bank of Minneapolis Quarterly Review.

9. Christiano, L. J., and Eichenbaum, M., 1990, *Unit Roots in Real GNP: Do We Know, and Do We Care?*, Carnegie-Rochester Conference Series on Public Policy, vol. 32, pp. 7-62.

10. Christiano, L. J., and Eichenbaum, M., 1992, *Liquidity Effects and the Monetary Transmission Mechanism*, American Economic Review, May, 346-353.

11. Christiano, L. J., and Eichenbaum, M., 1995, *Liquidity Effects, Monetary Policy, and the Business Cycle*, Journal of Money, Credit and Banking, vol. 27, no. 4, part 1, November.

12. Christiano, L. J., Eichenbaum, M., and Evans, C., 1997, *Sticky Price and Limited Participation Models: A Comparison*, European Economic Review.

13. Christiano, L. J., Eichenbaum, M., and Evans, C., 1997a, *Modeling Money*, unpublished manuscript, Northwestern University.

14. Christiano, L. J. and Fisher, J., 1997, *Algorithms for Solving Dynamic Models with Occasionally Binding Constraints*, National Bureau of Economic Research Technical Working Paper number 218.

15. Christiano, L. J. and Fisher, J., 1998, *Why Are Stock Prices Procyclical?*, manuscript, Northwestern University.

16. Cooley, Thomas F. and Edward C. Prescott, 1995, *Economic Growth and Business Cycles*, in Thomas F. Cooley, ed., Frontiers of Business Cycle Research, Princeton University Press.

17. Fisher, J., 1996, *Credit Market Implications and the Heterogeneous Response of Firms to Monetary Shocks*, Federal Reserve Bank of Chicago Working paper 96-23

18. Fisher, J., 1997, *Relative Prices, Complementarities and Co-movement Among Components of Aggregate Expenditures*, Journal of Monetary Economics, pp. 449-474.

19. Fuerst, T., 1992, *Liquidity, Loanable Funds, and Real Activity*, Journal of Monetary Economics, Vol. 29, No. 1, February, 3 - 24.

20. Gust, C., 1997, *Staggered Price Contracts and Factor Immobilities: The Persistence Problem Revisited*, unpublished manuscript November.

21. Judd, Ken. 1992, *Projection Methods for Solving Aggregate Growth Models*, Journal of Economic Theory, 58: 410-452.

22. King, Robert, and Sergio Rebelo,

23. King, Robert, Charles Plosser, and Sergio Rebelo, 1987, *Production, Growth and Business Cycles: Technical Appendix*, working paper, University of Rochester.

24. Luenberger, D. G., *Optimization by Vector Space Methods*, John Wiley and Sons, Inc., 1969.

25. McCallum, Bennet, 1983, *On Non-Uniqueness in Rational Expectations Models*, Journal of Monetary Economics, vol. 11, pp. 139-168.

26. Merz, Monika, 1996, *Heterogeneous Job-Matches and the Cyclical Behavior of Labor Turnover*, December, unpublished manuscript, Rice University, forthcoming, Journal of Monetary Economics.

27. Prescott, E., 1986, *Theory Ahead of Business-Cycle Measurement*, Carnegie-Rochester Conference on Public Policy, vol. 24, pp. 11-44, reprinted in Federal Reserve Bank of Minneapolis Quarterly Review, vol. 10, pp. 9-22.

28. Sargent, Thomas, 1987, *Macroeconomic Theory*, Academic Press, second edition.

29. Schlagenhauf, D. and Wrase, J. F., 1995 *Liquidity and Real Activity in a Simple Open Economy Model*, Journal of Monetary Economics 35(3), 431-461.

30. Sims, Christopher,

31. Stokey, Nancy, and Robert E. Lucas, Jr., with Edward Prescott, 1989, *Recursive Methods in Economic Dynamics*, Harvard University Press, third printing, 1993.

32. Uhlig, Harald, 1997, *A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily*, manuscript, CentER, University of Tilburg.

33. Woodford, Michael, 1986, *Stationary Sunspot Equilibria: The Case of Small Fluctuations Around a Deterministic Steady State*, unpublished manuscript, University of Chicago, September.