The Zero-Bound, Low Inflation, and Output Collapse*

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Abstract

I consider the example analyzed in Eggertsson and Woodford (2003a,b), which shows that the zero lower bound on the nominal rate of interest, in conjunction with a low-inflation policy by the central bank, can cause output to collapse in response to certain shocks. I present evidence which suggests that when capital is introduced into the model, the problem is less likely to occur. This analysis is carried out for the case when shocks are to the rate at which households discount future consumption. Additional analysis with other shocks (especially shocks to investment) and a wider range of economic frictions is of interest.

1 Introduction

Since the high inflation of the 1970s, central banks around the world have struggled to bring down inflation. With the fall in inflation, interest rates have come down so much that the zero lower bound might well bind sometime in the near future. Increasingly, there is a concern that this combination - nearly zero interest rates and a dedication to low inflation - exposes economies to a new threat. As Bernanke (2003) has recently emphasized, they create a possibility of a 'worst case scenario', in which an otherwise mild economic shock causes a collapse in output and a downward spiral in prices.[1]

These concerns raise two related questions. Can they be given a coherent intellectual foundation? If the answer is 'yes', then - given the best current econometric estimates of the economy - what is the likelihood that the worst case scenario will occur?

In two important papers, Eggertsson and Woodford (2003a,b) show that the answer to the first question is 'yes'. They present a fully articulated model which illustrates how the worst case scenario could occur.[2] In the example, the efficient response to a particular shock

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[1]For similar discussions, see also Sims (1999) and Svensson (2003).

[2]An important precursor to this work is Krugman (1998).

requires that the real rate of interest decline. However, this cannot happen because (i) the zero lower bound on the nominal interest rate binds, and (ii) the central bank's inability to commit to future monetary policy prevents it from being able to raise inflation expectations.[3] With the real interest rate unable to play its role in efficiently directing the allocation of resources, the result is the sort of welfare-reducing downward spiral in output and inflation that concerns observers such as Bernanke. Eggertsson and Woodford construct a numerical example of this which is dramatic and hard to ignore.[4] In the example there are welfare losses that exceed anything one can find in monetary economics, outside of models with sunspot fluctuations.

The design of Eggertsson and Woodford's example is heavily influenced by their desire for tractability. On this dimension, they have succeed abundantly. To reproduce the essence of their quantitative analysis all one needs is a cocktail napkin and a pencil. However, to determine whether the scenario illustrated by Eggertsson and Woodford's example is a practical concern, one must determine whether their conclusions are robust to relaxing some of their model simplifications. In my comment, I explore robustness to the introduction of investment, by embedding their model into the model with capital found in Woodford (2003, Chapter 5). I find that the introduction of investment reduces the likelihood that the downward-spiral scenario will occur. It now takes a bigger shock to trigger the scenario. However, the parameters in Woodford (2003) imply a very high Frisch elasticity of labor supply. When I use a more standard elasticity, the lower bound no longer binds, even for very large shocks.

I discuss the intuition behind the Eggertsson-Woodford results in the first section below. It should be familier to many economists because it closely resembles the logic of the Paradox of Thrift, discussed in standard undergraduate macroeconomics textbooks. Section 3 presents the models used in the analysis. Section 4 presents the results. Section 5 concludes. The computational algorithm used to obtain the results is described in the appendix.

[3] Eggertsson and Woodford (2003a,b) show that raising inflation expectations is the optimal policy when the central bank has the ability to commit to its future monetary policy. When the central bank cannot commit to future policy, everyone understands that ex post the central bank will always choose zero inflation if it can (more on this in section 3 below). This is because Eggertsson and Woodford assume that any sources of inflation bias in monetary policy have been removed by the appropriate setting of taxes and subsidies.

[4] Eggertsson and Woodford and others (see, for example, Svensson (2003)) describe solutions to the problem. I am not concerned with these here, and so I do not discuss them in the main text. Briefly, one feasible strategy is to shift to an average inflation rate that is significantly above zero, say 4 percent or so. Although a level of inflation like this generates welfare losses, economists have generally found that the losses associated with an inflation rate like this are quite small. However, Eggertsson and Woodford show that one can do better. They show that the optimal strategy involves very low inflation on average. The strategy involves increasing inflation by very small amounts in the periods after the realization of certain shocks. There may be difficulties with implementation of this policy if the shocks are not observable. Eggertsson and Woodford show, however, that the optimal policy is closely approximated by a policy of price level targetting. As noted in a previous footnote, to implement this policy in the Eggertsson-Woodford environment requires that the central bank be able to commit to its future policies. This may require institutional change.

2 Simple Intuition

Although the formal analysis on which my conclusions are based is dynamic (see the next section below), intuition can be obtained from a simple static analysis. Consider Figure 1. The real rate of interest,

$$\frac{1+i_t}{\pi_{t+1}},$$

appears on the vertical axis. Here, i_t denotes the nominal interest rate and π_{t+1} denotes the gross rate of change in the price level from the current period to the next. Also, investment and saving appear on the horizontal axis. The figure shows a shift to the right in the supply of savings (induced in the formal analysis by a shock to the discount rate). The demand for saving arises from firms seeking to finance investment. There are two savings demand curves in the figure, each exhibiting a different elasticity of investment with respect to the real interest rate. The inelastic investment case corresponds to the one considered by Eggertsson and Woodford, who exclude capital from their model. This case is captured by the vertical line. The negatively sloped line corresponds to the interest elastic case, modeled formally in the next section. There is a lower bound on the real interest rate because i_t cannot drop below zero and Eggertsson and Woodford assume that next period's monetary authority will do what it can to set π_{t+1} to unity.[5] The lower bound on the real interest rate is indicated by the horizontal line in the figure.

Consider the interest-inelastic case first. The shock induces people to cut spending and increase saving. Because the quantity of investment cannot change by hypothesis, the quantity saved also cannot change in equilibrium. Something must happen to cause the quantity saved to remain unchanged, despite the shock to the supply of saving. This can be accomplished efficiently by a drop in the real interest rate to point B. However, because of the lower bound on the real interest rate, this cannot happen. Something else must happen instead, to discourage a rise in the quantity saved. In the model, a fall in output and income does the job. The drop in output occurs naturally as households cut back on consumption in an effort to increase saving. Through a standard sticky price mechanism, this leads to a fall in income and employment (as well as the current price level). The decline in economic activity exerts downward pressure the quantity saved, for consumption-smoothing reasons. In effect, the decline in income causes the saving function to shift back to the left, so that the saving function intersects the vertical investment curve at the lower bound.

Actually, the scenario is worse than the reasoning just described suggests. In the formal analysis, the shift in the discount rate is assumed to be persistent, so that a similar thing is expected to occur in the future. That means that consumption and the price level are expected to fall in the future too. In fact, they fall so much that the expected gross inflation rate from the current period to the next, π_{t+1}, falls below unity. This has the effect of shifting the lower bound in Figure 1 up, increasing the distance that the saving function

[5] They conjecture that this is actually an equilibrium of a version of the model in which the monetary cannot commit to its future policy and it chooses inflation to maximize the utility of the representative household. The reason for this is that with their assumed price frictions, a policy which makes the current price level deviate from its previous period's level distorts the allocation of resources. Moreover (assuming a tax subsidy is used to eliminate monopoly distortions), there is no gain from a policy that makes the current price level deviate from its value in the previous period.

must be induced to shift back left. Put differently, the fall in expected inflation stimulates households' desire to save even further, increasing the amount by which income must fall to maintain equilibrium.[6]

The point of the example is that ordinarily the discount rate shock would be met by a suitable drop in the real interest rate and no change in output. In the formal analysis this is the welfare-efficient response. With the central bank's dedication to low inflation and with the lower bound on the interest rate, the real interest rate mechanism cannot do its job. Instead, households are prevented from increasing their saving by a welfare-reducing fall in income. In their numerical example, discussed below, Eggertsson and Woodford show that this drop in income can be quite large, and deserves to be called a "downward spiral".

Now consider the elastic investment case. In this case, the real interest rate only has to drop to point C to maintain equilibrium. This cut in the real interest rate is enough that the resulting expansion in investment absorbs the rise in investment. Evidently, the size of the saving shock that produces the downward spiral in the economy with inelastic investment does not do so in the economy with elastic investment. This is the sense in which the downward spiral is less likely when investment is elastic.

The intuitive analysis suggests various policy conclusions. For example, one way to reduce the likelihood of the downward spiral is to create the expectation that there will be higher inflation in the period after the shock. Eggertsson and Woodford show that this is the optimal monetary policy of a monetary authority that is able to commit to its future policies. Interestingly, they show that the quantity of extra inflation required is quite modest, especially by comparison to the large deflation that occurs in their downward-spiral equilibrium. However, in Eggertsson and Woodford's model, creating expectations of higher inflation requires commitment. Their environment has the property that once the shock is over, the monetary authority without commitment has no incentive to deliver on a promise of high inflation made in the past. Because absence of commitment forecloses this painless way to respond to a shock, Eggertsson and Woodford's analysis in effect provides another example of the value of commitment in monetary policy.[7]

3 Formal Models

The first part of this section replicates the example reported by Eggertsson and Woodford. After that, I embed their model into the model with capital in Woodford (2003, Chapter 5), to determine the impact of investment and other shocks on the analysis.

3.1 Eggertsson-Woodford Model

The algebra of the Eggertsson and Woodford example is strikingly simple. There are two equations that characterize equilibrium for the private sector in their economy, an intertemporal Euler equation associated with the household saving decision and the standard Calvo

[6]The reasoning here is close to that in DeLong and Summers (1986).

[7]For another, see Albanesi, Chari and Christiano (2003).

pricing equation:

$$x_t = E_t x_{t+1} - \sigma \left(\hat{\imath}_t - E_t \hat{\pi}_{t+1} - \hat{r}_t^n \right) \quad (1)$$
$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa x_t. \quad (2)$$

Here, x_t, $\hat{\imath}$, π_t and $\hat{r}_t^n$ denote consumption, the nominal rate of interest, gross inflation and the discount rate. Each is expressed in deviation from its value in a zero-inflation steady state. Also, $\beta^{-1} - 1$ is the steady state value of the discount rate and κ, σ are parameters. The model is completed with a description of monetary policy. Policy sets $\hat{\imath}_t$ to ensure $\hat{\pi}_t = 0$, subject to the constraint that the nominal rate of interest is non-negative. When the constraint is binding then the nominal rate of interest is simply set to zero and the inflation rate is determined by the equilibrium conditions. Since $\hat{\imath}_t$ is the percent deviation in the interest rate from its zero-inflation steady state, the lower bound on the actual nominal rate of interest translates into a negative lower bound for $\hat{\imath}_t : \hat{\imath}_t \geq \beta - 1$.[8] From (1) it is clear that if $\hat{\imath}_t = \hat{r}_t^n$ for all t, then there is an equilibrium in which consumption remains at its steady state value and inflation is always zero. However, if it should ever happen that $\hat{r}_t^n < \beta - 1$, then $\hat{\imath}_t = \hat{r}_t^n$ is not possible, as it would imply a violation of the zero lower bound on the interest rate.

To obtain a sense of what would happen in this case, Eggertsson and Woodford construct the following scenario. Suppose the economy is in a deterministic steady state with no shocks up until period 0. Then, unexpectedly in period 1 there is a shock to $\hat{r}_t^n$ which causes it to drop to $\hat{r}_l^n < \beta - 1$. In each subsequent period, with probability p, $\hat{r}_t^n$ remains low, and with probability $1 - p$ it returns to its steady state value of zero. Once zero, $\hat{r}_t^n$ can never change value again.

The variables to be solved for in equilibrium are the levels of output and inflation, $\hat{x}_l$ and $\hat{\pi}_l$, that prevail while $\hat{r}_t^n = \hat{r}_l^n$. To find these, just solve the versions of (1) and (2) that occur as long as $\hat{r}_t^n = \hat{r}_l^n : x_l = p x_l - \sigma((\beta - 1) - p\pi_l - \hat{r}_l^n)$ and $\pi_l = \beta p \pi_l + \kappa x_l$. This represents two equations in the two unknowns. Eggertsson and Woodford assume the time period in their model is one quarter, and adopt the following parameterization:

$$p = 0.9, \ \sigma = 0.5, \ \kappa = 0.02, \ \beta = 0.99, \ r_l^n = -.02/4.$$

It is easy to confirm that for these parameter values,

$$x_l = -0.14, \ \pi_l = -0.0263.$$

That is, output is 14 percent below steady state and inflation is -12 percent annualy as long as the discount rate remains low. The discount rate is expected to stay low for $1/(1-p) = 10$ periods, a notably long time. The intuition for these results was described in the previous section. Since any drop in output represents an inefficient response to the $\hat{r}_t^n$ shock, this represents a substantial inefficiency.

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[8] In particular, the

$$\hat{\imath}_t \equiv \frac{i_t - i}{1 + i},$$

where i_t is the nominal rate of interest and i is its steady state value. In a zero-inflation steady state, $1/(1+i) = \beta$, so that

$$\hat{\imath}_t = \beta \left(i_t + 1 \right) - 1,$$

so that when i_t is at its lower bound of zero, $\hat{\imath}_t = \beta - 1$.

3.2 A More General Model with Capital

I embed the Eggertsson and Woodford (2003a,b) model into the more general model with capital in Woodford (2003, section 5). In addition, to simplify the analysis the model is deterministic.

The preferences of households are as follows:

$$\sum_{t=0}^{\infty} \beta_t \left[u(C_t, M_t/P_t) - \int_0^1 v(H_t(j))dj \right],$$

where u is increasing and concave in its first argument, v is increasing and convex, and

$$\beta_t = \frac{1}{(1+r_0^n)(1+r_1^n)\cdots(1+r_{t-1}^n)},$$

for $t = 1, 2, ...$. Also, $\beta_0 \equiv 1$ and

$$\frac{\beta_{t+1}}{\beta_t} = \frac{1}{1+r_t^n}.$$

Each household supplies every type of labor, $j \in (0,1)$. Here, C_t denotes consumption and M_t denotes the household's end-of-period t stock of money. Finally, P_t denotes the price of the consumption good. The household's flow budget constraint is:

$$P_t C_t + M_t + B_{t+1} \leq M_{t-1} + B_t(1 + i_{t+1}) + \int_0^1 P_t w_t(j) H_t(j) dj + T_t, \tag{3}$$

where B_t denotes the beginning-of-period t stock of bonds, purchased in period $t-1$. Also, $w_t(j)$ denotes the real wage rate paid to type j labor, and T_t denotes lump sum transfers from the government. We suppose there is a lower bound constraint on B_t. Households are competitive in goods and labor markets.

Given this setup, a rise in the real rate of interest, $(1+i_t)/\pi_{t+1}$, induces a household to "tilt" its consumption profile towards the future. Doing so without violating its intertemporal budget constraint requires reducing consumption - hence, increasing saving - in the current period. In this way, the positively sloped saving function in Figure 1 captures a basic economic force in the model. Similarly, a drop in r_t^n induces households to save more for every value of the real interest rate, which corresponds to the right-shift displayed in Figure 1.

Final goods are produced using intermediate goods by a representative, competitive firm using the following Dixit-Stiglitz production function:

$$Y_t = \left[\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \ \theta > 1.$$

The first order condition for profit maximization by the final good firm is:

$$\left(\frac{Y_t}{y_t(j)} \right)^{\frac{1}{\theta}} = \frac{P_t(j)}{P_t}. \tag{4}$$

The i^{th} intermediate good is produced by a monopolist using the following technology:

$$y_t(i) = K_t(i) f\left(\frac{h_t(i)}{K_t(i)}\right). \tag{5}$$

Here, $K_t(i)$ is the capital owned by the monopolist, and $h_t(i)$ is the quantity of labor hired. The firm is competitive in the market for type i labor, and takes the wage rate, $w_t(i)$, as given.[9] Investment by the i^{th} monopolist produces new capital in the next period according to the following adjustment cost function:

$$I_t(i) = I\left(\frac{k_{t+1}(i)}{k_t(i)}\right) k_t(i). \tag{6}$$

Here, investment, $I_t(i)$, corresponds to purchases of the final good. Also, $I(1) = \delta$, $I'(1) = 1$, $I''(1) = \epsilon_\psi > 0$.

The i^{th} intermediate good firm faces frictions in the setting of its price, $P_t(i)$. With probability $1-\alpha$ it may set its price optimally, and with probability α is must set $P_t(i) = P_{t-1}(i)$.

The present discounted value of profits of the intermediate good firm are:

$$E_t \sum_{j=0}^{\infty} \beta_{t+j} \Lambda_{t+j} \left\{ (1+\tau) P_{t+j}(i) y_{t+j}(i) - P_{t+j} w_{t+j}(i) h_{t+j}(i) - P_{t+j} I_{t+j}(i) \right\}. \tag{7}$$

Here, Λ_t denotes the shadow value of a dollar to the household, the owner of the intermediate good firm. It is the multiplier on (3) in the Lagrangian representation of the household's problem. Also, the subsidy, τ, is designed to eliminate the distorying effects of monopoly power in the model. The i^{th} intermediate good firm chooses $P_{t+j}(i)$, $y_{t+j}(i)$, $h_{t+j}(i)$, $I_{t+j}(i)$ to maximize profits, subject to (4), (5), (6). It takes P_{t+j}, τ and $w_{t+j}(i)$ as given.

The resource constraint is:

$$C_t + I_t + G_t = Y_t,$$

where

$$I_t = \int_0^1 I_t(i) di.$$

Finally, the monetary authority controls i_t, which it sets with an objective of achieving $\pi_t = 1$. In periods when doing so would (infeasibly) imply a negative interest rate, it sets

[9] At first glance, this may seem odd, because according to the formalism in the text, the j^{th} intermediate good producer is the only employer of type j labor. This suggests that the producer must be a monopsonist. We follow an alternative interpretation suggested by Woodford (2003), which rationalizes competitive labor markets. Think of the j^{th} intermediate good producer as being a member of an industry composed of intermediate goods producers with indeces lying in a small neighborhood, J, of j. Suppose there is a finite, but large, number of such industries that do not intersect, but whose union is the unit interval. Imagine that instead of there being a continuum of labor types in the household utility function, there is a discrete number as in the Riemann approximation to the integral of the household utility function. Each of these labor types works in one of the industries. With this setup, there is a continuum of suppliers and demanders in the labor market corresponding to each industry, so that competition makes sense. This is the case even for intervals, J, whose length is very small. This is how we interpret the model. For further discussion, see Woodford (2003, pp. 148-149).

$i_t = 0$ and lets π_t be free to be determined by the equilibrium conditions. I assume that the fiscal authorities set lump sum taxes to guarantee that the government's budget constraint is balanced in each period.

When the curvature on investment adjustment costs, ϵ_ψ, is large, then the stock of capital is a constant. This is my representation of the model in Woodford and Eggertsson (2003a,b).[10] The elastic investment case corresponds to smaller values of ϵ_ψ.

I now summarize the parameters of the model. Since I study a log-linear approximation to the model economy, I need only specify its properties in a neighborhood of steady state. In particular, I parameterize the curvature, σ_u, σ_v, and σ_f of the utility of consumption, disutility of labor and production function, respectively, as follows:

$$\sigma_u = -\frac{u''C}{u'} > 0, \ \sigma_v = \frac{v''h}{v'} > 0, \ \sigma_f = -\frac{f''\bar{h}}{f'} > 0.$$

Here, one and two primes, '′', '″' denote the first and second derivatives, respectively, of the associated function. In each case, the second derivative is multiplied by the argument of the function, evaluated in steady state. Thus, C, h, and $\bar{h}$ denote consumption, hours worked and the hours to capital ratio, all evaluated in steady state. In addition, there is the parameter, ϵ_ψ, which controls the curvature of the investment adjustment cost function, which was mentioned above. In the case of the production function, I follow Woodford (2003) in parameterizing the (inverse of the) elasticity f with respect to its argument:

$$\phi = \frac{f}{f'\bar{h}}.$$

Finally, I require g, the steady state value of the ratio, G_t/Y_t. The complete set of nine parameters which need to be assigned values in order to solve a log-linear approximation of the model, is:

$$\sigma_u, \ \sigma_v, \ \sigma_f, \ \epsilon_\psi, \ \phi, \ \delta, \ \alpha, \ \theta, \ \beta, \ g.$$

I follow Woodford (2003) in setting these parameters. In particular, $\sigma_u = 1.0$, $\sigma_v = 0.11$, $\sigma_f = 0.25$, $\epsilon_\psi = 3$, $\phi = 1/0.75$, $\delta = 0.012$, $\alpha = 0.66$, $\theta = 1 + 1/0.15$, and $\beta = 0.99$. In addition, I set $g = 0.18$, roughly the ratio of actual government purchases of goods and services to gross output. If utility were a constant elasticity function of consumption, then $\sigma_u = 1$ corresponds to the log. Similarly, $\sigma_v = 0.11$ corresponds to a power of 1.11 on labor if v where a constant elasticity function. If the production function were Cobb-Douglas then σ_f and ϕ both indicate a power on labor equal to 0.75. The value of α implies that on average a price is fixed for 3 quarters. The values of δ and β are standard in the real business cycle literature. The value of θ corresponds to a steady-state markup over marginal cost of 15 percent, also not far from standard estimates. Except for σ_v, these parameter values are all fairly standard. The value of σ_v implies a Frisch labor supply elasticity of $9 = 1/0.11$, which is higher than is often used in the literature. For this reason, in my analysis I also consider the unit Frisch elasticity case, $\sigma_v = 1$.

[10] Even with $\epsilon_\psi = \infty$, the model differs slightly from the one in Eggertsson and Woodford (2003) because they set investment to zero while it is a positive constant here. The difference is relatively unimportant, because my results are very similar to theirs.

4 Results

I suppose that the economy has been in a deterministic steady state up until period 0. In period 1, r_t^n unexpectedly drops from its steady state annualized percent value of 4 percent. I consider three cases. In the "small shock", "medium shock", "large shock", and "larger shock" cases, r_t^n drops to -2, -4, -5, and -6 percent, respectively. In each case, r_t^n remains low until period 14. In period 15 it switches back to its steady state value. The small shock corresponds to the shock size used in the Eggertsson and Woodford example discussed in section 3.1.

Consider Figure 2 first. This corresponds closely to the Eggertsson and Woodford example, because it is the model without investment, i.e., with $\epsilon_\psi = \infty$.[11] Notice that for each shock size, the lower bound constraint on the nominal interest rate is binding, and output drops substantially. In the Eggertsson and Woodford small shock case, output drops nearly 20 percent in the period of the shock. After that the magnitude of the drop shrinks as the horizon over which the discount rate will remain low becomes shorter.[12] The results in Figure 2 match the intuition in section 2 of the paper.

Now consider Figure 3. The results here correspond to the same experiment as in Figure 2, with the sole exception that ϵ_ψ takes on the value suggested in Woodford (2003), namely, $\epsilon_\psi = 3$. Note that for the small shock the lower bound constraint is not binding. The real interest rate falls, and consumption is reallocated from the present to the future. This intertemporal reallocation is accomplished by way of a strong increase in investment. The rise in investment exceeds the fall in consumption, so that there is an increase in output in excess of 10 percent.

Interestingly, even though the lower bound on the interest rate is binding for the medium sized shock (see the starred line in Figure 3), the response of output is still strong, indeed, it is stronger than it is for the small sized shock. It seems fair to conclude that when investment is elastic, the scenario in which the zero lower bound binds and output and inflation collapse is less likely.

The zero lower bound on the interest is also binding for the large and larger shocks, however now output, consumption, investment and inflation all collapse. In the case of the larger shock, the collapse is enormous: the fall in output is 60 percent and deflation is in excess of 30 percent.

Figures 4 and 5 report the same experiments as in Figures 2 and 3, respectively, with the exception that now the Frisch labor supply elasticity is unity, with $\sigma_v = 1$. Note that in the no investment case (Figure 4), the drop in consumption is somewhat smaller than before (see Figure 2). However, qualitatively the results are not very different. In contrast, the results for the variable investment case are substantially different. Now the lower bound is never binding. In this case, the larger is the discount rate shock the more households intertemporally substitute consumption.

[11] Actually, I set ϵ_ψ =30000000000.

[12] A difference between the deterministic example here and the one in Eggertsson and Woodford is that the equilibrium values of variables in the latter case are independent of how long the discount rate has been low. This is because the stochastic process on $\hat{r}_t^n$ has the property that the horizon over which $\hat{r}_t^n$ is expected to remain low is independent of how long $\hat{r}_t^n$ has been down.

5 Conclusion

I incorporated the Eggertsson and Woodford (2003a,b) example into the more general model with capital studied in Woodford (2003). Using the parameters in Woodford (2003), I found that the likelihood of output collapse and price deflation is reduced. Still, it can happen. However, this parameterization incorporates a Frisch elasticity of labor supply equal to nearly 100, which is considerably higher than the values used in the literature. When I assign a value of unity to this elasticity, then the results in the Eggertsson and Woodford model with no capital are qualitatively unaffected. However, the model with investment is strikingly sensitive to this change. The lower bound does not bind, even for extremely large values of the shock.

The Eggertsson and Woodford example suggests that the stakes are very high. Even if we only assign a low probability to the output collapse scenario in their example, the welfare losses are so great that it behooves us to pay attention. The analysis in this paper suggests that the output-collapse scenario is less likely once we take into account investment. Still, considerably more work - with other frictions and other shocks - needs to be done before we can safely ignore the output collapse scenario.[13] One shock that would be particularly interesting to study is a negative disturbance to investment. This too would require a fall in the real interest rate to encourage households to cut back on saving. If the zero bound makes a big enough drop infeasible, the result could be output collapse and price deflation. Pursuing this further requires constructing stochastic, dynamic general equilibrium models based on solid empirical foundations. In addition, it requires developing model solution methods which can accommodate occasionally binding constraints. Devising appropriate solution methods is a particularly big challenge. Some progress has been made (see, e.g., Christiano and Fisher (2000)). However, these methods are very burdensome computationally.

A Appendix: Computational Algorithm

Here, I describe the shooting algorithm used to solve the model. The first part of this appendix explains the equations used to characterize the equilibrium. Following Eggertsson and Woodford, I log-linearize the equations which must always hold with equality in equilibrium. This discussion follows Woodford (2003, Chapter 5), and is included here for completeness. The second part of this appendix describes the actual shooting algorithm. The algorithm is constructed to ensure that the lower bound constraint on the interest rate is always satisfied.

A.1 Log-Linearized Equations Characterizing Equilibrium

I begin with a statement of the first-order necessary conditions for household and firm optimality. I then compute the features of the steady state that are required. After this, I derive three log-linear equations that I use to characterize equilibrium.

First-order Conditions

[13] Initial work on this may be found in Adam and Billi (2003).

The first order necessary conditions for household optimization include a transversality condition and:

$$\begin{aligned}\frac{v'(H_t(j))}{u'_t} &= w_t(j)\\ u'_t &= \frac{1}{1+r_t^n}u'_{t+1}\frac{1+i_t}{\pi_{t+1}}\\ \frac{u_{m,t}}{u'_t} &= \frac{i_t}{1+i_t}\end{aligned}$$

Here, u'_t denotes the marginal utility of consumption. The first equation is the intratemporal Euler equation for labor. The second equation is the intertemporal Euler equation associated with the household saving decision. By concavity of the utility function, current consumption falls - hence saving rises as in Figure 1 - when the real interest rate rises, for given values of r_t^n and u'_{t+1}. The third is the Euler equation associated with real balances. From here on, we assume u'_t is not a function of M_t/P_t, and we ignore the last first order condition.

The first order necessary condition associated with the optimal choice of capital by the i^{th} firm is:

$$I'\left(\frac{k_{t+1}(i)}{k_t(i)}\right) = \frac{1}{(1+i_t)/\pi_{t+1}}\left\{\rho_{t+1}(i) - I\left(\frac{k_{t+2}(i)}{k_{t+1}(i)}\right) + I'\left(\frac{k_{t+2}(i)}{k_{t+1}(i)}\right)\frac{k_{t+2}(i)}{k_{t+1}(i)}\right\}. \tag{8}$$

Here, we have made use of the household's intertemporal Euler equation and the fact, $\Lambda_t = u_{c,t}/P_t$. With some algebra it can be shown that $\rho_{t+1}(i)$ in (8) can be written:

$$\rho_{t+1}(i) = w_{t+1}(i)\frac{f(\bar{h}_{t+1}(i)) - \bar{h}_{t+1}(i)f'(\bar{h}_{t+1}(i))}{f'(\bar{h}_{t+1}(i))}. \tag{9}$$

One interpretation of $\rho_{t+1}(i)$ is that it is the real rental rate of capital that would rationalize the amount of capital used by the intermediate good firm in $t+1$, if there were a competitive capital rental market.[14]

Note that a rise in the real rate of interest, holding the objects in braces on the right of (8) fixed, leads to a fall in I'. Given the assumption, $I''(1) > 0$, it follows that $k_{t+1}(i)$ falls, i.e., investment falls, as in Figure 1.

If the i^{th} firm has the opportunity to reoptimize its price, then it does so, taking into account that it must satisfy its demand curve and that it will not be able to reoptimize again, with probability α. The firm's first order condition is, after some algebra,

$$\sum_{j=0}^{\infty}\alpha^j\beta_{t+j}\lambda_{t+j}P_{t+j}^{\theta}Y_{t+j}\left\{(1+\tau)p_{t+j}(i) - \frac{\theta}{\theta-1}s_{t+j}(i)\right\} = 0.$$

Setting $1+\tau = \theta/(\theta-1)$ and multiplying by $(1+\tau)^{-1}$,

$$\sum_{j=0}^{\infty}\alpha^j\beta_{t+j}\lambda_{t+j}P_{t+j}^{\theta}Y_{t+j}\left\{p_{t+j}(i) - s_{t+j}(i)\right\} = 0, \tag{10}$$

[14] See Woodford (2003, p. 355) for an alternative interpretation.

where

$$p_{t+j}(i) = \frac{P_t(i)}{P_{t+j}},$$

where $P_t(i)$ is the price set in period t. Also, $s_{t+j}(i)$ is the marginal cost of producing an extra unit of output:

$$s_t(i) = \frac{w_t(i)}{f'\left(\frac{h_t(i)}{K_t(i)}\right)}. \tag{11}$$

According to (10), the firm sets its price to marginal cost on average, given the setting of the revenue subsidy, τ.

Steady State

To compute the log-linear expansion about steady state, we require the capital-to-output ratio, k, and consumption-to-output ratio, c. We compute these first. In steady state, (8) reduces to:

$$\frac{1}{\beta} = \rho - \delta + 1,$$

where ρ is the steady state value of the first object in braces in (8). Then, $\rho = (1/\beta) + \delta - 1$. In steady state, all prices are equal, so that $p_t(i) = 1$. This, and the fact that the object in braces in (10) is zero in steady state imply $s = 1$. Combining this with (11), we obtain:

$$w = f'.$$

That is, in steady state the real wage is equated to the marginal product of labor. Combine these results with (9) to obtain, in steady state:

$$\begin{aligned}
\frac{1}{\beta} + \delta - 1 &= w\frac{f - \bar{h}f'}{f'} \\
&= f'\frac{f - \bar{h}f'}{f'} \\
&= f \times \left(\frac{\phi - 1}{\phi}\right) \\
&= \frac{1}{k}\left(\frac{\phi - 1}{\phi}\right),
\end{aligned}$$

where $k \equiv K/Y$. Then,

$$k = \frac{\frac{\phi-1}{\phi}}{\frac{1}{\beta} + \delta - 1}.$$

Finally, note that in steady state:

$$Y = C + \delta K + G,$$

so that

$$c \equiv \frac{C}{Y} = 1 - \delta k - g,$$

where

$$g = \frac{G}{Y}.$$

Log-Linear Expansion

Consider the national income identity:

$$Y_t = C_t + I_t + G_t,$$

where G_t denotes the exogenous level of government spending. Then,

$$\hat{Y}_t = \hat{C}_t + \hat{I}_t + \hat{G}_t.$$

Unless otherwise noted, a hat over a variable indicates deviation from steady state, as a fraction from steady state. That is, $\hat{z}_t \equiv dz_t/z$, where z is the steady state value of z_t and dz_t is a small deviation, $z_t - z$. However, in the case of the aggregate quantities appearing in the national income identity, a hat indicates deviation from steady state, expressed as a fraction of steady state aggregate output:

$$\hat{I}_t = \frac{dI_t}{Y},\ \hat{Y}_t = \frac{dY_t}{Y},\ \hat{C}_t = \frac{dC_t}{Y},\ \hat{G}_t = \frac{dG_t}{Y}.$$

Log-linearizing the household's intertemporal Euler equation:

$$\hat{u}_{c,t} = \hat{u}_{c,t+1} - \hat{r}_t^n + \hat{\imath}_t - \hat{\pi}_{t+1}, \tag{12}$$

where

$$\hat{\imath}_t \equiv \frac{i_t - i}{1+i},\ \hat{r}_t^n \equiv \frac{dr_t^n}{1+r^n}.$$

The non-negativity constraint on i_t implies, since $1 + i = 1/\beta$ in steady state,

$$\hat{\imath}_t \geq \hat{\imath}_l = \beta - 1.$$

It is useful to develop an expression for $\hat{u}_{c,t}$. We have

$$\begin{aligned} \hat{u}_{c,t} &\equiv \frac{du_{c,t}}{u_c} = \frac{u_{cc}Y}{u_c}\hat{C}_t \\ &= \frac{u_{cc}C}{u_c}\frac{Y}{C}\hat{C}_t \\ &= -\sigma^{-1}\hat{C}_t, \end{aligned}$$

where

$$\begin{aligned} \sigma &= \sigma_u^{-1}c > 0 \\ \sigma_u &= -\frac{u_{cc}C}{u_c} \end{aligned}$$

Then,

$$\hat{u}_{ct} = -\sigma^{-1}\left[\hat{Y}_t - \hat{I}_t - \hat{G}_t\right]. \tag{13}$$

We now turn to the firms. Integrating (4) over $j \in (0,1)$ and imposing the final goods firm technology, the following relationship between the aggregate and intermediate good prices must hold:

$$P_t = \left[\int P_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}}. \tag{14}$$

Loglinearizing (8) about steady state and integrating the result over $i \in (0,1)$, we obtain:[15]

$$\begin{aligned} \hat{u}_{c,t} + \epsilon_\psi \left[\hat{K}_{t+1} - \hat{K}_t \right] &= (1-\delta)\beta \hat{u}_{c,t+1} - \hat{r}_t^n \\ &\quad + [1-(1-\delta)\beta] \left[\rho_y \hat{Y}_{t+1} - \rho_k \hat{K}_{t+1} \right] \\ &\quad + \beta \epsilon_\psi \left[\hat{K}_{t+2} - \hat{K}_{t+1} \right], \end{aligned} \tag{15}$$

where

$$\begin{aligned} \rho_y &= \sigma_v \phi + \frac{\omega_p \phi}{\phi - 1}, \ \omega_p = \sigma_f \phi \\ \rho_k &= \rho_y - \sigma_v. \end{aligned}$$

That (15) only involves aggregate output and the aggregate capital stock reflects that we have integrated the linearized first order conditions of the firms in a neighborhood of a symmetric steady state equilibrium. Note that (15) implies $\hat{K}_{t+1} - \hat{K}_t = (1/\beta)^t \left[\hat{K}_1 - \hat{K}_0 \right]$ when $\epsilon_\psi = \infty$. So, the only $\hat{K}_1 - \hat{K}_0$ which implies a non-explosive path for the capital stock is one in which $\hat{K}_1 - \hat{K}_0 = 0$, in which case the capital stock is constant. This is what one would expect when adjustment costs are infinite.

Loglinearizing the investment equation, (6):

$$\hat{I}_t = k \left[\hat{K}_{t+1} - (1-\delta)\hat{K}_t \right], \tag{16}$$

where

$$k = \frac{K}{Y}.$$

Now consider the pricing decision of the firm. Log-linearizing (10) we obtain (after considerable algebra, see Woodford 2003, page 360):

$$\hat{\pi}_t = \xi_0 \hat{s}_t - \xi_1 \hat{s}_{t+1} + \psi_1 \hat{\pi}_{t+1} - \psi_2 \hat{\pi}_{t+2}, \tag{17}$$

where s_t is the real marginal cost of production for the firm. Here, s_t denotes real marginal cost and, after log-linearizing:

$$\hat{s}_t = \omega \left(\hat{Y}_t - \hat{K}_t \right) + \sigma_v \hat{K}_t - \hat{u}_{c,t}, \tag{18}$$

where

$$\omega = \sigma_v \phi + \omega_p.$$

[15] Expression (15) corresponds to Woodford (2003, equation 3.7, page 356).

In (17),

$$\xi_0 = \frac{1-\alpha}{\alpha}\frac{1}{a-b},\ \xi_1 = \mu_2^{-1}\xi_0,$$

and

$$\psi_1 = \frac{a\left(\beta+\mu_2^{-1}\right) - b(\alpha\beta+\alpha^{-1}\mu_2^{-1})}{a-b},\ \psi_2 = \beta\mu_2^{-1}.$$

Also,

$$\begin{aligned}
a &= \frac{1+\omega\theta}{1-\alpha\beta} + (\omega-\sigma_v)\frac{\alpha}{1-\alpha\beta}\frac{\Xi}{(1-\alpha\beta\mu_1)(1-\alpha\beta\mu_2)} \\
b &= (\omega-\sigma_v)\frac{\alpha}{1-\mu_2^{-1}}\frac{\Xi}{(1-\alpha\beta\mu_1)(1-\alpha\beta\mu_2)} \\
\Xi &= (1-\beta(1-\delta))\,\rho_y\theta\epsilon_\psi^{-1}
\end{aligned}$$

Finally, μ_2 is the root greater than β^{-1} in the equation

$$\beta\mu^2 - \left[1+\beta+(1-\beta(1-\delta))\,\rho_k\epsilon_\psi^{-1}\right]\mu + 1 = 0,$$

and μ_1 is the other root, which is less than unity.

I reduce the system of private economy equations to three equations, (12), (15), (17) in the four unknowns, $\hat{K}_{t+1}$, $\hat{Y}_t$, $\hat{\pi}_t$, $\hat{\imath}_t$. This is accomplished by substituting out for $\hat{u}_{c,t}$ using (13), $\hat{I}_t$ using (16), and $\hat{s}_t$ using (18). The inflation equation becomes:

$$\begin{aligned}
&\hat{\pi}_t - \xi_0\left\{\left(\omega+\sigma^{-1}\right)\hat{Y}_t - \left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right]\hat{K}_t - \sigma^{-1}k\hat{K}_{t+1} - \sigma^{-1}\hat{G}_t\right\} \\
&+\xi_1\left\{\left(\omega+\sigma^{-1}\right)\hat{Y}_{t+1} - \left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right]\hat{K}_{t+1} - \sigma^{-1}k\hat{K}_{t+2} - \sigma^{-1}\hat{G}_{t+1}\right\} \\
&\qquad -\psi_1\hat{\pi}_{t+1} + \psi_2\hat{\pi}_{t+2} = 0
\end{aligned}$$

Writing this in vector notation,

$$F_\pi z_t = 0, \tag{19}$$

where z_t is the 12×1 vector:

$$z_t = (\hat{\pi}_t, \hat{\pi}_{t+1}, \hat{\pi}_{t+2}, \hat{Y}_t, \hat{Y}_{t+1}, \hat{K}_t, \hat{K}_{t+1}, \hat{K}_{t+2}, \hat{G}_t, \hat{G}_{t+1}, \hat{r}_t^n, \hat{\imath}_t)'$$

and F_π is the 1×12 vector:

$$\begin{aligned}
F_{\pi 1} &= 1,\ F_{\pi 2} = -\psi_1,\ F_{\pi 3} = \psi_2 \\
F_{\pi 4} &= -\left(\omega+\sigma^{-1}\right)\xi_0,\ F_{\pi 5} = \left(\omega+\sigma^{-1}\right)\xi_1 \\
F_{\pi 6} &= \xi_0\left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right], \\
F_{\pi 7} &= \xi_0\sigma^{-1}k - \xi_1\left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right] \\
F_{\pi 8} &= -\xi_1\sigma^{-1}k,\ F_{\pi 9} = \xi_0(1+\sigma_v),\ F_{\pi 10} = -\xi_1(1+\sigma_v) \\
F_{\pi 11} &= \xi_0\sigma^{-1},\ F_{\pi 12} = -\xi_1\sigma^{-1},\ F_{\pi 13} = F_{\pi 14} = 0.
\end{aligned}$$

The intertemporal Euler equation associated with investment becomes:

$$\begin{aligned}\epsilon_\psi^{-1}\hat{Y}_t - \epsilon_\psi^{-1}k\left[\hat{K}_{t+1}-(1-\delta)\hat{K}_t\right] - \epsilon_\psi^{-1}\hat{G}_t - \sigma\left[\hat{K}_{t+1}-\hat{K}_t\right] \\ -\epsilon_\psi^{-1}(1-\delta)\beta\left(\hat{Y}_{t+1} - k\left[\hat{K}_{t+2}-(1-\delta)\hat{K}_{t+1}\right] - \hat{G}_{t+1}\right) - \epsilon_\psi^{-1}\sigma\hat{r}_t^n \\ +\epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right]\left[\rho_y\hat{Y}_{t+1} - \rho_k\hat{K}_{t+1} - (1+\sigma_v)\hat{A}_{t+1}\right] \\ +\sigma\beta\left[\hat{K}_{t+2}-\hat{K}_{t+1}\right] = 0,\end{aligned}$$

or,

$$F_k z_t = 0, \tag{20}$$

where the 1×12 vector F_k is:

$$\begin{aligned}F_{k1} &= F_{k2} = F_{k3} = 0,\ F_{k4} = \epsilon_\psi^{-1}, \\ F_{k5} &= -\epsilon_\psi^{-1}(1-\delta)\beta + \epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right]\rho_y \\ F_{k6} &= \epsilon_\psi^{-1}k(1-\delta)+\sigma, \\ F_{k7} &= -\epsilon_\psi^{-1}k - \sigma - \epsilon_\psi^{-1}(1-\delta)\beta(1-\delta)k - \epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right]\rho_k - \sigma\beta \\ F_{k8} &= \epsilon_\psi^{-1}(1-\delta)\beta k + \sigma\beta,\ F_{k9} = 0,\ F_{k10} = -\epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right](1+\sigma_v) \\ F_{k11} &= -\epsilon_\psi^{-1},\ F_{k12} = \epsilon_\psi^{-1}(1-\delta)\beta,\ F_{k13} = -\epsilon_\psi^{-1}\sigma,\ F_{k14} = 0.\end{aligned}$$

The household's intertemporal Euler equation is:

$$\begin{aligned} & \hat{Y}_t - k\left[\hat{K}_{t+1}-(1-\delta)\hat{K}_t\right] - \hat{G}_t \\ = & \ \hat{Y}_{t+1} - k\left[\hat{K}_{t+2}-(1-\delta)\hat{K}_{t+1}\right] - \hat{G}_{t+1} - \sigma\left(\hat{\imath}_t - \hat{\pi}_{t+1} - \hat{r}_t^n\right)\end{aligned} \tag{21}$$

or,

$$F_i z_t = 0, \tag{22}$$

where the 1×12 vector F_i is:

$$\begin{aligned}F_{i,1} &= 0,\ F_{i2} = -\sigma,\ F_{i3} = 0,\ F_{i4} = 1,\ F_{i5} = -1,\ F_{i6} = k(1-\delta) \\ F_{i7} &= -k - k(1-\delta),\ F_{i8} = k,\ F_{i9} = F_{i10} = 0,\ F_{i11} = -1,\ F_{i12} = 1 \\ F_{i13} &= -\sigma,\ F_{i14} = \sigma.\end{aligned}$$

Let

$$F = \begin{bmatrix} F_\pi \\ F_k \\ F_i \end{bmatrix},$$

so that the equations are written:

$$Fz_t = 0,\ \text{for all } t.$$

It is convenient to write this as:

$$Fz_t = F_1 z_{1t} + F_2 z_{2t},$$

where F_1 includes all but the 5^{th} and 8^{th} columns of F and F_2 is composed just of these two. In addition, z_{1t} includes all but the 5^{th} and 8^{th} elements of z_t, while z_{2t} includes the 5^{th} and 8^{th} only. So, F_1 is 3×10 and F_2 is 3×2. It is useful to consider two 2×2 pieces of F_2: let F_2^1 denote the first two rows, and let F_2^2 denote the second two rows. Define F_1^1 and F_2^2 analogously.

Write:

$$\begin{aligned} z_{1t} &= (\pi_t, \pi_{t+1}, \pi_{t+2}, \hat{Y}_t, \hat{K}_t, \hat{K}_{t+1}, \hat{G}_t, \hat{G}_{t+1}, \hat{r}_t^n, \hat{\imath}_t)' \\ z_{2t} &= \left(\hat{Y}_{t+1}, \hat{K}_{t+2}\right)'. \end{aligned}$$

A.2 A Shooting Algorithm

Suppose that the economy has been in a deterministic, zero-inflation steady state until period 0, so that $\hat{K}_1 = 0$. Then, in $t = 1$, r_t^n, drops unexpectedly to $\hat{r}_l^n < \beta - 1$. The discount rate, r_t^n, remains low $T - 1$ periods, and returns to its steady state value in the Tth period, where it remains forever after. I study the equilibrium allocations in response to the savings shock, r_t^n. Government spending, G_t, is allowed to change values in the same periods.

Monetary policy has the property that $\pi_t = 0$, unless this implies the interest rate violates its lower bound, $\hat{\imath}_t \geq \beta - 1$, in which case $\hat{\imath}_t = \beta - 1$. I conjecture (and then verify) that the zero bound becomes binding in one particular period, $t_1^* \geq 1$, and then continues to bind in each period until period t_2^*, when it ceases to bind. Thus,

$$\begin{array}{cc} \hat{\imath}_t = \beta - 1 & t_1^* \leq t < t_2^* \\ \hat{\imath}_t > \beta - 1,\ \pi_t = 0 & t \geq t_2^*,\ t < t_1^*. \end{array}.$$

The algorithm described below finds t_1^*, t_2^*, as well as the other equilibrium variables of the model.

The basic idea of the algorithm is as follows. There is one initial condition for the economy, $\hat{K}_1 = 0$. I then simulate the equations that characterize equilibrium forward in time. Because of the dimension of the equations, to initiate the simulations I need to assign values to four endogenous variables, aside from the given value of $\hat{K}_1$. I adjust the values of these four variables to ensure that the system eventually converge to steady state, and that another side condition is satisfied.

Suppose that $t_1^* = 1$, so that the system is assumed to bind in period 1. Arbitrarily, assign a set of values to the four variables, $\hat{Y}_1$, $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{K}_2$. Temporarily normalize $\hat{\pi}_3 = 0$, and set $\hat{\imath}_1 = \beta - 1$. We can now construct z_{1t} for $t = 1$. Use (20)-(22) for $t = 1$ to compute $\hat{Y}_2$, $\hat{K}_3$:

$$z_{2,1} = -\left[F_2^2\right]^{-1} F_1^2 z_{1,1}. \tag{23}$$

Note that $\hat{\pi}_3$ does not appear in (20)-(22) for $t = 1$, so that setting $\hat{\pi}_3 = 0$ has no impact on the calculations in (23). I now use (19) to compute $\hat{\pi}_3$ and I insert this into the third element of $z_{1,1}$.

Now consider $t = 2$. We have enough information to determine the values of z_{1t} for $t = 2$, as long as we normalize $\hat{\pi}_4 = 0$. Use (20)-(22) to compute $\hat{Y}_3$ and $\hat{K}_4$ using:

$$z_{2,t} = -\left[F_2^2\right]^{-1} F_1^2 z_{1,t},$$

for $t = 2$. Then, use (19) to compute $\hat{\pi}_4$. Proceed in this way until $t = t_2^* - 1$.

Consider $t \geq t^*$. I set $\hat{\pi}_t = 0$ for $t \geq t^*$. As before, we have enough information to put together the values of z_{1t}. Now we use (19)-(20) to compute $\hat{Y}_{t^*+1}$, $\hat{K}_{t^*+2}$:

$$z_{2,t} = -\left[F_2^1\right]^{-1} F_1^1 z_{1,t}. \tag{24}$$

Then, (22) is used to solve for $\hat{\imath}_t$. Simulate this forward for many periods, say until $t = T^* > T$.

Adjust the values of $\hat{Y}_1$, $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{K}_2$ until

$$\hat{Y}_{T^*} = 0, \ \hat{K}_{T^*+1} = 0, \tag{25}$$

and the second and third elements of z_{1,t_2^*-1} are zero.

Now consider the case, $t_1^* \geq 2$. For $t = 1, ..., t_1^* - 1$, I use (24) to simulate the system. To initiate these simulations, I assign values to $\hat{Y}_1$, $\hat{K}_2$. In addition, I require values for $\hat{\pi}_{t_1^*}$ and $\hat{\pi}_{t_1^*+1}$. It is necessary to include these in z_{1,t_1^*-2} and z_{1,t_1^*-1} when evaluating (24) for $t = t_1^* - 2$, $t_1^* - 1$. I assign values to these arbitrarily. I adjust the values of $\hat{Y}_1$, $\hat{K}_2$, $\hat{\pi}_{t_1^*}$ and $\hat{\pi}_{t_1^*+1}$ until the same criterion used for the case, $t_1^* = 1$ is satisfied. The MATLAB routine, FSOLVE.M, was able to solve this problem without incident in less than 2 seconds.

To determine values of t_1^* and t_2^*, I proceeded as follows. First, I solved the model without imposing the lower bound constraint on $\hat{\imath}_t$. I then determined the first date when the lower bound constraint was violated, and made that my initial guess of t_1^*. I then determined the first date thereafter when the lower bound constraint ceased to bind and made that my guess of t_2^*. For the model without investment, i.e., $\epsilon_\psi = \infty$, this procedure led to an initial guess, $t_1^* = 1$, and I did not deviate from that. The initial guess for t_2^* was 15. I kept that value because the lower bound constraint was violated when I dropped t_2^* to a lower value.

For the model with investment, $\epsilon_\psi = 3$, my procedure led to an initial guess, $t_1^* = 7$ and $t_2^* = 15$. When I solved the model with these values, I found that the lower bound constraint was violated. I then found that it was also violated for values of t_1^* that exceeded 7 and for every admissible value of t_1^* below 7 except $t_1^* = 1$. So, I fixed $t_1^* = 1$. I then tried $t_2^* = 14$, and found that the lower bound constraint was violated. So, I remained with $t_1^* = 1$ and $t_2^* = 15$ for the $\epsilon_\psi = 3$ model.

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Figure 1: Consequence of Increase in Saving When there is Lower Bound on Real Interest Rate. For Two Investment Elasticities

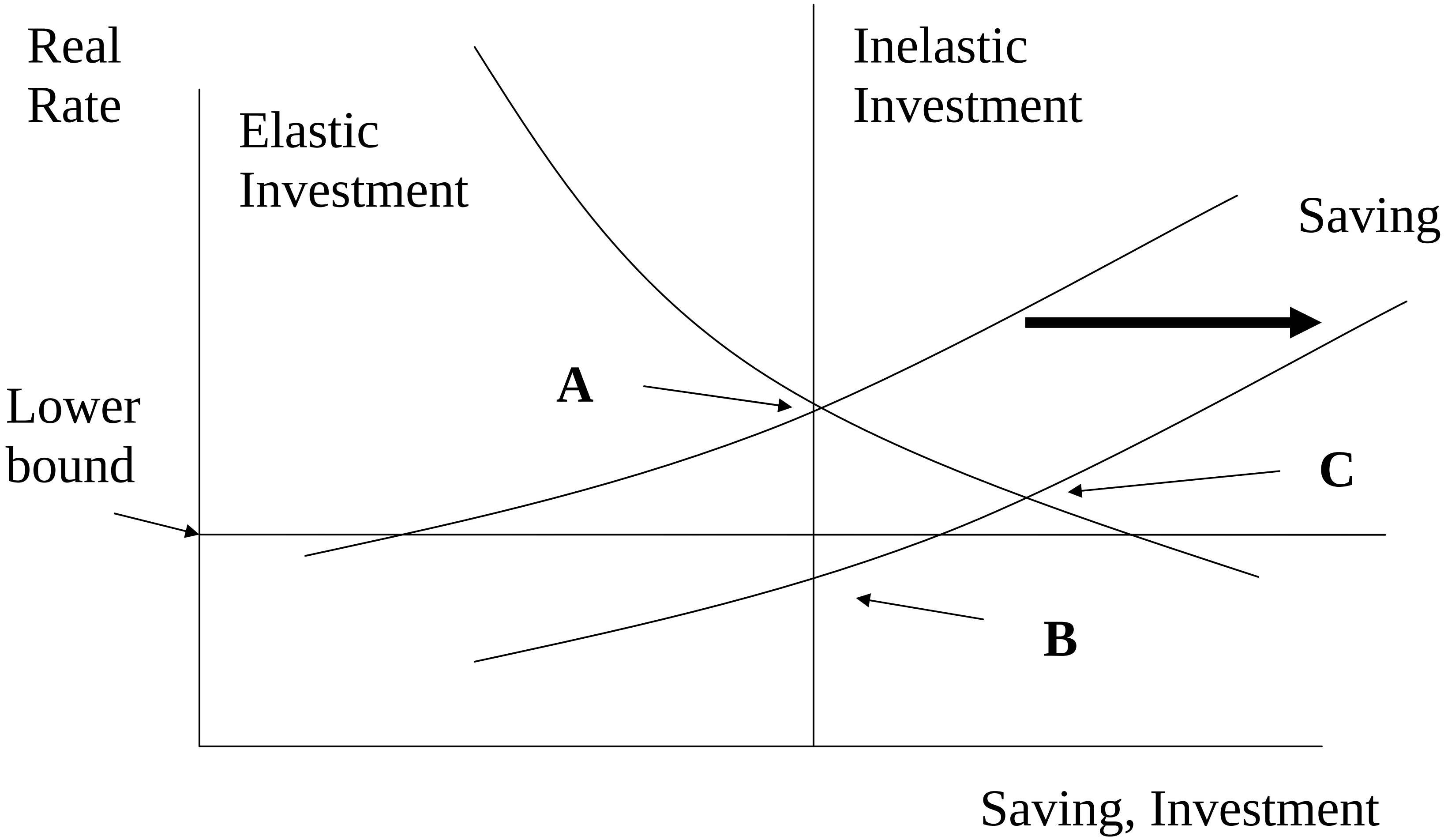

Figure 2: Discount Rate Shock in Model without Investment, Three Discount Rate Shocks

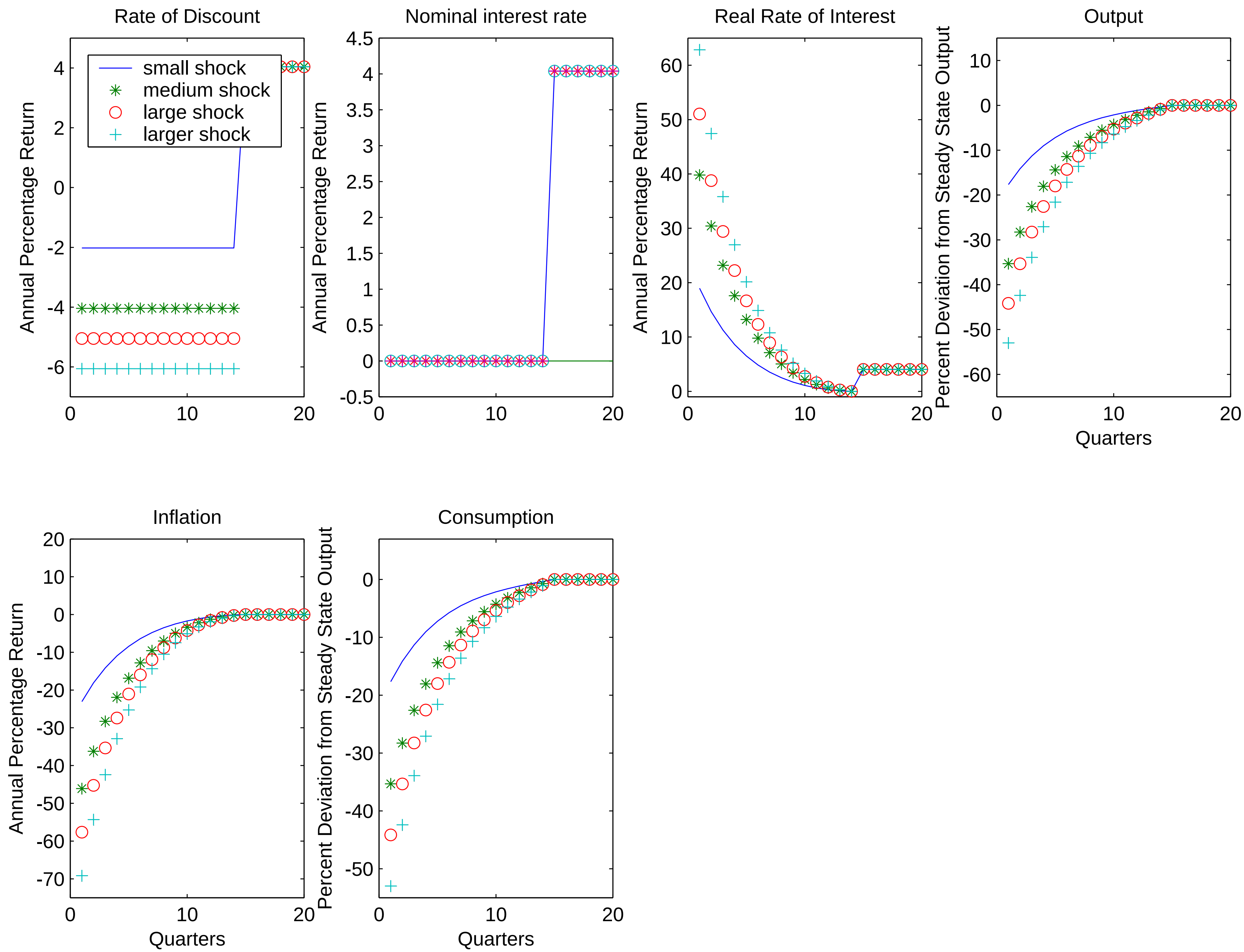

Figure 3: Discount Rate Shock in Model with Investment, Three Discount Rate Shocks

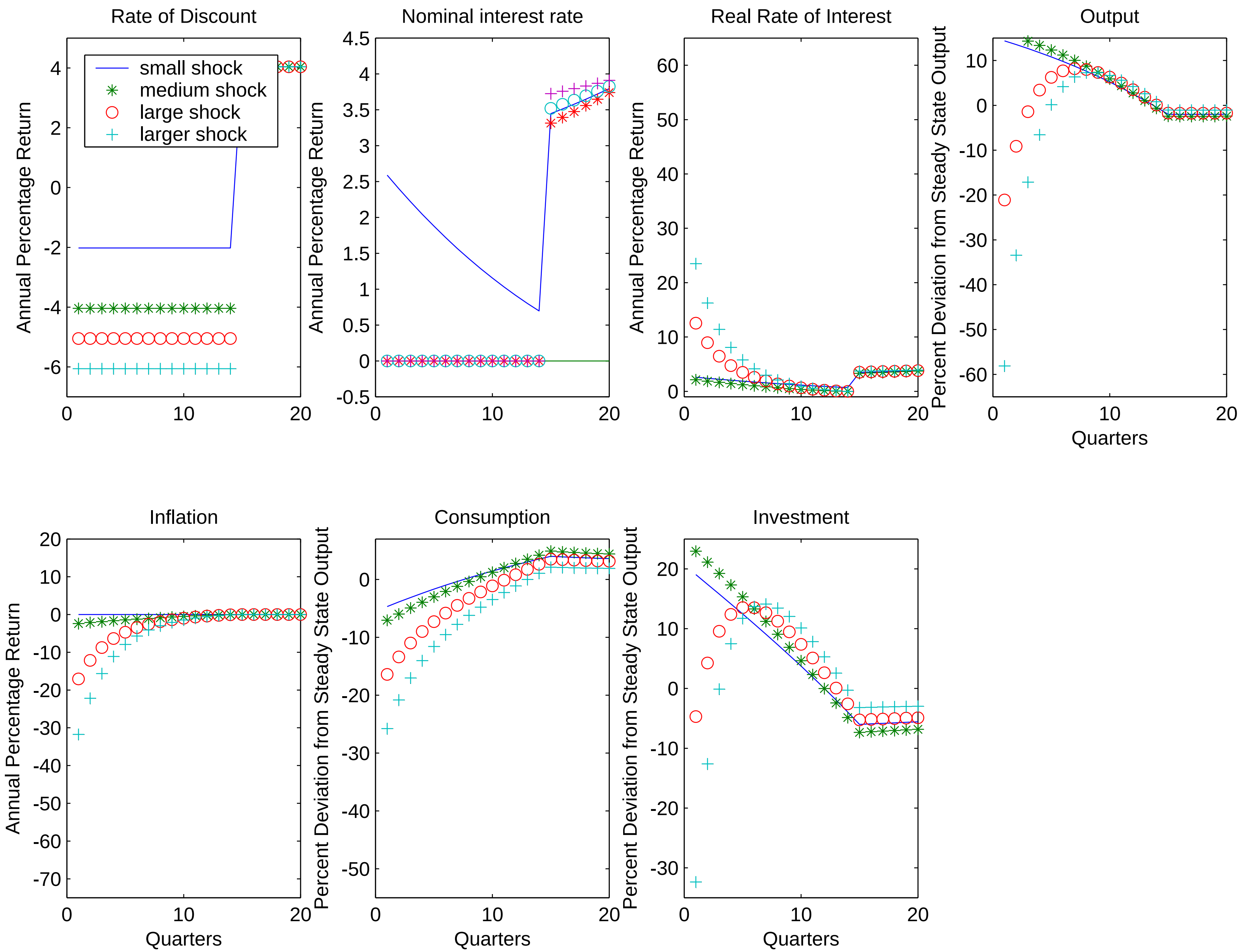

Figure 4: Discount Rate Shock in Model without Investment, Low Labor Supply Elasticity, Three Discount Rate Shocks

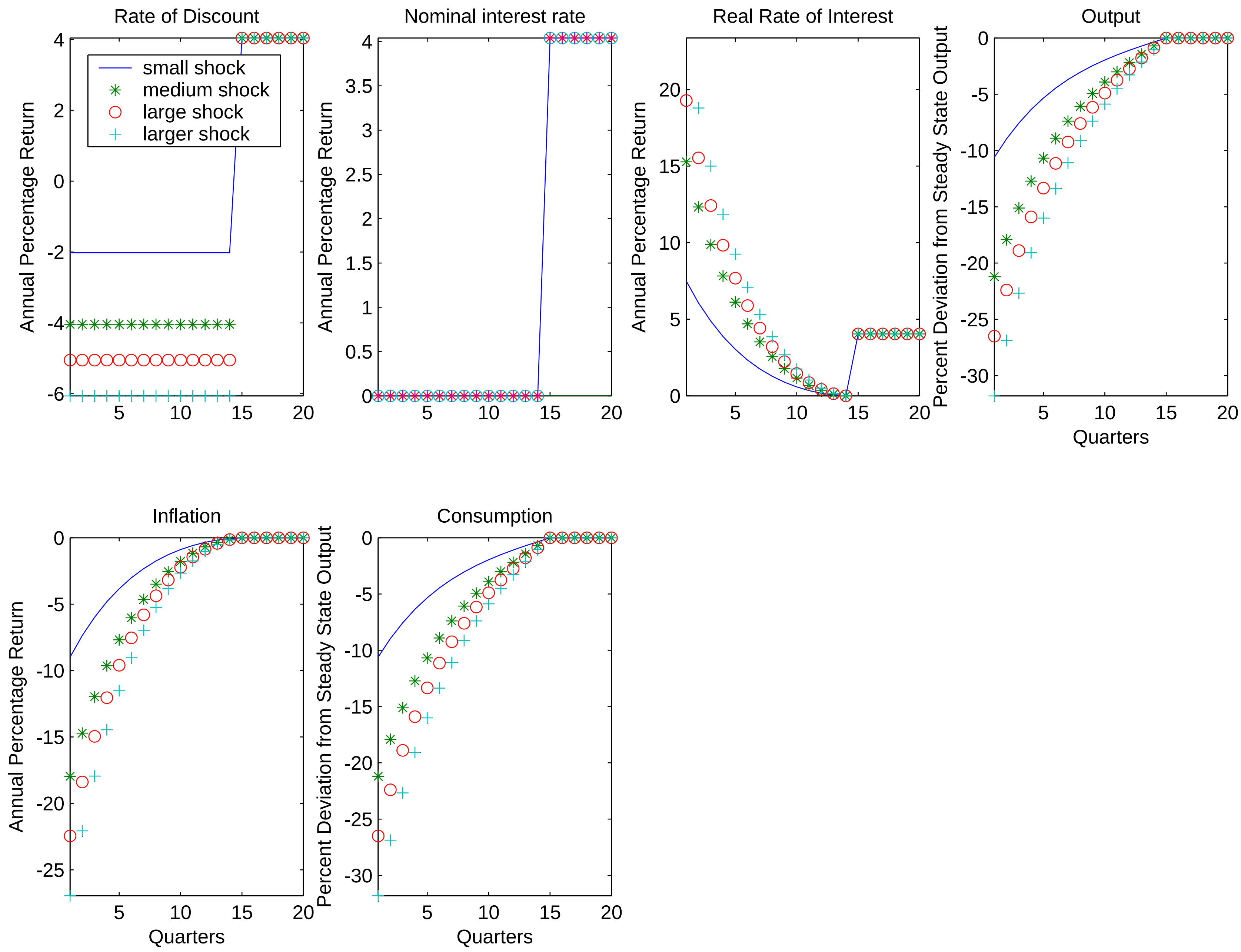

Figure 5: Discount Rate Shock in Model with Investment, Low Labor Supply Elasticity, Three Discount Rate Shocks