The Zero-Bound, Zero-Inflation Targetting, and Output Collapse*
(Incomplete and Still Under Revision!)

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Abstract

I consider the example analyzed in Eggertsson and Woodford (2003a,b), which shows that the zero lower bound on the nominal rate of interest, in conjunction with a zero-inflation policy by the central bank, can cause output to collapse in response to certain shocks. I show that when capital and government spending are introduced into the analysis, there is much less reason to fear the combination of a zero lower bound and zero-inflation targeting. The zero bound is not likely to bind, and if it does the consequences may not be severe. Moreover, the multiplier on government spending is predicted to be very large in the event of a binding zero bound, so that in the worst case scenario and increase in government spending should help substantially.

An examination of historical U.S. data suggests that this model analysis may provide an overly optimistic assessment of the operating characteristics of a monetary policy that targets zero inflation. This suggests that additional model analysis which allows for a broader range of shocks, a wider range of economic frictions and open-economy considerations, is of interest.

1 Introduction

Since the high inflation of the 1970s, central banks around the world have successfully brought down inflation. As a result, interest rates are so low that the zero lower bound might well bind sometime in the near future. Increasingly, there is a concern that this combination - nearly zero interest rates and a monetary policy that targets zero or low inflation - exposes economies to a new threat. As Bernanke (2003) has recently emphasized, targetting zero

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inflation creates the possibility of a 'worst case scenario', in which an otherwise mild economic shock causes a collapse in output and a downward spiral in prices.

These concerns raise two related questions. Can they be given a coherent intellectual foundation? If the answer is 'yes', then what is the likelihood that the worst case scenario will occur?

Several papers, including most recently Eggertsson and Woodford (2003a,b), show convincingly that the 'worst case scenario' can be articulated in a coherent economic model.[1] Eggertsson and Woodford construct an example in which the efficient response to a particular shock requires that the real rate of interest be negative. However, this is prevented from occuring because the zero lower bound on the nominal interest rate binds, and the central bank can do nothing to raise inflation expectations. This is because ex post, a zero-inflation-targetting central bank will always choose to produce zero inflation, if it can.[2] With the real interest rate unable to play its role in efficiently directing the allocation of resources, the result is the sort of welfare-reducing downward spiral in output and inflation that concerns observers such as Bernanke. Eggertsson and Woodford's example is dramatic and hard to ignore. In the example there are welfare losses that exceed anything that one can find in monetary economics, outside of models with sunspot fluctuations.

The policy issues at stake are substantial. Now that many countries have won the battle to bring down inflation, there is a debate over what to do to ensure that the victory endures. One possibility is for central banks to formally adopt a policy of targetting zero, or at least low, inflation. Until recently, most economists would have taken it for granted that this is a desirable thing. However, analyses such as those of Eggertsson and Woodford suggest that this policy may actually be counterproductive. And, the problems they point to may not just exist in theory. Some worry that the weak economic performance of Japan in the 1990s and the disasterous output collapse in the U.S. in the 1930s show that the dangers highlighted by Eggertsson and Woodford's example are very real.

Given the stakes, it is important to understand whether the Eggertsson and Woodford results are sensitive to relaxing some of their simplifying assumptions. Their model does not allow a role for capital. I introduce capital along the lines proposed in Woodford (2003, Chapter 5). I find that allowing investment to be endogenous changes the results in two ways. First, the introduction of investment reduces the likelihood that the zero lower bound on the interest rate binds. It now takes a bigger shock for the lower bound to bind. Also, the parameters in Woodford (2003, Chapter 5) imply a very high Frisch elasticity of labor supply. When I use a more standard elasticity, the lower bound fails to bind for even for very large shocks.[3] Second, I find that if the zero lower bound does bind, there is a large

[1] An early, informal statement of the problem appears in Summers (1991) (see also De Long 1999, Sims 1999 and Svensson 2003.) More formal statements of the problem appear in Krugman (1998), Orphanides and Wieland (1998), Uhlig (2000) and Wolman (1998,2003). Two distinguishing features of the Eggertsson and Woodford (2003a,b) analysis are that (i) their analysis provides a more fully articulated statement of the problem, by providing a natural explanation for why the central bank has cannot move inflation expectations; and (ii) they compute the optimal monetary policy.

[2] In Eggertsson and Woodford's model zero inflation targetting is rationalized as what an optimizing central bank unable to commit to future policy will do. Eggertsson and Woodford assume away the sources of inflation bias present in the models of Kydland and Prescott (1977) and Barro and Gordon (1983), and the literature that followed.

[3] For implausibly large shocks, the lower bound still binds, even with the low Frish labor supply elasticity.

range of values for the shocks when the consequences of a binding lower bound are relatively mild.

I also introduce government consumption into the analysis. I find that the multiplier effect on output of a rise in government spending is less than unity when the lower bound on the interest rate is not binding, but very large when it is binding. As a result, in the event that the lower bound does bind, a relatively small rise in government spending can substantially reduce the likelihood of the 'worst case scenario'.

In sum, the model analysis with investment and government spending presents a less foreboding picture of the operating characteristics of a zero-inflation targetting rule than does Eggertsson and Woodford's example. I then turn to the U.S. historical data to assess the likelihood that the lower bound on the nominal rate would bind in the event that the Fed were to adopt a zero-inflation monetary policy target. For this I estimate the frequency of times that the lower bound would have bound, if a zero inflation targetting policy had been adopted in the past. The model analysis suggests estimating this by the frequency of times that the real interest rate was negative in the past. The data suggests that the zero bound would have been binding a nontrivial number of times. Exactly how often depends on what tax rate one uses to compute after-tax returns, and what interest rate one uses. I conclude that although the formal model analysis does not by itself justify alarm, the empirical results and Eggertsson and Woodford's dramatic example do give one pause. So, the prudent thing to do is to study the consequences of hitting the lower bound in a much broader range of models and shocks before writing this off as a problem not worth worrying about. I discuss what range of models and shocks ought to be examined in my concluding remarks.

In the first section below, I discuss the intuition behind the Eggertsson-Woodford results. It should be familier to many economists because it closely resembles the logic of the Paradox of Thrift, discussed in standard undergraduate macroeconomics textbooks. Section 3 presents the Eggertsson-Woodford analysis. This is carried out using a Calvo-sticky price model which implies that inflation is entirely forward looking. I verify that the results are at least qualitatively robust to introducing backward-looking elements. Section 4 describes the model with investment, and displays the results of my numerical experiments. Finally, section 5 reviews the data on real interest rates. Section 6 concludes. The computational algorithm used to obtain the results is described in the appendix.

2 Simple Intuition

Although the formal analysis on which my conclusions are based is dynamic (see the next two sections below), intuition can be obtained from a simple static analysis. Consider Figure 1. The real rate of interest,

$$\frac{1+i_t}{\pi_{t+1}},$$

appears on the vertical axis. Here, i_t denotes the nominal interest rate and π_{t+1} denotes the gross rate of change in the price level from the current period to the next. Also, investment and saving appear on the horizontal axis. The figure shows a shift to the right in the supply of savings (induced in the formal analysis by a shock to the discount rate). The demand for saving arises from firms seeking to finance investment. There are two savings demand curves in the figure, each exhibiting a different elasticity of investment with respect to the real interest rate. The inelastic investment case corresponds to the one considered by Eggertsson and Woodford, who exclude capital from their model. This case is captured by the vertical line. The negatively sloped line corresponds to the interest elastic case, modeled formally in section 4. There is a lower bound on the real interest rate because i_t cannot drop below zero and under a zero-inflation targetting next period's monetary authority will do what it can to set π_{t+1} to unity. The lower bound on the real interest rate is indicated by the horizontal line in the figure.

Consider the interest-inelastic case first. The shock induces people to cut spending and increase saving. Because the quantity of investment cannot change by hypothesis, the quantity saved also cannot change in equilibrium. Something must happen to cause the quantity saved to remain unchanged, despite the shock to the supply of saving. This can be accomplished efficiently by a drop in the real interest rate to point B. However, because of the lower bound on the real interest rate, this cannot happen. Something else must happen instead, to discourage a rise in the quantity saved. In the model, a fall in output and income does the job. The drop in output occurs naturally as households cut back on consumption in an effort to increase saving. Through a standard sticky price mechanism, this leads to a fall in income and employment (as well as the current price level). The decline in economic activity exerts downward pressure the quantity saved, for consumption-smoothing reasons. In effect, the decline in income causes the saving function to shift back to the left, so that the saving function intersects the vertical investment curve at the lower bound.

Actually, the scenario is worse than the reasoning just described suggests. In the formal analysis, the shift in the discount rate is assumed to be persistent, so that a similar thing is expected to occur in the future. That is, consumption and the price level are expected to fall in the future too. In fact, they fall so much that the expected gross inflation rate from the current period to the next, π_{t+1}, falls below unity. This has the effect of shifting the lower bound in Figure 1 up, increasing the distance that the saving function must be induced to shift back left. Put differently, the fall in expected inflation stimulates households' desire to save even further, increasing the amount by which income must fall to maintain equilibrium.[4]

The point of the example is that ordinarily the discount rate shock would be met by a suitable drop in the real interest rate and no change in output. In the formal analysis this is the welfare-efficient response. With a zero-inflation target and the lower bound on the interest rate, the real interest rate is prevented from doing its job. Instead, households are prevented from increasing their saving by a welfare-reducing fall in income. In their numerical example, discussed below, Eggertsson and Woodford show that this drop in income can be quite large, and deserves to be called a "downward spiral".

Now consider the elastic investment case. In this case, the real interest rate only has to drop to point C to maintain equilibrium. This cut in the real interest rate is enough that the resulting expansion in investment absorbs the rise in investment. Evidently, the size of the saving shock that produces the downward spiral in the economy with inelastic investment does not do so in the economy with elastic investment. This is the sense in which

[4]See DeLong and Summers (1986) for a discussion of how a drop in the price level (which normally stabilizes the output effect of a spending drop) can actually be destabilizing if the drop is associated with an even bigger drop in the next period.

the downward spiral is less likely when investment is elastic.

The intuitive analysis suggests various policy conclusions. For example, one way to reduce the likelihood of the downward spiral is to create the expectation that there will be higher inflation in the period after the shock. Eggertsson and Woodford show that this is the optimal monetary policy of a monetary authority that is able to commit to its future policy.[5] Interestingly, they show that the quantity of extra inflation required is quite modest, especially by comparison with the large deflation that occurs in their downward-spiral equilibrium. However, in Eggertsson and Woodford's model, creating expectations of higher inflation requires that the monetary authority be able to commit to its future monetary policy. Their environment has the property that once the shock is over, the monetary authority without commitment has no incentive to deliver on a promise of high inflation made in the past. Because absence of commitment forecloses this painless way to respond to a shock, Eggertsson and Woodford's analysis in effect provides another example of the value of commitment in monetary policy.[6]

3 The Eggertsson-Woodford Model

The first part of this section replicates the example reported by Eggertsson and Woodford. This model incorporates the standard version of Calvo-sticky prices, in which inflation is purely forward looking. To investigate whether their results are sensitive to this assumption, I also examine the version of the Calvo model proposed by Christiano, Eichenbaum and Evans (2004), in which price level indexation introduces a backward-looking component to inflation. I find that the Eggertsson-Woodford results are qualitatively similar with this change. If anything, the downward spiral in inflation and output is made slightly worse.

3.1 Eggertsson-Woodford Model

The algebra of the Eggertsson and Woodford example is strikingly simple. There are two equations that characterize equilibrium for the private sector in their economy, an intertemporal Euler equation associated with the household saving decision and the standard Calvo pricing equation:

$$x_t = E_t x_{t+1} - \sigma \left(\hat{\imath}_t - E_t \hat{\pi}_{t+1} - \hat{r}_t^n \right) \tag{1}$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa x_t, \tag{2}$$

where $0 < \beta < 1$, $\sigma, \kappa > 0$. Here, x_t, $\hat{\imath}$, π_t and $\hat{r}_t^n$ denote consumption, the nominal rate of interest, gross inflation and the discount rate. Each is expressed in deviation from its value in a zero-inflation steady state. Also, $\beta^{-1} - 1$ is the steady state rate at which the households in the economy discount future utility. The model is completed with a description of monetary policy. Policy sets $\hat{\imath}_t$ to ensure $\hat{\pi}_t = 0$, subject to the requirement of equilibrium that the nominal rate of interest is non-negative. When this constraint is binding then the nominal

[5] Christiano, Motto and Rostagno (2004) argue that this sort of policy, if followed in the 1930s, would have greatly reduced the severity of the Great Depression.

[6] For another, see Albanesi, Chari and Christiano (2003).

rate of interest is set to zero and the inflation rate is free to be determined by the equilibrium conditions. Since $\hat{\imath}_t$ is the deviation in the interest rate from its zero-inflation steady state, the zero lower bound on the nominal rate of interest translates into a negative lower bound for $\hat{\imath}_t : \hat{\imath}_t \geq \beta - 1$.[7] From (1) it is clear that if $\hat{\imath}_t = \hat{r}_t^n$ for all t, then there is an equilibrium in which consumption remains at its steady state value and inflation is always zero. However, if it should ever happen that $\hat{r}_t^n < \beta - 1$, then $\hat{\imath}_t = \hat{r}_t^n$ is not possible, as it would imply a violation of the zero lower bound on the interest rate.

To obtain a sense of what would happen in this case, Eggertsson and Woodford construct the following scenario. Suppose the economy is in a deterministic steady state with no shocks up until period 0. Then, unexpectedly in period 1 there is a shock to $\hat{r}_t^n$ which causes it to drop to $\hat{r}_l^n < \beta - 1$. In each subsequent period, with probability p, $\hat{r}_t^n$ remains low, and with probability $1-p$ it returns to its steady state value of zero. Once zero, $\hat{r}_t^n$ can never change value again.

The variables to be solved for in equilibrium are the levels of output and inflation, $\hat{x}_l$ and $\hat{\pi}_l$, that prevail while $\hat{r}_t^n = \hat{r}_l^n$. To find these, just solve the versions of (1) and (2) that occur as long as $\hat{r}_t^n = \hat{r}_l^n : x_l = px_l - \sigma((\beta - 1) - p\pi_l - \hat{r}_l^n)$ and $\pi_l = \beta p \pi_l + \kappa x_l$. This represents two equations in the two unknowns. Eggertsson and Woodford assume the time period in their model is one quarter, and adopt the following parameterization:

$$p = 0.9,\ \sigma = 0.5,\ \kappa = 0.02,\ \beta = 0.99,\ r_l^n = -.02/4.$$

It is easy to confirm that for these parameter values,

$$x_l = -0.14,\ \pi_l = -0.0263.$$

That is, output is 14 percent below steady state and inflation is -12 percent annualy as long as the discount rate remains low. The discount rate is expected to stay low for $1/(1-p) = 10$ periods, a notably long time. The intuition for these results was described in the previous section. Since any drop in output represents an inefficient response to the $\hat{r}_t^n$ shock, this represents a substantial inefficiency.

3.2 Inflation Indexation

With inflation indexation, the inflation term in (2) is replaced by its first difference. The two equations of the model are this adjusted version of (2) and (1). To simplify the analysis, I consider a particular deterministic experiment, in which the economy is in a zero inflation steady state up until period 1, whereupon r_t^n unexpectedly drops to -2 percent, at an annual rate. The discount rate remains low from periods 1 to period 14, and then returns to its previous value of 4 percent in period 15. It remains at that value forever after.

[7] In particular, the

$$\hat{\imath}_t \equiv \frac{i_t - i}{1+i},$$

where i_t is the nominal rate of interest and i is its steady state value. In a zero-inflation steady state, $1/(1+i) = \beta$, so that

$$\hat{\imath}_t = \beta\,(i_t + 1) - 1,$$

so that when i_t is at its lower bound of zero, $\hat{\imath}_t = \beta - 1$.

The solid line in Figure 2 displays the dynamic behavior of the Eggertsson-Woodford model in response to this shock.[8] Note that deflation is a little over 2 percent in the period of the shock, and after this it converges monotonically back to its steady state value in period 15. Output falls nearly 5 percent in the period of the shock and then converges monotonically back to steady state too. Now consider the version of the model with inflation indexation. The drop in output is nearly twice as large in the period of the shock. Output then oscillates back to steady state, with output rising above steady state after period 10. The drop in inflation is also more severe. Finally, the zero bound constraint is binding longer in the version of the model with indexation.

According to the evidence in Figure 2, the results qualitatively robust to the presence of indexation. They is some quantitative sensitivity, with output falling relatively more at the beginning and less at the end.

4 Introducing Variable Investment into the Eggertsson-Woodford Model

I first embed the Eggertsson and Woodford (2003a,b) model into the more general model with capital in Woodford (2003, section 5). To simplify the analysis, I consider a deterministic environment. I then report the quantitative results based on the same experiment discussed in section 3.2.

4.1 The Model

The preferences of households are as follows:

$$\sum_{t=0}^{\infty} \beta_t \left[u(C_t, M_t/P_t) - \int_0^1 v(H_t(j))dj \right],$$

where u is increasing and concave in its first argument, v is increasing and convex, and

$$\beta_t = \frac{1}{\left(1+r_0^n\right)\left(1+r_1^n\right)\cdots\left(1+r_{t-1}^n\right)},$$

for $t = 1, 2, ...$. Also, $\beta_0 \equiv 1$ and

$$\frac{\beta_{t+1}}{\beta_t} = \frac{1}{1+r_t^n}.$$

Each household supplies every type of labor, $j \in (0,1)$. Here, C_t denotes consumption and M_t denotes the household's end-of-period t stock of money. Finally, P_t denotes the price of

[8] I solved the models using a shooting algorithm. I guessed the date, t^*, when the zero bound on the nominal rate of interest ceases to bind. I set the period 0 inflation rate to its steady state value, and I put the interest rate at its lower bound for periods $t = 1, ..., t^*$. I then guessed a value for the output gap and inflation rate in period 1. It is now possible to use (1) and (2) to simulate the output gap and inflation. I adjusted the period 1 output gap and rate of inflation to guarantee that the system arrives in a steady state. I then assigned t^* the smallest value that produced a sequence of interest rates consistent with the lower bound constraint on the nominal rate of interest..

the consumption good. The household's flow budget constraint is:

$$P_tC_t + M_t + B_{t+1} \leq M_{t-1} + B_t(1 + i_{t+1}) + \int_0^1 P_t w_t(j) H_t(j) dj + T_t, \tag{3}$$

where B_t denotes the beginning-of-period t stock of bonds, purchased in period $t-1$. Also, $w_t(j)$ denotes the real wage rate paid to type j labor, and T_t denotes lump sum profits and transfers from the government. We suppose there is a lower bound constraint on B_t. Households are competitive in goods and labor markets.

Given this setup, a rise in the real rate of interest, $(1+i_t)/\pi_{t+1}$, induces a household to "tilt" its consumption profile towards the future. Doing so without violating its intertemporal budget constraint requires reducing consumption - hence, increasing saving - in the current period. In this way, the positively sloped saving function in Figure 1 captures a basic economic force in the model. Similarly, a drop in r_t^n induces households to save more for every value of the real interest rate, which corresponds to the right-shift displayed in Figure 1.

Final goods are produced using intermediate goods by a representative, competitive firm using the following Dixit-Stiglitz production function:

$$Y_t = \left[\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}}, \ \theta > 1.$$

The first order condition for profit maximization by the final good firm is:

$$\left(\frac{Y_t}{y_t(j)}\right)^{\frac{1}{\theta}} = \frac{P_t(j)}{P_t}. \tag{4}$$

The i^{th} intermediate good is produced by a monopolist using the following technology:

$$y_t(i) = K_t(i) f\left(\frac{h_t(i)}{K_t(i)}\right). \tag{5}$$

Here, $K_t(i)$ is the capital owned by the monopolist, and $h_t(i)$ is the quantity of labor hired. The firm is competitive in the market for type i labor, and takes the wage rate, $w_t(i)$, as given.[9] Investment by the i^{th} monopolist produces new capital in the next period according to the following adjustment cost function:

$$I_t(i) = I\left(\frac{k_{t+1}(i)}{k_t(i)}\right) k_t(i). \tag{6}$$

[9] At first glance, this may seem odd, because according to the formalism in the text, the j^{th} intermediate good producer is the only employer of type j labor. This suggests that the producer must be a monopsonist. We follow an alternative interpretation suggested by Woodford (2003), which rationalizes competitive labor markets. Think of the j^{th} intermediate good producer as being a member of an industry composed of intermediate goods producers with indeces lying in a small neighborhood, J, of j. Suppose there is a finite, but large, number of such industries that do not intersect, but whose union is the unit interval. Imagine that instead of there being a continuum of labor types in the household utility function, there is a discrete number as in the Riemann approximation to the integral of the household utility function. Each of these labor types works in one of the industries. With this setup, there is a continuum of suppliers and demanders in the labor market corresponding to each industry, so that competition makes sense. This is the case even for intervals, J, whose length is very small. This is how we interpret the model. For further discussion, see Woodford (2003, pp. 148-149).

Here, investment, $I_t(i)$, corresponds to purchases of the final good. Also, $I(1) = \delta$, $I'(1) = 1$, $I''(1) = \epsilon_\psi > 0$.

The i^{th} intermediate good firm faces frictions in the setting of its price, $P_t(i)$. With probability $1 - \alpha$ it may set its price optimally, and with probability α is must set $P_t(i) = P_{t-1}(i)$.

The present discounted value of profits of the intermediate good firm are:

$$E_t \sum_{j=0}^{\infty} \beta_{t+j} \Lambda_{t+j} \left\{ (1+\tau) P_{t+j}(i) y_{t+j}(i) - P_{t+j} w_{t+j}(i) h_{t+j}(i) - P_{t+j} I_{t+j}(i) \right\}. \tag{7}$$

Here, Λ_t denotes the shadow value of a dollar to the household, the owner of the intermediate good firm. It is the multiplier on (3) in the Lagrangian representation of the household's problem. Also, the subsidy, τ, is designed to eliminate the distoring effects of monopoly power in the model. I assume

$$1 + \tau = \frac{\theta}{\theta - 1}.$$

The i^{th} intermediate good firm chooses $P_{t+j}(i)$, $y_{t+j}(i)$, $h_{t+j}(i)$, $I_{t+j}(i)$ to maximize profits, subject to (4), (5), (6). It takes P_{t+j}, τ and $w_{t+j}(i)$ as given.

The resource constraint is:

$$C_t + I_t + G_t = Y_t,$$

where

$$I_t = \int_0^1 I_t(i) di.$$

Finally, the monetary authority controls i_t, which it sets with an objective of achieving $\pi_t = 1$. In periods when doing so would (infeasibly) imply a negative interest rate, it sets $i_t = 0$ and lets π_t be free to be determined by the equilibrium conditions. I assume that the fiscal authorities set lump sum taxes to guarantee that the government's budget constraint is balanced in each period.

When the curvature on investment adjustment costs, ϵ_ψ, is large, then the stock of capital is a constant. This is my representation of the model in Woodford and Eggertsson (2003a,b).[10] The elastic investment case corresponds to smaller values of ϵ_ψ.

I now summarize the parameters of the model. Since I study a log-linear approximation to the model economy, I need only specify its properties in a neighborhood of steady state. In particular, I parameterize the curvature, σ_u, σ_v, and σ_f of the utility of consumption, disutility of labor and production function, respectively, as follows:

$$\sigma_u = -\frac{u'' C}{u'} > 0, \ \sigma_v = \frac{v'' h}{v'} > 0, \ \sigma_f = -\frac{f'' \bar{h}}{f'} > 0.$$

Here, one and two primes, '′', '″' denote the first and second derivatives, respectively, of the associated function. In each case, the second derivative is multiplied by the argument

[10] Even with $\epsilon_\psi = \infty$, the model differs slightly from the one in Eggertsson and Woodford (2003) because they set investment to zero while it is a positive constant here. The difference is relatively unimportant, because my results are very similar to theirs.

of the function, evaluated in steady state. Thus, C, h, and $\bar{h}$ denote consumption, hours worked and the hours to capital ratio, all evaluated in steady state. In addition, there is the parameter, ϵ_ψ, which controls the curvature of the investment adjustment cost function, which was mentioned above. In the case of the production function, I follow Woodford (2003) in parameterizing the (inverse of the) elasticity f with respect to its argument:

$$\phi = \frac{f}{f'\bar{h}}.$$

Finally, I require g, the steady state value of the ratio, G_t/Y_t. The complete set of nine parameters which need to be assigned values in order to solve a log-linear approximation of the model, is:

$$\sigma_u,\ \sigma_v,\ \sigma_f,\ \epsilon_\psi,\ \phi,\ \delta,\ \alpha,\ \theta,\ \beta,\ g. \tag{8}$$

I follow Woodford (2003) in setting these parameters. In particular, $\sigma_u = 1.0$, $\sigma_v = 0.11$, $\sigma_f = 0.25$, $\epsilon_\psi = 3$, $\phi = 1/0.75$, $\delta = 0.012$, $\alpha = 0.66$, $\theta = 1 + 1/0.15$, and $\beta = 0.99$. In addition, I set $g = 0.18$, roughly the ratio of actual government purchases of goods and services to gross output. If utility were a constant elasticity function of consumption, then $\sigma_u = 1$ corresponds to the log. Similarly, $\sigma_v = 0.11$ corresponds to a power of 1.11 on labor if v where a constant elasticity function. If the production function were Cobb-Douglas then σ_f and ϕ both indicate a power on labor equal to 0.75. The value of α implies that on average a price is fixed for 3 quarters. The values of δ and β are standard in the real business cycle literature. The value of θ corresponds to a steady-state markup over marginal cost of 15 percent, also not far from standard estimates. Except for σ_v, these parameter values are all fairly standard. The value of σ_v implies a Frisch labor supply elasticity of $9 = 1/0.11$, which is higher than is often used in the literature. For this reason, in my analysis I also consider the unit Frisch elasticity case, $\sigma_v = 1$.

4.2 Quantitative Results

As in the deterministic experiment in section 3, I suppose that the economy has been in a deterministic steady state up until period 0. In period 1, r_t^n unexpectedly drops from its steady state annualized percent value of 4 percent. I consider three cases. In the "small shock", "medium shock", "large shock", and "larger shock" cases, r_t^n drops to -2, -3, -4, and -5 percent, respectively. In each case, r_t^n remains low until period 14. In period 15 it switches back to its steady state value. The small shock corresponds to the shock size used in the Eggertsson and Woodford example discussed in section 3.1.

Consider Figure 3 first. This corresponds closely to the Eggertsson and Woodford example, because it is the model without investment, i.e., with $\epsilon_\psi = \infty$.[11] Notice that for each shock size, the lower bound constraint on the nominal interest rate is binding, and output drops substantially. In the Eggertsson and Woodford small shock case, output drops nearly 20 percent in the period of the shock. After that the magnitude of the drop shrinks as the horizon over which the discount rate will remain low becomes shorter.[12] With larger shock

[11] Actually, I set ϵ_ψ =30000000000.

[12] A difference between the deterministic example here and the one in Eggertsson and Woodford is that

sizes, the magnitude of the drop in output increases. For example, in the larger shock case, output drops over 40 percent in the period of the shock.

The results in Figure 3 match the intuition in section 2 of the paper. The increased desire to save in the wake of the discount rate shock exerts a negative effect on inflation. To keep inflation on target would require a negative nominal interest rate. Since the zero lower bound now binds, the nominal rate of interest is set to zero. But, with the nominal interest rate at its lower bound and future policy understood to be oriented to keeping inflation at zero if possible, this means that the real interest rate is now also at its lower bound.[13] In this way, the real interest rate - the most efficient mechanism for ensuring that households do not act on their increased desire to save - is 'short-circuited'. Something else must occur to prevent households from increasing their saving in equilibrium. This is so, because there simply is no place for additional saving to go. What happens to discourage households from increasing their saving is a fall in income. This is brought about as the desire to increase saving leads households to reduce consumption. This cutback in spending leads to a drop in the price level. Because the future price level is driven down by more than the current price level, this implies drop in anticipated inflation. Perversely, the resulting increase in the real interest rate further amplifies households' incentive to save, so that an even larger drop in output is required to ensure that saving in fact does not rise in equilibrium. As we can see in the figure, the reductions in output that occur are very large.

Now consider Figure 4. The results here correspond to the same experiment as in Figure 3, with the sole exception that ϵ_ψ takes on the value suggested in Woodford (2003), namely, $\epsilon_\psi = 3$. Note that now the zero lower bound on the interest rate is less likely to bind. For the small and medium-sized shocks, it does not bind at all. As explained in section 2, when investment is elastic, the real interest rate does not have to fall by much to maintain equilibrium. The rise in investment associated with a lower real rate implies that saving does not have to be driven back to its pre-shock level. As one would expect, in the small and medium shock cases consumption drops in the period of the shock, and then increases later on. This intertemporal reallocation of consumption is accomplished by a rise in investment in the initial periods, and a fall in later periods. The initial rise in investment is much larger than the drop in consumption, so that output rises strongly in response to the shock. With the small shock, the rise in output is nearly 20 percent.

The zero lower bound does bind for the large and larger shocks. In the case of the large shock, it begins to bind in period 10, and then ceases to bind in period 15. Interestingly, the binding zero bound does not produce the dramatic reduction in output and inflation that we saw in the inelastic investment case in Figure 3. The dynamic response of the economy to the large shock is similar to the response in the case of the small and medium shocks. In the case of the larger shock, the lower bound constraint binds in each of periods 1 to 14. There is a substantial drop in inflation in the first few periods after the shock. However, there is no drop in output. By comparison with what happens after the small and medium shocks,

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the equilibrium values of variables in the latter case are independent of how long the discount rate has been low. This is because the stochastic process on $\hat{r}_t^n$ has the property that the horizon over which $\hat{r}_t^n$ is expected to remain low is independent of how long $\hat{r}_t^n$ has been down.

[13] In the numerical experiments, it has always been the case that the inflation rate was negative whenever the zero bound on the interest rate was binding. Thus, under the monetary policy studied in thie paper, the inflation rate was always zero or negative.

output is low for the first three periods. After this, all responses look similar.

Overall, the simulations in Figure 4 are not consistent with the dire implications for the zero bound which emerge from Figure 3. Of course, if the shock is big enough then output goes into a free-fall in the elastic investment case, just as it does in the inelastic case depicted in Figure 3. For example, when r_t^n drops to 6 percent, I found that output drops 40 percent in the period of the shock when investment is elastic, while it drops over 50 percent in the inelastic case.

Figures 5 and 6 report the same experiments as in Figures 3 and 4, respectively, with the exception that now the Frisch labor supply elasticity is unity, with $\sigma_v = 1$. Note that in the inelastic investment case (Figure 5), the drop in consumption is somewhat smaller than before (see Figure 3). However, qualitatively the results are not very different, since the magnitude of the drop still qualifies it to be called a 'worst case scenario'. In contrast, the results for the elastic investment case are substantially different. Now the lower bound is never binding. In results not reported here, I found that the lower bound is not even binding when when r_t^n drops to 6 percent.

5 The Effects of Government Spending

I now investigate the effects of increasing government spending when there is a small and a larger discount rate shock. Throughout, I assume that monetary policy targets a zero inflation rate. My choice of shock sizes allows me to compare the magnitude of the government spending multiplier under 'normal' circumstances and when the interest rate lower bound binds. I find that the government spending multiplier is larger when the non-negativity constraint on the interest rate is binding, than when it is not. I first show this in the version of the model with inflexible investment, when the result is easily established analytically. I then simulate the model with flexible investment, to show that the government spending multiplier is very much bigger when the lower bound binds, than when it does not.

5.1 Model With Inelastic Investment

In Appendix 1, I show that after linearization about the zero inflation steady state, the household's intertemporal Euler equation is:

$$\hat{Y}_t = \hat{Y}_{t+1} + \hat{G}_t - \hat{G}_{t+1} + \sigma \left(\hat{\pi}_{t+1} + \hat{r}_t^n - \hat{\imath}_t \right). \tag{9}$$

Here, $\hat{Y}_t = (Y_t - Y)/Y$, where Y denotes output in the zero inflation steady state. Also, $\hat{G}_t = (G_t - G)/Y$, where G denotes government spending in steady state. Similarly, the appendix shows that inflation equation is:

$$\hat{\pi}_t = \beta \hat{\pi}_{t+1} + \frac{A}{\sigma} \left\{ (\sigma\omega + 1) \hat{Y}_t - \hat{G}_t \right\}, \tag{10}$$

where

$$\omega = \sigma_v \phi + \omega_p, \;\; \omega_p = \sigma_f \phi, \;\; A = \frac{1-\alpha}{\alpha} \frac{1-\alpha\beta}{1+\omega\theta},$$

and σ_v, σ_f, ϕ were defined in the previous subsection. Equations (9) and (10) are standard and can be found, for example, in Woodford (2003). Consider first the case in which it is feasible for monetary policy to keep the inflation rate at zero in each period, without violating the lower bound in the interest rate. In this case it is immediate from (10) that the output multiplier associated with government spending, dY_t/dG_t, is:

$$\frac{dY_t}{dG_t} = \frac{1}{\sigma\omega + 1} < 1. \tag{11}$$

Note from this expression that the government spending multiplier is necessarily less than unity. That is, a one unit increase in government spending leads to less than one unit increase in output. Interestingly, the impact of a government spending shock on current output is independent of the degree of persistence in the shock.

I now consider the case when the lower bound on the interest rate is binding. By recursive substitution, and using (10) to substitute out for $\hat{\pi}_t$ in (9), I obtain:

$$\hat{Y}_t = \sum_{j=1}^{\infty} h_j^G \hat{G}_{t+j-1} + \sum_{j=1}^{\infty} h_j \left(\hat{r}_{t+j-1}^n - \hat{\imath}_{t+j-1}\right). \tag{12}$$

The objects of interest to us here are the h_j^G's. In Appendix 2, I show:

$$\begin{aligned} h_1^G &= 1 \\ h_2^G &= A\sigma\omega \\ h_j^G &= \frac{A\sigma\omega}{\lambda_2 - \lambda_1}\left(\lambda_2^{j-1} - \lambda_1^{j-1}\right),\ j \geq 3. \end{aligned} \tag{13}$$

where

$$\lambda_2 > 1,\ \beta > \lambda_1 > 0,\ \lambda_1\lambda_2 = \beta,\ \lambda_1 + \lambda_2 = 1 + \beta + A(\sigma\omega + 1). \tag{14}$$

As in the previous section, I assume $\hat{r}_t^n$ drops in periods $t = 1, ..., T-1$, and the drop is sufficiently large to cause the interest rate lower bound to bind under the zero-inflation targetting monetary policy. I assume $\hat{r}_t^n$ returns to its steady state value in period T, and that all other variables return to their steady states then too (this is a feature of the equilibria with inflexible investment studied in the previous section.) It is clear from (12) that the output effect of a jump in government spending depends on how persistent that jump it. At one extreme, consider the effect of a jump in government spending in period 1 which is completely temporary, so that $\hat{G}_1 = \hat{G}_h > 0$ and $\hat{G}_t = 0$ for $t \geq 2$. According to (13) the multiplier effect of this on period 1 output is unity. Evidently, this exceeds the value of the multiplier when the lower bound is not binding (see (11)). Suppose, at the other extreme, that the jump in government spending lasts as long the interest rate lower bound is binding, so that $\hat{G}_t = \hat{G}_h > 0$ for $t = 1, 2, ..., T-1$, and $\hat{G}_t = 0$ for $t \geq T$. In this case, the multiplier on period 1 output is obtained by summing over the relevant h_j^G's in (13):

$$\sum_{j=1}^{T-1} h_j^G = 1 + A\sigma\omega + \frac{A\sigma\omega}{\lambda_2 - \lambda_1}\left(\lambda_2^2 \frac{1 - \lambda_2^{T-3}}{1 - \lambda_2} - \lambda_1^2 \frac{1 - \lambda_1^{T-3}}{1 - \lambda_1}\right).$$

This definitely exceeds unity, because $h_j^G > 0$ for each $j > 1$. When the parameter values after (8) are used, this multiplier is:

$$\sum_{j=1}^{T} h_j^G = 3.2,$$

well in excess of unity. At the same time, the multiplier in (11) is 0.75. So, in this example, the government spending multiplier jumps by a factor of 4 when the zero bound is encountered.

5.2 Model With Elastic Investment

I now compute the government spending multiplier a version of the model with elastic investment and inelastic labor supply. This corresponds to the model underlying the analysis in Figure 4. I do two experiments, which are differentiated according to the size of the discount rate shock. In each case, I consider the output effect of an increase in government spending which remains in place as long as r_t^n is low. To understand the magnitude of the government spending multiplier when the lower bound on the interest rate is not binding, I consider the small shock to r_t^n. To understand the magnitude of the multiplier when the bound does bind, I also consider an experiment with the larger shock. In both experiments, government purchases, G_t, are increased to 19.25 percent of steady state output in periods 1 to 14. In period 15 they are returned to their steady state value of 18 percent of steady state output.

Consider the small shock case first, which is reported in Figure 7. As a convenient benchmark, the solid line displays the response of the economy to the shock when there is no G_t response. This reproduces the solid line in Figure 4. The line indicated by stars shows the response of the economy when there is a G_t response. Note that the two lines virtually coincide, indicating a very small multiplier. I also report the difference in equilibrium output across the two experiments, divided by the difference in government purchases, for periods 1 to 14. This multiplier, dY_t/dG_t, is very small. It is 0.76 in the period of the shock and then converges monotonically to 0.41 by period 14.

Now consider the large shock case in Figure 8. Again, for convenience the solid line reproduces the lines indicated by '+' in Figure 4, when there is no G_t response. The line indicated by stars shows the response of the economy when there is a G_t response. Notice that now the response of the economy to the increase in government purchases is very large. In the period of the shock, output rises by nearly 25 percent of its steady state value, in contrast to no rise at all when there is no increase in government puchases. Similarly, there is an increase (relative to the benchmark) in consumption and investment. Inflation shows only a very small decline. The increase in government purchases in effect makes the zero bound constraint less binding. A larger increase than the one reported here was found to cause the zero bound constraint to cease binding altogether.

Figure 8 also reports the multiplier, dY_t/dG_t. Note that the multiplier in the first period is nearly 20. Evidently, it is much larger than the multiplier in the model with inelastic investment.

6 An Informal Look at US Data

If a country adopts a monetary policy of targetting low or zero inflation, how likely is the 'worst case scenario' in which output and inflation collapse? That is, (i) how often will the zero bound prevent the real interest rate from being negative, when efficient resource allocations require it?, and when this happens, (ii) how severe are the consequences? Using the model analysis as a informal guide, I attempt to answer these questions using the past 150 years' observations on interest rates and inflation in the U.S.

A zero-inflation targetting policy can cause economic disruption by preventing the real interest rate from becoming negative at times when the efficient allocation of resources requires it. So, one way to answer (i) is to compute the frequency of times that the real interest rate has been negative in the past. I do this in the first subsection below. A potential pitfall of this approach occurs if there are periods in the past when the monetary authorities pursued a zero-inflation targeting policy. In such periods, the real interest rate would never be negative, even if efficient resource allocations required it. If there are such periods, then the frequency of observed negative real rates may be a severe underestimate of the likelihood that a zero-inflation targetting policy would lead to trouble. The second subsection below addresses this concern, and addresses question (ii) as well. The final subsection summarizes.

6.1 Real Interest Rates in the Past 150 Years

I used two data sets. One covers a 'long sample', 1867-1996 and the other covers a 'short sample', 1953-2002. For the long sample, data from 1867 to 1971 on the interest rate corresponds to 60-90 day commercial paper rates in New York city and data from 1972 to 1996 are observations on the 3-month nonfinancial commerical rate.[14] In addition, annual data on the consumer price index were obtained from Historical Abstracts of the United States. The inflation rate for a particular year was computed at the percent change in the price level from that year to the next (using the log transform).

Consider the long sample first. The inflation and interest data are displayed in Figure 9a. Figure 9b reports the (ex post) real interest rate. The frequencies of negative real rates are reported in Table 1 for various time periods. The real interest rate is negative 19 percent of the years in the long sample. Because of price controls and other distortions, it is useful to exclude the years of the two world wars, and the year immediately after. When I do this, I find that the frequency of negative real interest rates falls to 14 percent. When I consider only the period after World War II, 1946-1996, the fraction is up to 20 percent (see Table 1).

It is interesting that the probability of negative real rates is greater in the high inflation period at the end of the sample. Note, for example, that the real interest rate is never negative in the period before World War I. To see this more formally, consider the three solid horizontal lines in Figure b. They indicate the average inflation rate over three intervals: 1867-1916, 1921-1928 and 1947-1996. The numbers in parentheses indicate the mean value of the real rate over the period indicated by the solid, horizontal line. Note that in the first

[14]Data for 1857-1971 were taken from the historical database on the NBER website. The NBER series number is 13002. Data for the other years are the 3 month nonfinancial commercial paper rate taken from the economagic website, www.economagic.com. Both series are monthly and were annualized by averaging.

two samples, when inflation is near zero, the average real rate of interest is quite high. In the late sample, the real interest rate is low.[15] This suggests the possibility that adoption of a zero inflation target may lead to a higher average real rate and, hence, a lower likelihood that the real rate turns negative. Thus, the observed historical frequency of negative real interest rates may overstate the likelihood of a binding lower bound in case the monetary authority were to adopt a zero-inflation monetary target. A factor working against this conclusion is that inflation in recent years has been relatively low, and yet the real interest rate has been quite low as well.

For the short sample, I considered nominal returns on 1-Year Treasury Bills. These returns were converted to ex post real returns adjusting by a measure of inflation based on the CPI (see Figure c), and by a measure of inflation based on the personal consumption expenditure deflator (PCE). The frequencies with which these real returns are negative is 18 and 14 percent of the time, respectively. Under the theory in this paper, the real interest rate corresponds better to the ex ante real interest rate rather than the ex post real return considered up to now. To gain a sense of what happens when something closer to an ex ante real interest rate is computed, I adjusted the 1-year Treasury bill return by the measure of one-year ahead (CPI) inflation expectations obtained from the Michigan survey of families. The frequency with which this is negative, 18 percent, corresponds roughly to what I obtained with ex post real returns.

The preceding analysis does not adjust for taxes. Obviously, the frequency of negative interest rates increases when taxes are taken into account. For example, when it is assumed that the tax rate on interest income is 30 percent, then the frequency of negative real rates jumps substantially. These fractions are provided in parentheses in Table 1.

Table 1: Percent of Times That Indicated (After Tax) Real Rate is Negative

Interest Rate	Inflation	Long Sample	Long Sample, Excluding Wars	Post WWII
3-Month Commercial Paper	CPI	19 (32)	14 (27)	20 (41)
1-Year Treasury Bill	CPI			18 (40)
1-Year Treasury Bill	PCE			14 (29)
1-Year Treasury Bill	Michigan Expectation			18 (48)

6.2 Interest Rates and Inflation in the 1930s

Outside of the 1930s, it seems unlikely that the Federal Reserve ever implemented the sort of zero-inflation targeting policy studied in this paper. Before the 1930s the U.S. was on the gold standard, and this is better characterized as a policy of targetting the price level rather than inflation targetting. As shown in Eggertsson and Woodford (2003a,b) and Wolman (1998, 2003), under this type of monetary policy the zero bound on the interest rate is unlikely to pose a significant impediment to achieving a negative real interest rate. The period since the 1930s also seems unlikely to be characterized by a policy of zero inflation

[15] For another discussion of the fact that at low frequencies, the real interest rate appears to be negatively correlated with the inflation rate, see Kandel, Ofer, and Sarig (1996).

targetting. This is a time when monetary policy, as other government policies, were focused not just on inflation but also on output stabilization.

Now consider the period of the 1930s, a time when the U.S. transitioned away from the gold standard. At the beginning of the decade, there was a dramatic collapse in the price level. Could this have reflected the sort of 'worst case scenario' that some are worried about? The evidence in Figure 9a indicates that the answer is 'no'. The figure shows inflation, $100\times \log(P_{t+1}/P_t)$, for each year, $t = 1928, 1929, ..., 1940$, where P_t is the consumer price index. The figure also displays two interest rates, a short-term U.S. government securities rate and a 3-6 month commercial paper rate. In each case, the interest rate is averaged over the year and is expressed in annual percent terms. Notice the collapse in inflation that begins in 1929. But, the collapse occurs at a time when both measures of the nominal rate of interest are well above zero. Evidently, the fall in inflation during the contraction phase of the Great Depression could not have been triggered by a shock that caused the lower bound on the nominal interest rate to bind.

Now consider the expansion phase of the Great Depression, which began in 1933. A recent review of speeches and other documents of Federal Reserve officials in this period by Orphanides (2003) suggests that there was considerable fear of inflation at the Fed. This is consistent with the notion that it was policy at the Fed to target zero or low inflation. At the same time, the economic expansion was anemic in the sense that per capita hours worked remained more than 20 percent below its level in 1929 (see Figure 10). Since the interest rate on short-term government securities was nearly zero after 1933, this raises the possibility that the low level of employment was a consequence of a binding lower bound on the interest rate. Still, this episode does not bear all the trappings of the sort of liquidity trap analyzed in this paper, since inflation was on average positive. Moreover, other explanations have been advanced for the slow recovery of output and employment in the 1930s. For example, Cole and Ohanian (2003) argue that the New Deal policies that improved the bargaining power of workers can go a long way towards explaining the fact that total hours worked in 1939 was still around 20 percent below its 1929 level.

6.3 Summary

A quick examination of the data suggests that the real rate of interest is negative a substantial fraction of the time, say 20 percent. However, there is considerable uncertainty about this estimate. It is an underestimate, to the extent that taxes on interest rates matter. It overestimates the likelihood of falling into a liquidity trap if it is the case that real rates systematically rise when inflation is brought down. The data do suggest this, but there may be other factors, apart from monetary policy that account for it.

7 Conclusion

[still under construction]

I incorporated the Eggertsson and Woodford (2003a,b) example into the more general model with capital studied in Woodford (2003). Using the parameters in Woodford (2003), I found that the likelihood that the zero bound on the interest rate binds is reduced. Still, it can happen. However, this parameterization incorporates a Frisch elasticity of labor supply equal to nearly 100, which is considerably higher than the values used in the literature. When I assign a value of unity to this elasticity, then the results in the Eggertsson and Woodford model with no capital are qualitatively unaffected. However, the model with investment is strikingly sensitive to this change. The lower bound does not bind, even for extremely large values of the shock. I also find that conditional on the zero lower bound binding, the consequences are less severe in the model with flexible investment. Morever, I found that the multiplier effects on output of government spending are very large - in a neighborhood of 20 - when the lower bound on the interest rate binds. This suggests that in the event the zero bound binds severely, the output effects can be minimized by a modest increase in government spending.

I studied the U.S. data .....

The Eggertsson and Woodford example suggests that the stakes are very high. Even if we only assign a low probability to the output collapse scenario in their example, the welfare losses are so great that it behooves us to pay attention. The analysis in this paper suggests that the output-collapse scenario is less likely once we take into account investment. Still, considerably more work - with other frictions and other shocks - needs to be done before we can safely ignore the output collapse scenario.[16] One shock that would be particularly interesting to study is a negative disturbance to investment. This too would require a fall in the real interest rate to encourage households to cut back on saving. If the zero bound makes a big enough drop infeasible, the result could be output collapse and price deflation. Pursuing this further requires constructing stochastic, dynamic general equilibrium models based on solid empirical foundations. In addition, it requires developing model solution methods which can accommodate occasionally binding constraints. Devising appropriate solution methods is a particularly big challenge. Some progress has been made (see, e.g., Christiano and Fisher (2000)). However, these methods are very burdensome computationally.

A Appendix 1: Computational Algorithm

Here, I describe the shooting algorithm used to solve the model. The first part of this appendix explains the equations used to characterize the equilibrium. Following Eggertsson and Woodford, I log-linearize the equations which must always hold with equality in equilibrium. This discussion follows Woodford (2003, Chapter 5), and is included here for completeness. The second part of this appendix describes the actual shooting algorithm. The algorithm is constructed to ensure that the lower bound constraint on the interest rate is always satisfied.

A.1 Log-Linearized Equations Characterizing Equilibrium

I begin with a statement of the first-order necessary conditions for household and firm optimality. I then compute the features of the steady state that are required. After this, I derive three log-linear equations that I use to characterize equilibrium.

First-order Conditions

[16] Initial work on this may be found in Adam and Billi (2003).

The first order necessary conditions for household optimization include a transversality condition and:

$$\begin{aligned} \frac{v'(H_t(j))}{u'_t} &= w_t(j) \\ u'_t &= \frac{1}{1+r_t^n} u'_{t+1} \frac{1+i_t}{\pi_{t+1}} \\ \frac{u_{m,t}}{u'_t} &= \frac{i_t}{1+i_t} \end{aligned}$$

Here, u'_t denotes the marginal utility of consumption. The first equation is the intratemporal Euler equation for labor. The second equation is the intertemporal Euler equation associated with the household saving decision. By concavity of the utility function, current consumption falls - hence saving rises as in Figure 1 - when the real interest rate rises, for given values of r_t^n and u'_{t+1}. The third is the Euler equation associated with real balances. From here on, we assume u'_t is not a function of M_t/P_t, and we ignore the last first order condition.

The first order necessary condition associated with the optimal choice of capital by the i^{th} firm is:

$$I'\left(\frac{k_{t+1}(i)}{k_t(i)}\right) = \frac{1}{(1+i_t)/\pi_{t+1}} \left\{ \rho_{t+1}(i) - I\left(\frac{k_{t+2}(i)}{k_{t+1}(i)}\right) + I'\left(\frac{k_{t+2}(i)}{k_{t+1}(i)}\right)\frac{k_{t+2}(i)}{k_{t+1}(i)} \right\}. \tag{15}$$

Here, we have made use of the household's intertemporal Euler equation and the fact, $\Lambda_t = u_{c,t}/P_t$. With some algebra it can be shown that $\rho_{t+1}(i)$ in (15) can be written:

$$\rho_{t+1}(i) = w_{t+1}(i) \frac{f(\bar{h}_{t+1}(i)) - \bar{h}_{t+1}(i) f'(\bar{h}_{t+1}(i))}{f'(\bar{h}_{t+1}(i))}. \tag{16}$$

One interpretation of $\rho_{t+1}(i)$ is that it is the real rental rate of capital that would rationalize the amount of capital used by the intermediate good firm in $t+1$, if there were a competitive capital rental market.[17]

Note that a rise in the real rate of interest, holding the objects in braces on the right of (15) fixed, leads to a fall in I'. Given the assumption, $I''(1) > 0$, it follows that $k_{t+1}(i)$ falls, i.e., investment falls, as in Figure 1.

If the i^{th} firm has the opportunity to reoptimize its price, then it does so, taking into account that it must satisfy its demand curve and that it will not be able to reoptimize again, with probability α. The firm's first order condition is, after some algebra,

$$\sum_{j=0}^{\infty} \alpha^j \beta_{t+j} \lambda_{t+j} P_{t+j}^{\theta} Y_{t+j} \left\{ (1+\tau) p_{t+j}(i) - \frac{\theta}{(\theta-1)} s_{t+j}(i) \right\} = 0.$$

Multiplying by $(1+\tau)^{-1}$,

$$\sum_{j=0}^{\infty} \alpha^j \beta_{t+j} \lambda_{t+j} P_{t+j}^{\theta} Y_{t+j} \left\{ p_{t+j}(i) - \frac{\theta}{(\theta-1)(1+\tau)} s_{t+j}(i) \right\} = 0, \tag{17}$$

[17] See Woodford (2003, p. 355) for an alternative interpretation.

where

$$p_{t+j}(i) = \frac{P_t(i)}{P_{t+j}},$$

where $P_t(i)$ is the price set in period t. Also, $s_{t+j}(i)$ is the marginal cost of producing an extra unit of output:

$$s_t(i) = \frac{w_t(i)}{f'\left(\frac{h_t(i)}{K_t(i)}\right)}. \tag{18}$$

According to (17), the firm sets its price to an after-tax markup over marginal cost on average.

Steady State

To compute the log-linear expansion about steady state, we require the capital-to-output ratio, k, and consumption-to-output ratio, c. We compute these first. In steady state, (15) reduces to:

$$\frac{1}{\beta} = \rho - \delta + 1,$$

where ρ is the steady state value of the first object in braces in (15). Then, $\rho = (1/\beta) + \delta - 1$, which, when combined with (16) evaluated in steady state, yields:

$$\frac{1}{\beta} + \delta - 1 = w\frac{f - \bar{h}f'}{f'}.$$

In steady state, all prices are equal, so that $p_t(i) = 1$. This, and the fact that the object in braces in (17) is zero in steady state imply

$$s = \frac{(\theta - 1)(1 + \tau)}{\theta}.$$

Combining this with (18), we obtain:

$$w = f'\frac{(\theta - 1)(1 + \tau)}{\theta},$$

which says that the real wage is an after tax markup over the marginal product of labor. Combining these results and rearranging, we obtain:

$$k = \frac{(\theta - 1)(1 + \tau)}{\theta}\left(\frac{\phi - 1}{\phi}\right)\left(\frac{1}{\frac{1}{\beta} + \delta - 1}\right).$$

where $k \equiv K/Y$. Finally, note that in steady state:

$$Y = C + \delta K + G,$$

so that

$$c \equiv \frac{C}{Y} = 1 - \delta k - g,$$

where

$$g = \frac{G}{Y}.$$

Log-Linear Expansion

Consider the national income identity:

$$Y_t = C_t + I_t + G_t,$$

where G_t denotes the exogenous level of government spending. Then,

$$\hat{Y}_t = \hat{C}_t + \hat{I}_t + \hat{G}_t.$$

Unless otherwise noted, a hat over a variable indicates deviation from steady state, as a fraction from steady state. That is, $\hat{z}_t \equiv dz_t/z$, where z is the steady state value of z_t and dz_t is a small deviation, $z_t - z$. However, in the case of the aggregate quantities appearing in the national income identity, a hat indicates deviation from steady state, expressed as a fraction of steady state aggregate output:

$$\hat{I}_t = \frac{dI_t}{Y},\ \hat{Y}_t = \frac{dY_t}{Y},\ \hat{C}_t = \frac{dC_t}{Y},\ \hat{G}_t = \frac{dG_t}{Y}.$$

Log-linearizing the household's intertemporal Euler equation:

$$\hat{u}_{c,t} = \hat{u}_{c,t+1} - \hat{r}_t^n + \hat{\imath}_t - \hat{\pi}_{t+1}, \tag{19}$$

where

$$\hat{\imath}_t \equiv \frac{i_t - i}{1+i},\ \hat{r}_t^n \equiv \frac{dr_t^n}{1+r^n}.$$

The non-negativity constraint on i_t implies, since $1+i = 1/\beta$ in steady state,

$$\hat{\imath}_t \geq \hat{\imath}_l = \beta - 1.$$

It is useful to develop an expression for $\hat{u}_{c,t}$. We have

$$\begin{aligned}
\hat{u}_{c,t} &\equiv \frac{du_{c,t}}{u_c} = \frac{u_{cc}Y}{u_c}\hat{C}_t \\
&= \frac{u_{cc}C}{u_c}\frac{Y}{C}\hat{C}_t \\
&= -\sigma^{-1}\hat{C}_t,
\end{aligned}$$

where

$$\begin{aligned}
\sigma &= \sigma_u^{-1} c > 0 \\
\sigma_u &= -\frac{u_{cc}C}{u_c}
\end{aligned}$$

Then,

$$\hat{u}_{ct} = -\sigma^{-1}\left[\hat{Y}_t - \hat{I}_t - \hat{G}_t\right]. \tag{20}$$

We now turn to the firms. Integrating (4) over $j \in (0,1)$ and imposing the final goods firm technology, the following relationship between the aggregate and intermediate good prices must hold:

$$P_t = \left[\int P_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}}. \tag{21}$$

Loglinearizing (15) about steady state and integrating the result over $i \in (0,1)$, we obtain:[18]

$$\begin{aligned} \hat{u}_{c,t} + \epsilon_\psi \left[\hat{K}_{t+1} - \hat{K}_t \right] &= (1-\delta)\beta \hat{u}_{c,t+1} - \hat{r}_t^n \\ &\quad + [1-(1-\delta)\beta] \left[\rho_y \hat{Y}_{t+1} - \rho_k \hat{K}_{t+1} \right] \\ &\quad + \beta \epsilon_\psi \left[\hat{K}_{t+2} - \hat{K}_{t+1} \right], \end{aligned} \tag{22}$$

where

$$\begin{aligned} \rho_y &= \sigma_v \phi + \frac{\omega_p \phi}{\phi - 1}, \ \omega_p = \sigma_f \phi \\ \rho_k &= \rho_y - \sigma_v. \end{aligned}$$

That (22) only involves aggregate output and the aggregate capital stock reflects that we have integrated the linearized first order conditions of the firms in a neighborhood of a symmetric steady state equilibrium. Note that (22) implies $\hat{K}_{t+1} - \hat{K}_t = (1/\beta)^t \left[\hat{K}_1 - \hat{K}_0 \right]$ when $\epsilon_\psi = \infty$. So, the only $\hat{K}_1 - \hat{K}_0$ which implies a non-explosive path for the capital stock is one in which $\hat{K}_1 - \hat{K}_0 = 0$, in which case the capital stock is constant. This is what one expects when adjustment costs are infinite.

Loglinearizing the investment equation, (6):

$$\hat{I}_t = k \left[\hat{K}_{t+1} - (1-\delta) \hat{K}_t \right], \tag{23}$$

where

$$k = \frac{K}{Y}.$$

Now consider the pricing decision of the firm. Log-linearizing (17) we obtain (after considerable algebra, see Woodford 2003, page 360):

$$\hat{\pi}_t = \xi_0 \hat{s}_t - \xi_1 \hat{s}_{t+1} + \psi_1 \hat{\pi}_{t+1} - \psi_2 \hat{\pi}_{t+2}, \tag{24}$$

where s_t is the real marginal cost of production for the firm. Here, s_t denotes real marginal cost and, after log-linearizing:

$$\hat{s}_t = \omega \left(\hat{Y}_t - \hat{K}_t \right) + \sigma_v \hat{K}_t - \hat{u}_{c,t}, \tag{25}$$

where

$$\omega = \sigma_v \phi + \omega_p.$$

[18] Expression (22) corresponds to Woodford (2003, equation 3.7, page 356).

In (24),

$$\xi_0 = \frac{1-\alpha}{\alpha}\frac{1}{a-b},\ \xi_1 = \mu_2^{-1}\xi_0,$$

and

$$\psi_1 = \frac{a\left(\beta+\mu_2^{-1}\right) - b(\alpha\beta+\alpha^{-1}\mu_2^{-1})}{a-b},\ \psi_2 = \beta\mu_2^{-1}.$$

Also,

$$\begin{aligned}
a &= \frac{1+\omega\theta}{1-\alpha\beta} + (\omega-\sigma_v)\frac{\alpha}{1-\alpha\beta}\frac{\Xi}{(1-\alpha\beta\mu_1)(1-\alpha\beta\mu_2)} \\
b &= (\omega-\sigma_v)\frac{\alpha}{1-\mu_2^{-1}}\frac{\Xi}{(1-\alpha\beta\mu_1)(1-\alpha\beta\mu_2)} \\
\Xi &= (1-\beta(1-\delta))\,\rho_y\theta\epsilon_\psi^{-1}
\end{aligned}$$

Finally, μ_2 is the root greater than β^{-1} in the equation

$$\beta\mu^2 - \left[1+\beta+(1-\beta(1-\delta))\,\rho_k\epsilon_\psi^{-1}\right]\mu + 1 = 0,$$

and μ_1 is the other root, which is less than unity.

I reduce the system of private economy equations to three equations, (19), (22), (24) in the four unknowns, $\hat{K}_{t+1}$, $\hat{Y}_t$, $\hat{\pi}_t$, $\hat{\imath}_t$. This is accomplished by substituting out for $\hat{u}_{c,t}$ using (20), $\hat{I}_t$ using (23), and $\hat{s}_t$ using (25). The inflation equation becomes:

$$\begin{aligned}
&\hat{\pi}_t - \xi_0\left\{\left(\omega+\sigma^{-1}\right)\hat{Y}_t - \left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right]\hat{K}_t - \sigma^{-1}k\hat{K}_{t+1} - \sigma^{-1}\hat{G}_t\right\} \\
&+\xi_1\left\{\left(\omega+\sigma^{-1}\right)\hat{Y}_{t+1} - \left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right]\hat{K}_{t+1} - \sigma^{-1}k\hat{K}_{t+2} - \sigma^{-1}\hat{G}_{t+1}\right\} \\
&\qquad\qquad -\psi_1\hat{\pi}_{t+1} + \psi_2\hat{\pi}_{t+2} = 0
\end{aligned}$$

Writing this in vector notation,

$$F_\pi z_t = 0, \tag{26}$$

where z_t is the 12×1 vector:

$$z_t = (\hat{\pi}_t, \hat{\pi}_{t+1}, \hat{\pi}_{t+2}, \hat{Y}_t, \hat{Y}_{t+1}, \hat{K}_t, \hat{K}_{t+1}, \hat{K}_{t+2}, \hat{G}_t, \hat{G}_{t+1}, \hat{r}_t^n, \hat{\imath}_t)'$$

and F_π is the 1×12 vector:

$$\begin{aligned}
F_{\pi 1} &= 1,\ F_{\pi 2} = -\psi_1,\ F_{\pi 3} = \psi_2 \\
F_{\pi 4} &= -\left(\omega+\sigma^{-1}\right)\xi_0,\ F_{\pi 5} = \left(\omega+\sigma^{-1}\right)\xi_1 \\
F_{\pi 6} &= \xi_0\left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right], \\
F_{\pi 7} &= \xi_0\sigma^{-1}k - \xi_1\left[(\omega-\sigma_v)-\sigma^{-1}k(1-\delta)\right] \\
F_{\pi 8} &= -\xi_1\sigma^{-1}k,\ F_{\pi 9} = \xi_0(1+\sigma_v),\ F_{\pi 10} = -\xi_1(1+\sigma_v) \\
F_{\pi 11} &= \xi_0\sigma^{-1},\ F_{\pi 12} = -\xi_1\sigma^{-1},\ F_{\pi 13} = F_{\pi 14} = 0.
\end{aligned}$$

The intertemporal Euler equation associated with investment becomes:

$$\begin{aligned}\epsilon_\psi^{-1}\hat{Y}_t - \epsilon_\psi^{-1}k\left[\hat{K}_{t+1} - (1-\delta)\hat{K}_t\right] - \epsilon_\psi^{-1}\hat{G}_t - \sigma\left[\hat{K}_{t+1} - \hat{K}_t\right] \\ -\epsilon_\psi^{-1}(1-\delta)\beta\left(\hat{Y}_{t+1} - k\left[\hat{K}_{t+2} - (1-\delta)\hat{K}_{t+1}\right] - \hat{G}_{t+1}\right) - \epsilon_\psi^{-1}\sigma\hat{r}_t^n \\ +\epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right]\left[\rho_y\hat{Y}_{t+1} - \rho_k\hat{K}_{t+1} - (1+\sigma_v)\hat{A}_{t+1}\right] \\ +\sigma\beta\left[\hat{K}_{t+2} - \hat{K}_{t+1}\right] = 0,\end{aligned}$$

or,

$$F_k z_t = 0, \tag{27}$$

where the 1×12 vector F_k is:

$$\begin{aligned}F_{k1} &= F_{k2} = F_{k3} = 0,\ F_{k4} = \epsilon_\psi^{-1}, \\ F_{k5} &= -\epsilon_\psi^{-1}(1-\delta)\beta + \epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right]\rho_y \\ F_{k6} &= \epsilon_\psi^{-1}k(1-\delta) + \sigma, \\ F_{k7} &= -\epsilon_\psi^{-1}k - \sigma - \epsilon_\psi^{-1}(1-\delta)\beta(1-\delta)k - \epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right]\rho_k - \sigma\beta \\ F_{k8} &= \epsilon_\psi^{-1}(1-\delta)\beta k + \sigma\beta,\ F_{k9} = 0,\ F_{k10} = -\epsilon_\psi^{-1}\sigma\left[1-(1-\delta)\beta\right](1+\sigma_v) \\ F_{k11} &= -\epsilon_\psi^{-1},\ F_{k12} = \epsilon_\psi^{-1}(1-\delta)\beta,\ F_{k13} = -\epsilon_\psi^{-1}\sigma,\ F_{k14} = 0.\end{aligned}$$

The household's intertemporal Euler equation is:

$$\begin{aligned}&\hat{Y}_t - k\left[\hat{K}_{t+1} - (1-\delta)\hat{K}_t\right] - \hat{G}_t \\ =\ &\hat{Y}_{t+1} - k\left[\hat{K}_{t+2} - (1-\delta)\hat{K}_{t+1}\right] - \hat{G}_{t+1} - \sigma\left(\hat{\imath}_t - \hat{\pi}_{t+1} - \hat{r}_t^n\right)\end{aligned} \tag{28}$$

or,

$$F_i z_t = 0, \tag{29}$$

where the 1×12 vector F_i is:

$$\begin{aligned}F_{i,1} &= 0,\ F_{i2} = -\sigma,\ F_{i3} = 0,\ F_{i4} = 1,\ F_{i5} = -1,\ F_{i6} = k(1-\delta) \\ F_{i7} &= -k - k(1-\delta),\ F_{i8} = k,\ F_{i9} = F_{i10} = 0,\ F_{i11} = -1,\ F_{i12} = 1 \\ F_{i13} &= -\sigma,\ F_{i14} = \sigma.\end{aligned}$$

Let

$$F = \begin{bmatrix} F_\pi \\ F_k \\ F_i \end{bmatrix},$$

so that the equations are written:

$$Fz_t = 0,\ \text{for all } t.$$

It is convenient to write this as:

$$Fz_t = F_1 z_{1t} + F_2 z_{2t},$$

where F_1 includes all but the 5^{th} and 8^{th} columns of F and F_2 is composed just of these two. In addition, z_{1t} includes all but the 5^{th} and 8^{th} elements of z_t, while z_{2t} includes the 5^{th} and 8^{th} only. So, F_1 is 3×10 and F_2 is 3×2. It is useful to consider two 2×2 pieces of F_2: let F_2^1 denote the first two rows, and let F_2^2 denote the second two rows. Define F_1^1 and F_2^2 analogously.

Write:

$$\begin{aligned} z_{1t} &= (\pi_t, \pi_{t+1}, \pi_{t+2}, \hat{Y}_t, \hat{K}_t, \hat{K}_{t+1}, \hat{G}_t, \hat{G}_{t+1}, \hat{r}_t^n, \hat{\imath}_t)' \\ z_{2t} &= \left(\hat{Y}_{t+1}, \hat{K}_{t+2}\right)'. \end{aligned}$$

A.2 A Shooting Algorithm

Suppose that the economy has been in a deterministic, zero-inflation steady state until period 0, so that $\hat{K}_1 = 0$. Then, in $t = 1$, r_t^n, drops unexpectedly to $\hat{r}_l^n < \beta - 1$. The discount rate, r_t^n, remains low $T-1$ periods, and returns to its steady state value in the Tth period, where it remains forever after. I study the equilibrium allocations in response to the savings shock, r_t^n. Government spending, G_t, is allowed to change values in the same periods.

Monetary policy has the property that $\pi_t = 0$, unless this implies the interest rate violates its lower bound, $\hat{\imath}_t \geq \beta - 1$, in which case $\hat{\imath}_t = \beta - 1$. I conjecture (and then verify) that the zero bound becomes binding in one particular period, $t_1^* \geq 1$, and then continues to bind in each period until period t_2^*, when it ceases to bind. Thus,

$$\begin{array}{cc} \hat{\imath}_t = \beta - 1 & t_1^* \leq t < t_2^* \\ \hat{\imath}_t > \beta - 1, \ \pi_t = 0 & t \geq t_2^*, \ t < t_1^*. \end{array}.$$

The algorithm described below finds t_1^*, t_2^*, as well as the other equilibrium variables of the model.

The basic idea of the algorithm is as follows. There is one initial condition for the economy, $\hat{K}_1 = 0$. I then simulate the equations that characterize equilibrium forward in time. Because of the dimension of the equations, to initiate the simulations I need to assign values to four endogenous variables, aside from the given value of $\hat{K}_1$. I adjust the values of these four variables to ensure that the system eventually converge to steady state, and that another side condition is satisfied.

Suppose that $t_1^* = 1$, so that the zero bound is assumed to bind in period 1. Initially, I assign an arbitrary set of values to the four variables, $\hat{Y}_1$, $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{K}_2$. I temporarily normalize $\hat{\pi}_3 = 0$, and set $\hat{\imath}_1 = \beta - 1$. We can now construct z_{1t} for $t = 1$. Use (27)-(29) for $t = 1$ to compute $\hat{Y}_2$, $\hat{K}_3$:

$$z_{2,1} = -\left[F_2^2\right]^{-1} F_1^2 z_{1,1}. \tag{30}$$

Note that $\hat{\pi}_3$ does not appear in (27)-(29) for $t = 1$, so that setting $\hat{\pi}_3 = 0$ has no impact on the calculations in (30). I now use (26) to compute $\hat{\pi}_3$ and I insert this into the third element of $z_{1,1}$.

Now consider $t = 2$. We have enough information to determine the values of z_{1t} for $t = 2$, as long as we normalize $\hat{\pi}_4 = 0$. Use (27)-(29) to compute $\hat{Y}_3$ and $\hat{K}_4$ using:

$$z_{2,t} = -\left[F_2^2\right]^{-1} F_1^2 z_{1,t},$$

for $t = 2$. Use (26) to compute $\hat{\pi}_4$, and insert the result into the third element of $z_{1,2}$. Proceed in this way until $t = t_2^* - 1$.

Consider $t \geq t_2^*$. I set $\hat{\pi}_t = 0$ for $t \geq t_2^*$. As before, we have enough information to construct z_{1t}. Now we use (26)-(27) to compute $\hat{Y}_{t+1}$, $\hat{K}_{t+2}$:

$$z_{2,t} = -\left[F_2^1\right]^{-1} F_1^1 z_{1,t}. \tag{31}$$

Then, (29) is used to solve for $\hat{\imath}_t$. Simulate this forward for many periods, say until $t = T^* > T$.

Adjust the values of $\hat{Y}_1$, $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{K}_2$ until

$$\hat{Y}_{T^*} = 0,\ \hat{K}_{T^*+1} = 0, \tag{32}$$

and the second and third elements of z_{1,t_2^*-1} are zero.

Now consider the case, $t_1^* \geq 2$. For $t = 1, ..., t_1^* - 1$, I use (31) to simulate the system. To initiate these simulations, I assign arbitrary values to $\hat{Y}_1$, $\hat{K}_2$. In addition, I require values for $\hat{\pi}_{t_1^*}$ and $\hat{\pi}_{t_1^*+1}$. It is necessary to include these in z_{1,t_1^*-2} and z_{1,t_1^*-1} when evaluating (31) for $t = t_1^* - 2$, $t_1^* - 1$. Initially, I assign arbitrary values to $\hat{\pi}_{t_1^*}$ and $\hat{\pi}_{t_1^*+1}$. I then adjust the values of $\hat{Y}_1$, $\hat{K}_2$, $\hat{\pi}_{t_1^*}$ and $\hat{\pi}_{t_1^*+1}$ until the same criterion used for the case, $t_1^* = 1$ is satisfied. The MATLAB routine, FSOLVE.M, was able to solve this problem without incident in less than 2 seconds.

To determine values of t_1^* and t_2^*, I proceeded as follows. First, I solved the model without imposing the lower bound constraint on $\hat{\imath}_t$. I then determined the first date when the lower bound constraint was violated, and made that my initial guess of t_1^*. I then determined the first date thereafter when the lower bound constraint ceased to bind and made that my guess of t_2^*. For the model without investment, i.e., $\epsilon_\psi = \infty$, this procedure led to an initial guess, $t_1^* = 1$, and I did not deviate from that. The initial guess for t_2^* was 15. I kept that value because the lower bound constraint was violated when I dropped t_2^* to a lower value.

For the model with investment, $\epsilon_\psi = 3$, my procedure led to an initial guess, $t_1^* = 7$ and $t_2^* = 15$. When I solved the model with these values, I found that the lower bound constraint was violated. I then found that it was also violated for values of t_1^* that exceeded 7 and for every admissible value of t_1^* below 7 except $t_1^* = 1$. So, I fixed $t_1^* = 1$. I then tried $t_2^* = 14$, and found that the lower bound constraint was violated. So, I remained with $t_1^* = 1$ and $t_2^* = 15$ for the $\epsilon_\psi = 3$ model.

B Appendix 2: Government Spending Multiplier

To solve for $\hat{Y}_t$, it is convenient to express (9) and (10) in lag-operator form:

$$\hat{Y}_t = \hat{G}_t + \frac{\sigma}{1 - L^{-1}}\left(L^{-1}\hat{\pi}_t + \hat{r}_t^n - \hat{\imath}_t\right) \tag{33}$$

$$\hat{\pi}_t = \frac{1}{1 - \beta L^{-1}}\frac{A}{\sigma}\left\{(\sigma\omega + 1)\hat{Y}_t - \hat{G}_t\right\}, \tag{34}$$

where $L^{-j}w_t \equiv w_{t+j}$ for integer values of j. Substituting out for $\hat{\pi}_t$ from (34) into (33) and rearranging, one obtains:

$$\hat{Y}_t = h_1(L)\hat{G}_t + h_2(L)\left(\hat{r}_t^n - \hat{\imath}_t\right),$$

where

$$
\begin{aligned}
h_1(L) &= \frac{(1-\beta L^{-1})(1-L^{-1}) - AL^{-1}}{(1-\beta L^{-1})(1-L^{-1}) - A(\omega+\sigma^{-1})\sigma L^{-1}} = \frac{(1-\zeta_1 L^{-1})(1-\zeta_2 L^{-1})}{(1-\lambda_1 L^{-1})(1-\lambda_2 L^{-1})} \\
h_2(L) &= \frac{\sigma(1-\beta L^{-1})}{(1-\beta L^{-1})(1-L^{-1}) - A(\omega+\sigma^{-1})\sigma L^{-1}} = \frac{\sigma(1-\beta L^{-1})}{(1-\lambda_1 L^{-1})(1-\lambda_2 L^{-1})}.
\end{aligned}
$$

Here, $\lambda_1 < \beta$, $\lambda_2 > 1$ are the roots of the polynomial in the denominator of $h_1(L)$ and $h_2(L)$. In addition, ζ_1, ζ_2 are the roots of the polynomial in the numerator of $h_1(L)$ and it is easily verified that

$$
\begin{aligned}
\lambda_1 &< \zeta_1 < \beta, \lambda_2 > \zeta_2 > 1, \\
\zeta_2 + \zeta_1 &= 1 + \beta + A,\ \zeta_2\zeta_1 = \beta.
\end{aligned} \tag{35}
$$

Using standard methods to expand these polynomials (see, e.g., Sargent (1979)), I obtain:

$$
\begin{aligned}
h_1(L) &= \frac{A_1}{\sigma}\left(1 + (\lambda_1 - \zeta_2 - \zeta_1)L^{-1} + \frac{(\lambda_1-\zeta_2)(\lambda_1-\zeta_1)}{1-\lambda_1 L^{-1}} L^{-2}\right) \\
&\quad + \frac{A_2}{\sigma}\left(1 + (\lambda_2 - \zeta_2 - \zeta_1)L^{-1} + \frac{(\lambda_2-\zeta_2)(\lambda_2-\zeta_1)}{1-\lambda_2 L^{-1}} L^{-2}\right),
\end{aligned} \tag{36}
$$

where

$$
A_1 = \frac{\sigma}{1-\lambda_2/\lambda_1},\ A_2 = \frac{\sigma}{1-\lambda_1/\lambda_2}.
$$

Similarly,

$$
h_2(L) = A_1\left[1 + \frac{(\lambda_1-\beta)L^{-1}}{1-\lambda_1 L^{-1}}\right] + A_2\left[1 + \frac{(\lambda_2-\beta)L^{-1}}{1-\lambda_2 L^{-1}}\right]. \tag{37}
$$

Our objective is to derive h_j^G in the:

$$
h_1(L) = \sum_{j=1}^{\infty} h_j^G L^{-(j-1)}.
$$

It is immediate from (36) that $h_1^G = 1$. Also,

$$
\begin{aligned}
h_2^G &= \frac{A_1}{\sigma}(\lambda_1 - \zeta_2 - \zeta_1) + \frac{A_2}{\sigma}(\lambda_2 - \zeta_2 - \zeta_1) \\
&= \frac{\lambda_1}{\lambda_1-\lambda_2}(\lambda_1 - [\zeta_2+\zeta_1]) + \frac{\lambda_2}{\lambda_2-\lambda_1}(\lambda_2 - [\zeta_2+\zeta_1]) \\
&= \frac{1}{\lambda_2-\lambda_1}\left[\lambda_2^2 - \lambda_1^2 - (\lambda_2-\lambda_1)(\zeta_2+\zeta_1)\right] \\
&= \frac{1}{\lambda_2-\lambda_1}\left[(\lambda_2-\lambda_1)(\lambda_2+\lambda_1) - (\lambda_2-\lambda_1)(\zeta_2+\zeta_1)\right] \\
&= (\lambda_2+\lambda_1) - (\zeta_2+\zeta_1) \\
&= A\sigma\omega,
\end{aligned}
$$

using (14) and (35). For $j \geq 3$,

$$\begin{aligned}
h_j^G &= \frac{A_1}{\sigma}\left(\lambda_1-\zeta_2\right)\left(\lambda_1-\zeta_1\right)\lambda_1^{j-3}+\frac{A_1}{\sigma}\left(\lambda_2-\zeta_2\right)\left(\lambda_2-\zeta_1\right)\lambda_2^{j-3} \\
&= \frac{1}{1-\lambda_2/\lambda_1}\left(\lambda_1-\zeta_2\right)\left(\lambda_1-\zeta_1\right)\lambda_1^{j-3}+\frac{1}{1-\lambda_1/\lambda_2}\left(\lambda_2-\zeta_2\right)\left(\lambda_2-\zeta_1\right)\lambda_2^{j-3} \\
&= \frac{\lambda_1^2}{\lambda_1-\lambda_2}\left[\lambda_1+\beta\lambda_1^{-1}-\left(\zeta_1+\zeta_2\right)\right]\lambda_1^{j-3}-\frac{\lambda_2^2}{\lambda_1-\lambda_2}\left[\lambda_2+\beta\lambda_2^{-1}-\left(\zeta_1+\zeta_2\right)\right]\lambda_2^{j-3} \\
&= \frac{\left[\lambda_1+\lambda_2-\left(\zeta_1+\zeta_2\right)\right]}{\lambda_2-\lambda_1}\left(\lambda_2^{j-1}-\lambda_1^{j-1}\right) \\
&= \frac{A\sigma\omega}{\lambda_2-\lambda_1}\left(\lambda_2^{j-1}-\lambda_1^{j-1}\right)
\end{aligned}$$

Summarizing, we obtain:

$$h_1(L) = 1 + A\sigma\omega L^{-1} + \frac{A\sigma\omega}{\lambda_2-\lambda_1}\sum_{j=3}^{\infty}\left(\lambda_2^{j-1}-\lambda_1^{j-1}\right)L^{-(j-1)}$$

If $\hat{G}_t = \hat{G}_h$ for $t = 1, ..., T-1$, then

$$\begin{aligned}
h_1(L)\hat{G}_1 &= \left[1 + A\sigma\omega L^{-1} + \frac{A\sigma\omega}{\lambda_2-\lambda_1}\sum_{j=3}^{T-1}\left(\lambda_2^{j-1}-\lambda_1^{j-1}\right)L^{-(j-1)}\right]\hat{G}_1 \\
&= \left[1 + A\sigma\omega + \frac{A\sigma\omega}{\lambda_2-\lambda_1}\left(\lambda_2^2\frac{1-\lambda_2^{T-3}}{1-\lambda_2}-\lambda_1^2\frac{1-\lambda_1^{T-3}}{1-\lambda_1}\right)\right]\hat{G}_h,
\end{aligned}$$

as claimed in the text.

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Figure 1: Consequence of Increase in Saving When there is Lower Bound on Real Interest Rate. For Two Investment Elasticities

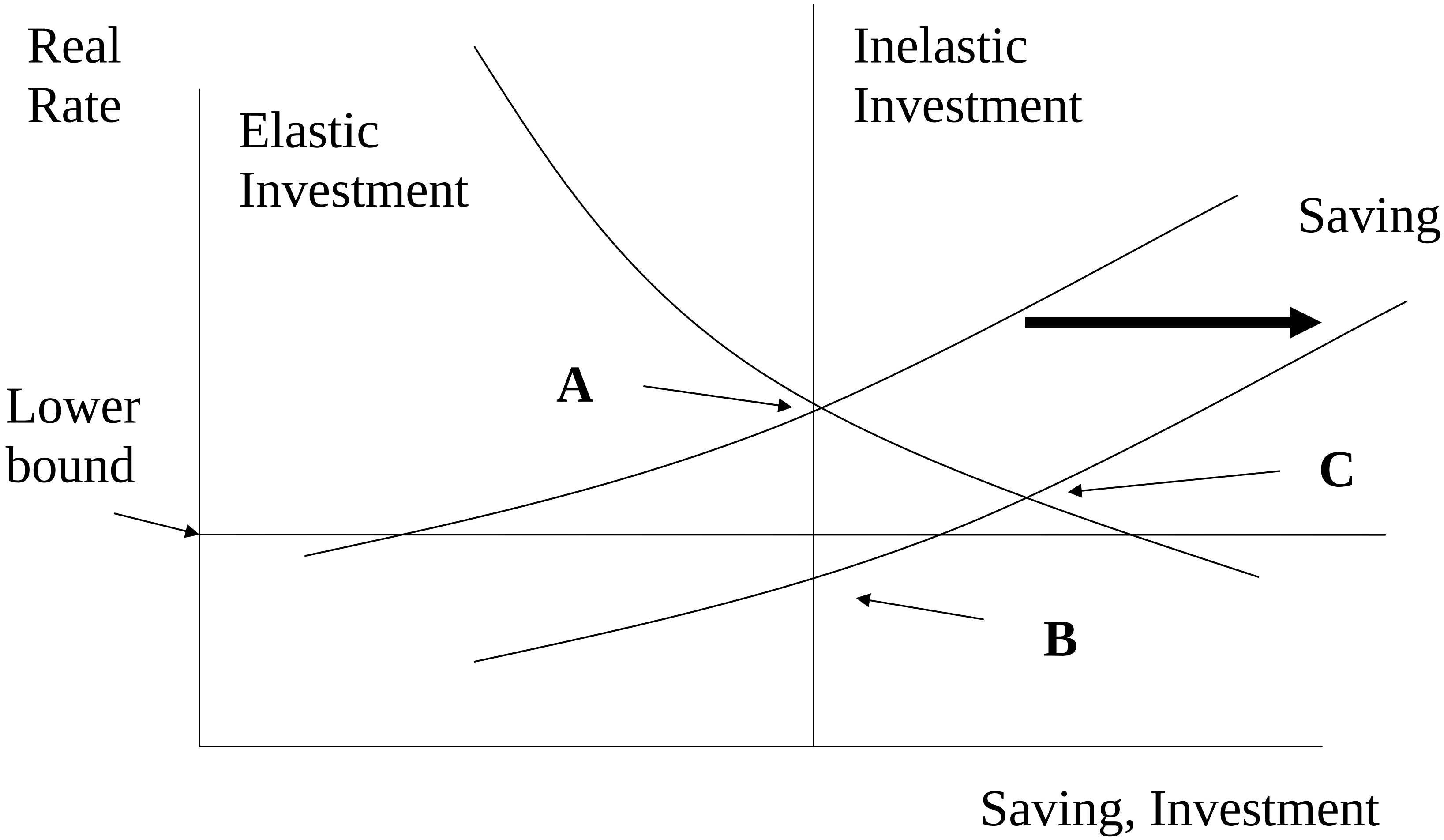

Figure 2: Response to Discount Rate Shock in Model With and Without Inflation Indexation

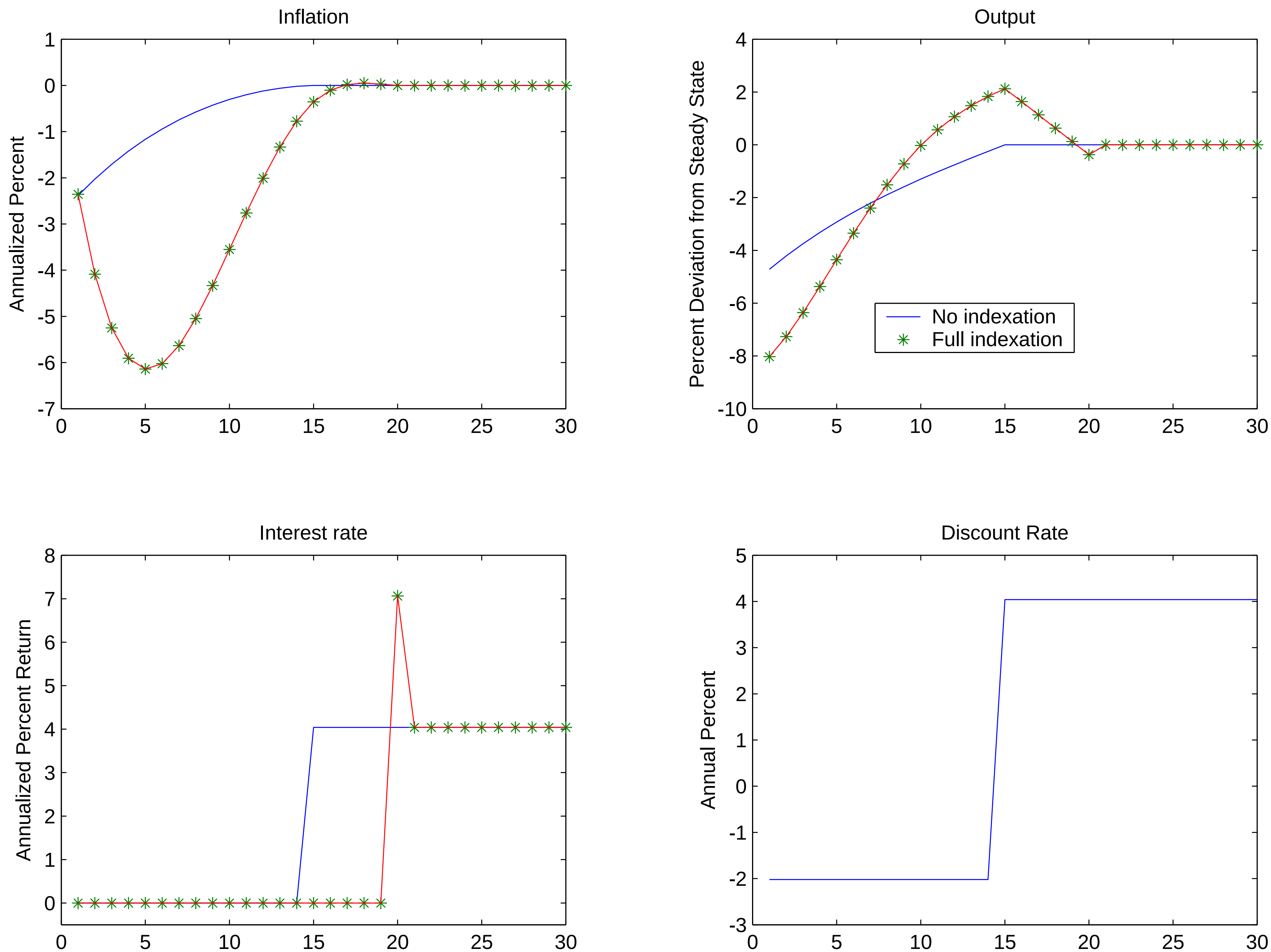

Figure 3: Discount Rate Shock in Model without Investment, Three Discount Rate Shocks

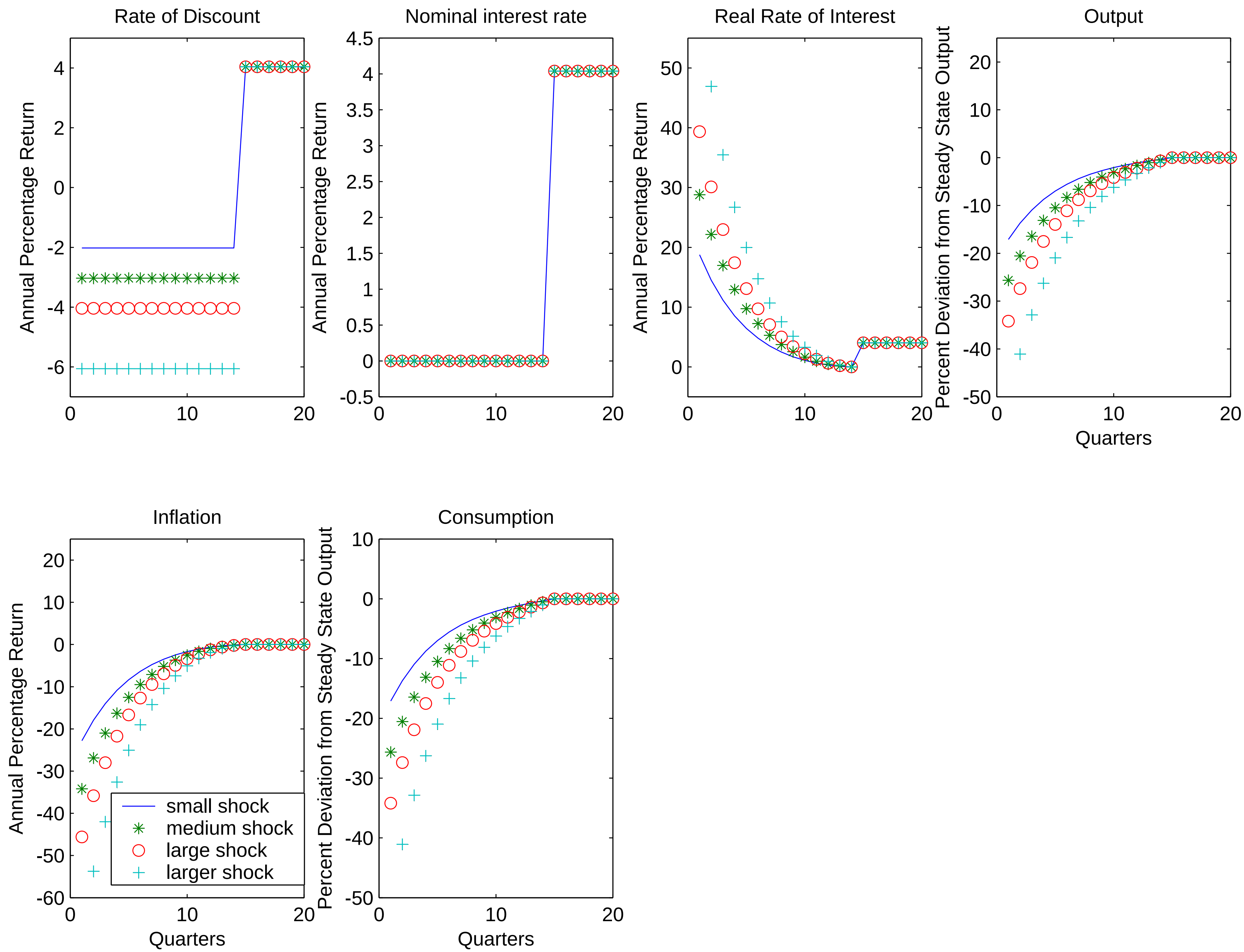

Figure 4: Discount Rate Shock in Model with Investment, Three Discount Rate Shocks

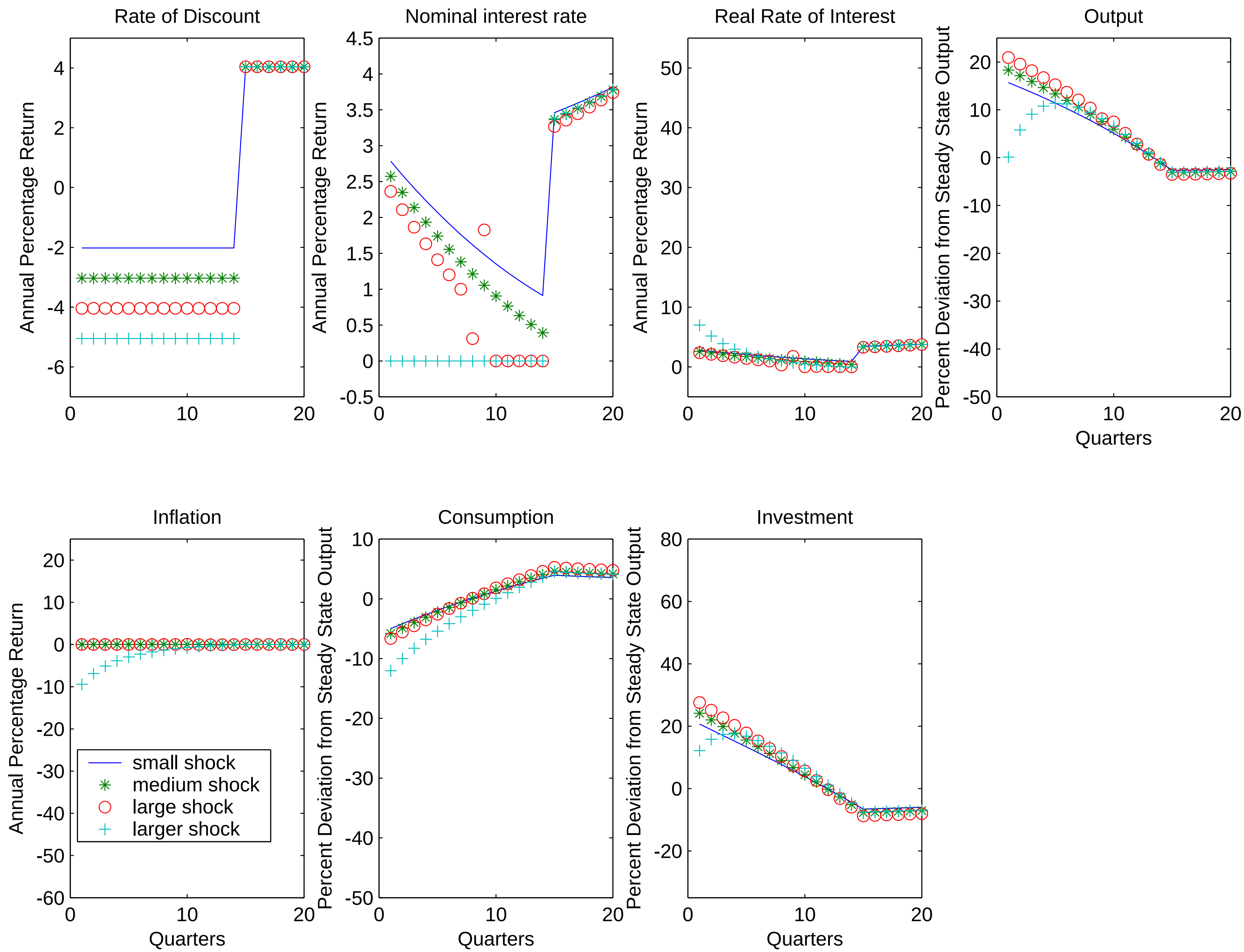

Figure 5: Discount Rate Shock in Model without Investment, Low Labor Supply Elasticity, Three Discount Rate Shocks

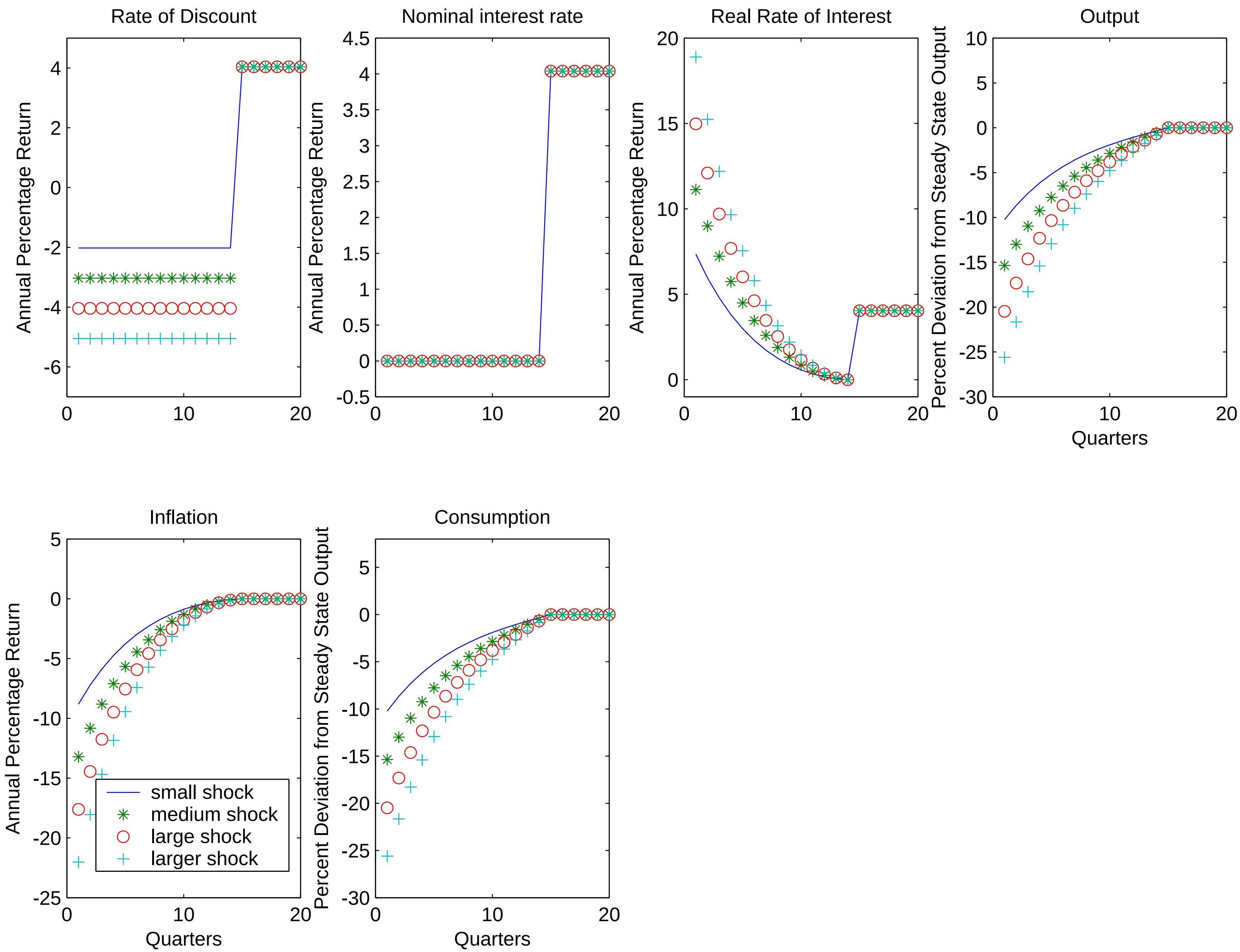

Figure 6: Discount Rate Shock in Model with Investment, Low Labor Supply Elasticity, Three Discount Rate Shocks

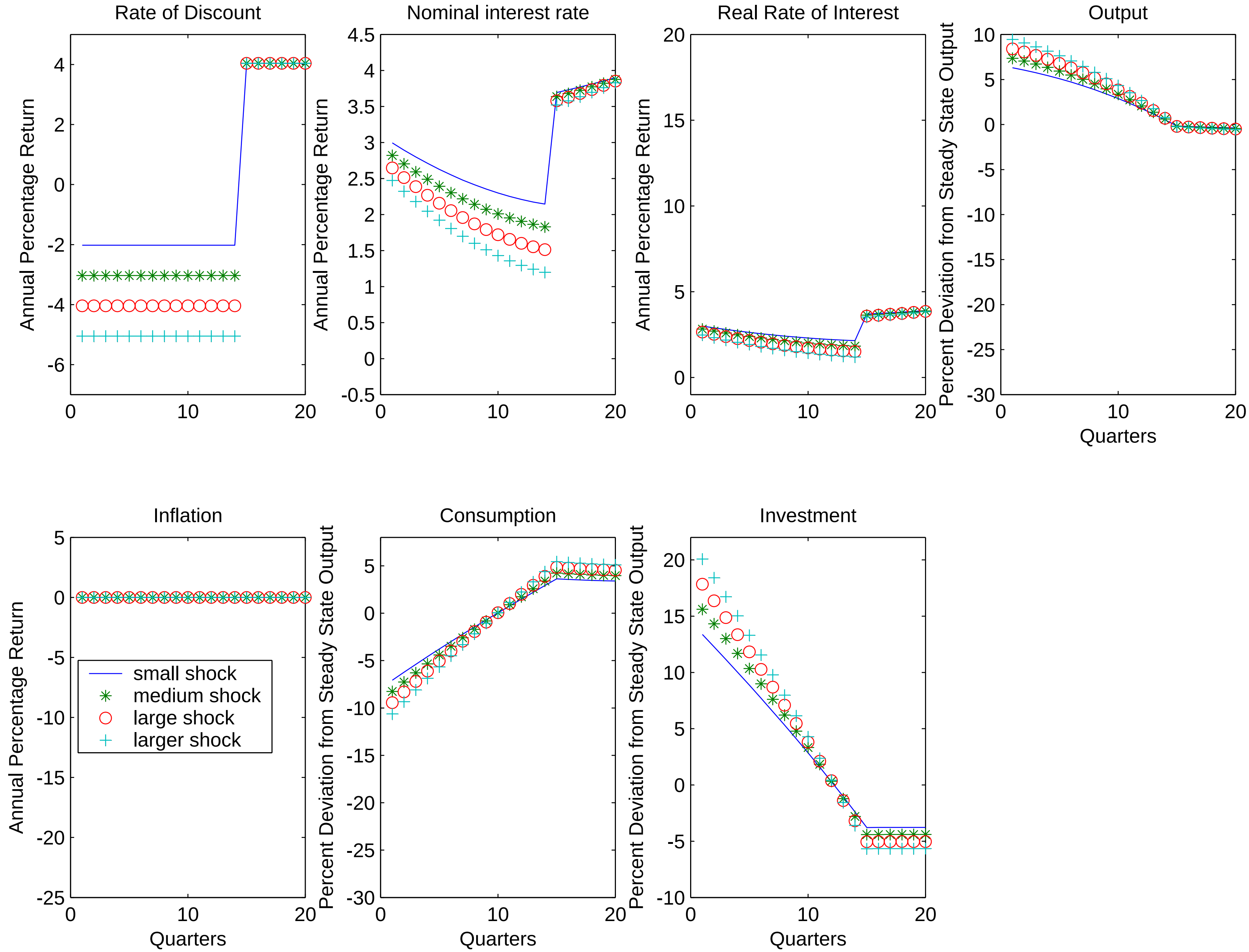

Figure 7: Dynamic Response to Small Shock, With (*) and Without (-) Increase in Gov't Spending

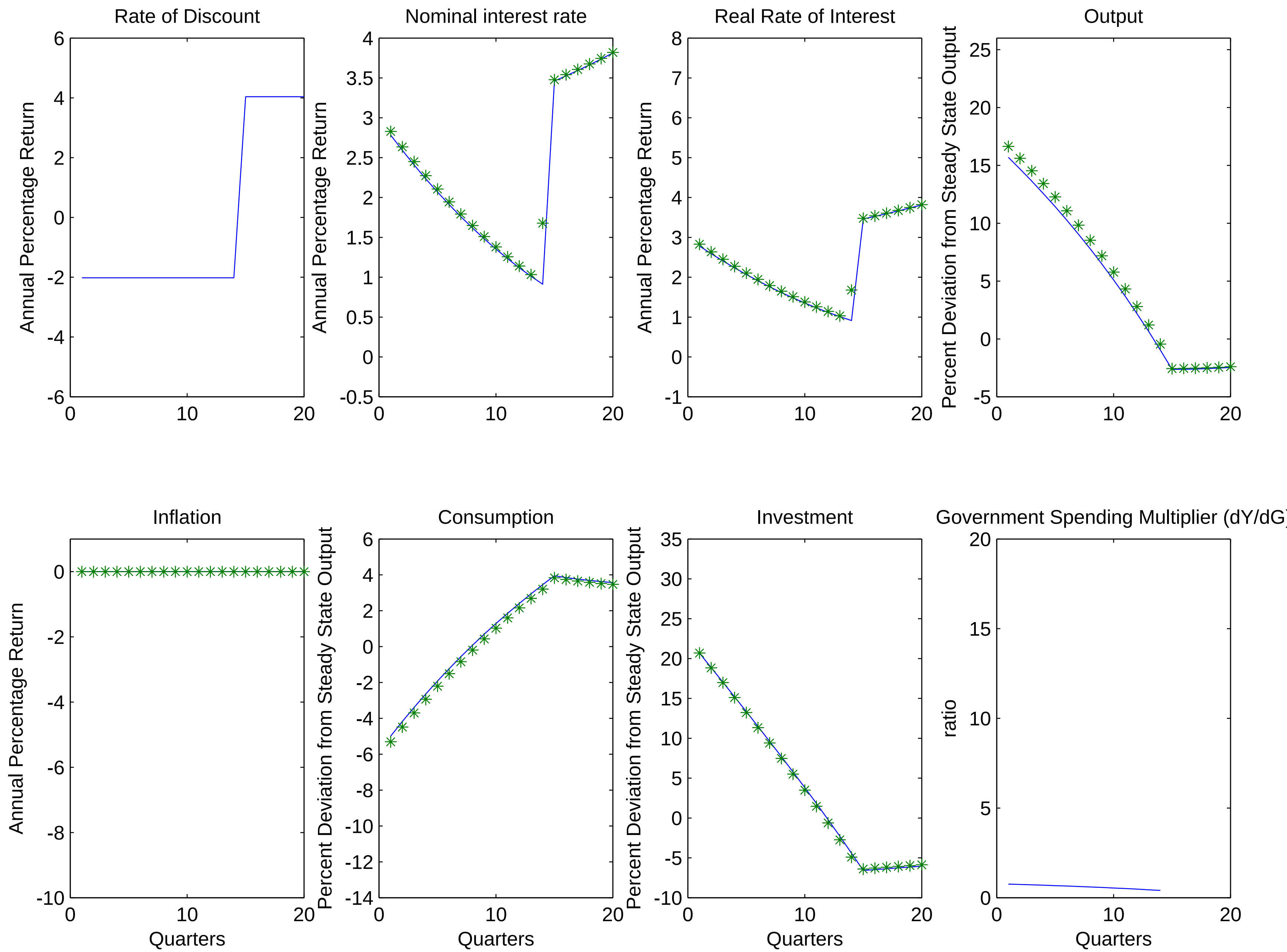

Figure 8: Dynamic Response to Larger Shock, With (*) and Without (-) Increase in Gov't Spending

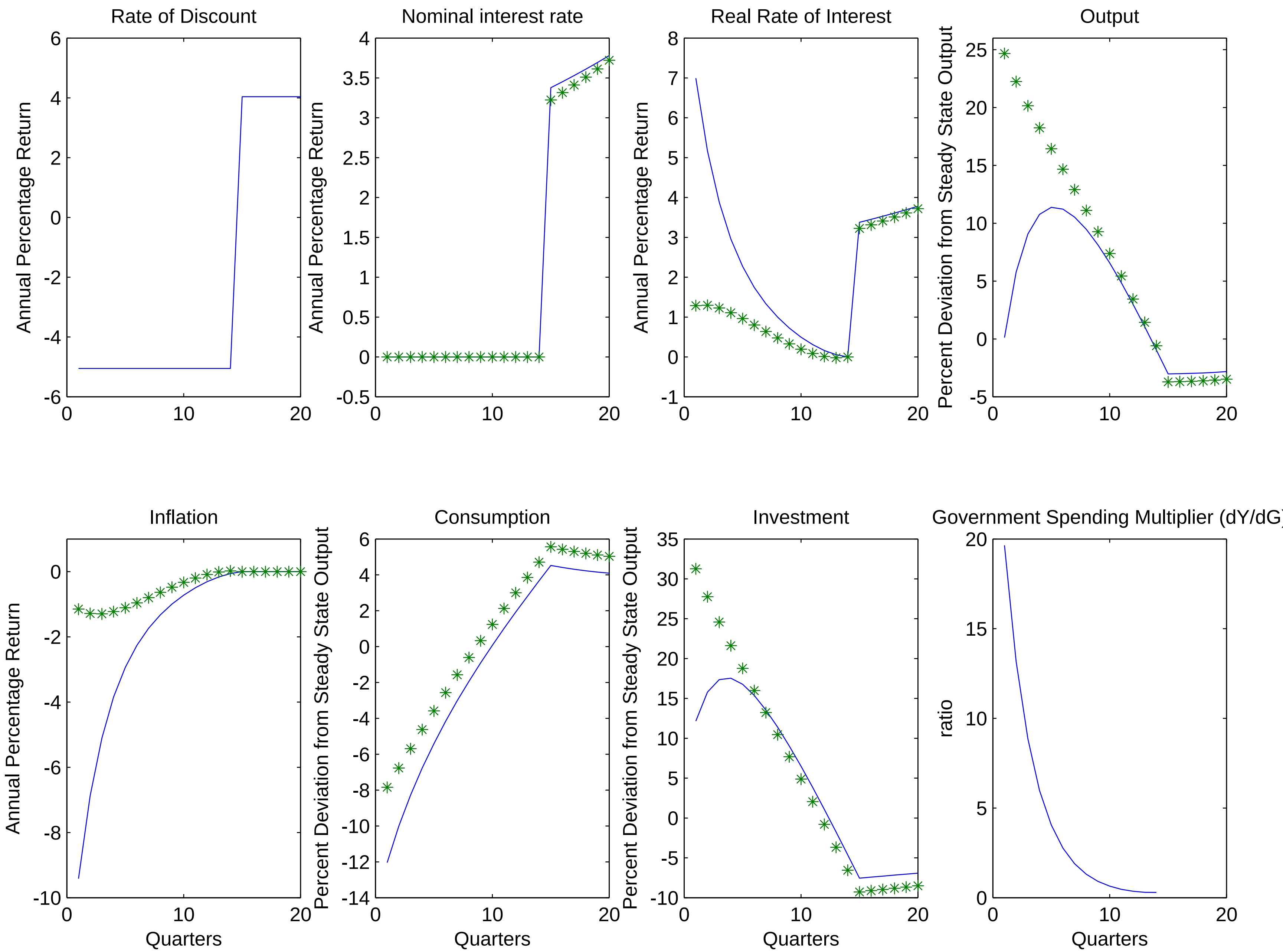

Figure 9: Real Interest Rate in 20th Century

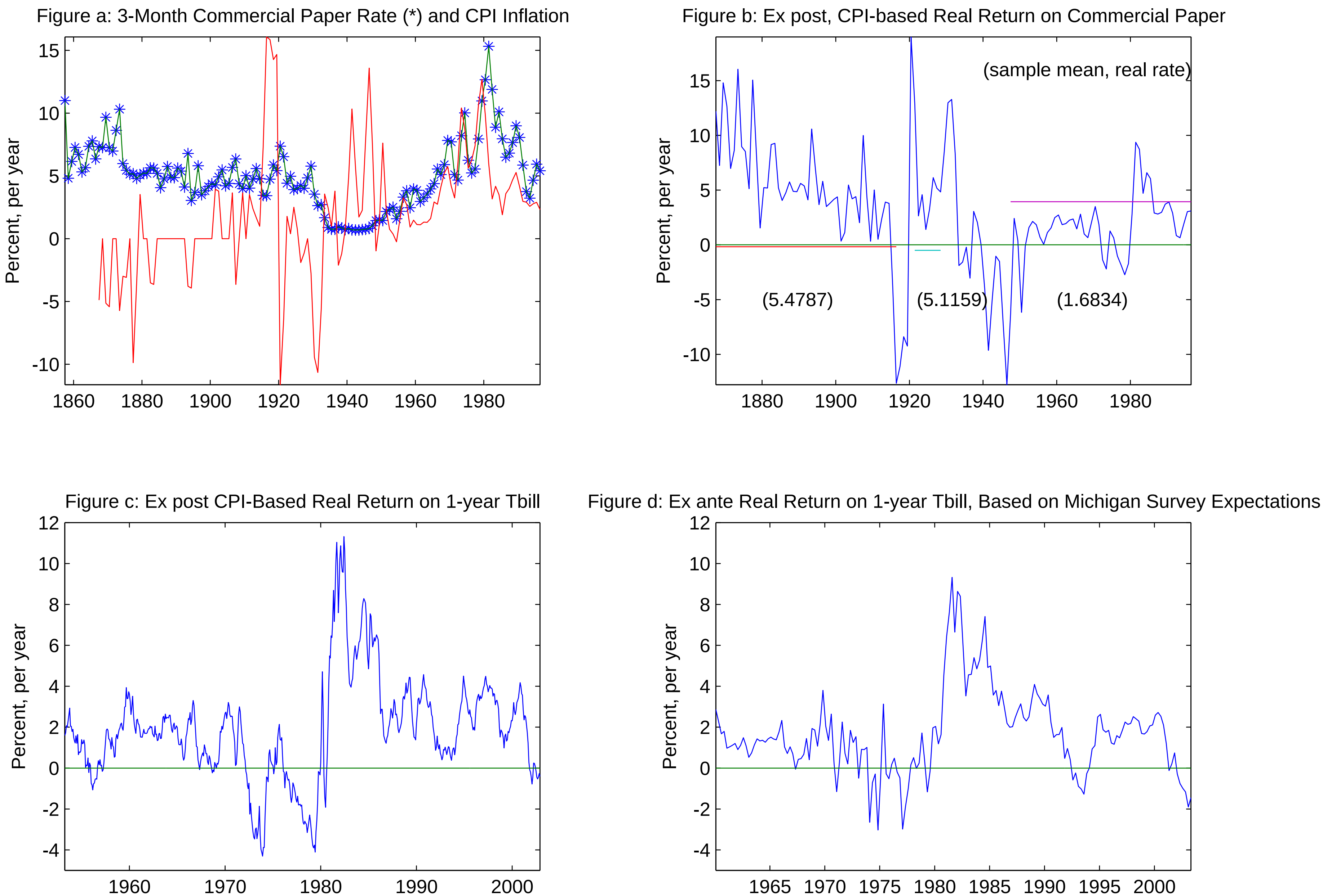

Figure 10: Inflation, Nominal and Real Interest Rates, 1930's

Figure a: Inflation, Nominal Interest Rates

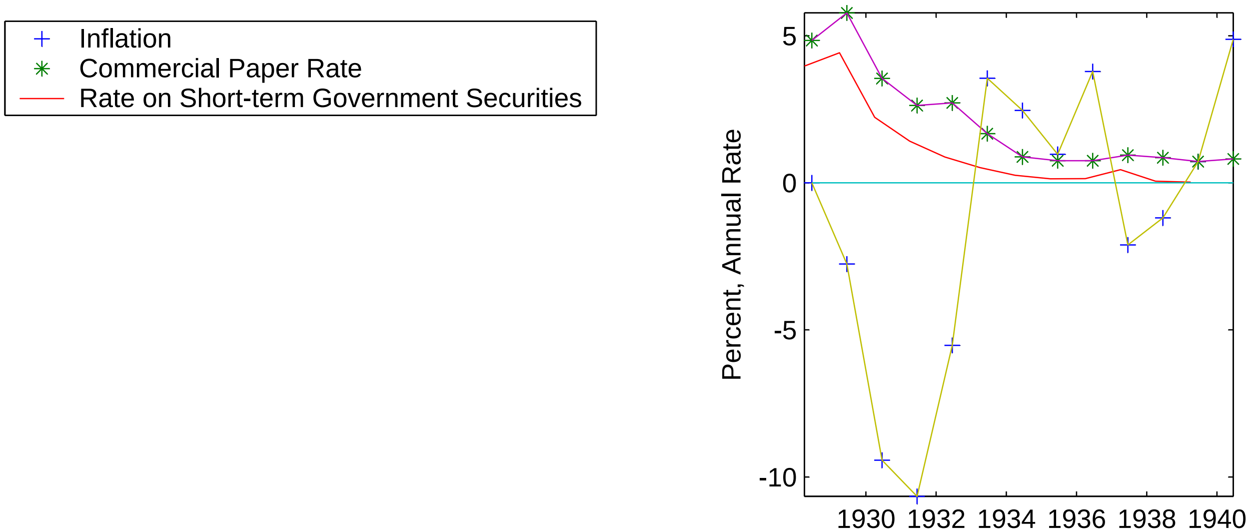

Figure b: Per Capita Hours Worked (incl. military), Per Capita GDP