Dynamic Mechanism Design with Hidden Income and Hidden Actions *

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Abstract

We develop general recursive methods to solve for optimal contracts in dynamic principal-agent models with both hidden income and hidden actions. The principal, or the community as a whole, observes nothing other than transfers. Nevertheless, optimal incentive-constrained insurance can be attained. Starting from a general mechanism with arbitrary communication, randomization, and full history dependence, we show that the optimal contract can be implemented with a number of direct recursive mechanisms, all established to be equivalent to one another. The state variable for the most natural recursive formulation is a vector of utility promises conditional on the realized endowment. However, this formulation suffers from a curse of dimensionality which arises from the interaction of hidden income and hidden investment. The curse can be overcome either by reporting unobserved income twice, or by introducing judiciously chosen utility bounds for deviation behavior off the equilibrium path. In an application to insurance with hidden storage, the optimal information-constrained contract can improve upon the utility the agent would obtain by self-insuring via borrowing and lending in a credit market if either the return on storage is low, or if the return on storage is high but uncertain with unobserved returns.

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1 Introduction

We consider a class of dynamic principal-agent problems in which a risk-averse agent receives unobserved income shocks and can take unobserved actions which influence future income realizations. The principal wants to provide optimal incentive-compatible insurance against the income shocks. We formulate a general planning problem allowing for history dependence and unrestricted communication and prove that this problem can be reduced to a recursive version with direct mechanisms and vectors of utility promises as the state variable. In this recursive formulation, however, a "curse of dimensionality" leads to an excessively large number of incentive constraints, which makes it impossible to compute optimal allocations numerically. We solve this problem by providing alternative equivalent formulations of the planning problem in which the planner can specify behavior off the equilibrium path. This leads to a dramatic reduction in the number of constraints that need to be imposed when computing the optimal contract. With the methods developed in this paper, a wide range of dynamic mechanism design problems can be analyzed which were previously considered to be intractable.

The design of optimal incentive-compatible mechanisms in environments with privately observed income shocks has been studied by a number of previous authors, including Townsend (1979), Green (1987), Thomas and Worrall (1990), Wang (1995), and Phelan (1995). In such environments, the principal extracts information from the agents about their endowments and uses this information to provide optimal incentive-compatible insurance. The existing literature has concentrated on environments in which the agent is not able to take unobserved actions that influence the probability distribution over future endowments. The reason for this limitation is mainly technical; with hidden actions, the agent and the planner do not have common knowledge over probabilities of future states, which renders standard methods of computing optimal mechanisms inapplicable.

Theoretical considerations suggest that the presence of hidden actions can have large effects on the constrained-optimal mechanism. To induce truthful reporting of endowments, the planner needs control over the agent's intertemporal rate of substitution. When hidden actions are introduced, the planner has less control over rates of substitution, so that providing insurance becomes more difficult. Indeed, there are some special cases where the presence of hidden actions causes insurance to break down completely. Allen (1985) shows that if an agent with hidden income shocks has unobserved access to a perfect credit market (both borrowing and lending), the principal cannot offer any insurance beyond the borrowing-lending solution. The reason is that any agent will choose the reporting scheme that yields the highest net discounted transfer regardless of what the actual endowment is.[1] Cole and Kocherlakota (2001b) consider a related closed economy environment in which the agent can only save, but not borrow. The hidden savings technology is assumed to have the same return as a public storage technology that the planner

[1]The issue of credit-market access is also analyzed in Fudenberg, Holmstrom, and Milgrom (1990), who show that if principal and agent can access the credit market on equal terms, the optimal dynamic incentive contracts can be implemented as a sequence of short-term contracts.

has access to. Remarkably, under additional assumptions, Cole and Kocherlakota obtain the same result: the optimal solution is equivalent to unrestricted access to credit markets. Ljungqvist and Sargent (2003) extend Cole and Kocherlakota to an open economy and establish the equivalence result with otherwise relatively few assumptions.

The methods developed in this paper can be used to compute optimal incentive-constrained mechanisms in more general environments with hidden actions in which self-insurance via investment or storage is an imperfect substitute for saving in an outside financial market, either because the return on own storage is lower than in the financial market, or the return on own investment is high but with uncertain and unobserved returns. In such environments optimal information-constrained insurance can improve over the borrowing-and-lending solution despite severe information problems. As in Townsend (1982), the agent can be given incentives to correctly announce the underlying state, since he is aware of his own intertemporal rate of substitution. Unobserved investment subject to moral hazard with unobserved random returns does not undercut this intuition. The optimal outcome reduces to pure borrowing and lending only for a relatively narrow range of returns where the expected return on uncertain investment is roughly equal to (slightly above) the return in outside financial markets.

The existing literature on dynamic mechanism design has been restricted to environments where the curse of dimensionality that arises in our model is avoided from the outset. The paper most closely related to ours is Fernandes and Phelan (2000). Fernandes and Phelan develop recursive methods to deal with dynamic incentive problems with Markov income links between the periods, and as we do, they use a vector of utility promises as the state variable. There are three key differences between our paper and the approach by Fernandes and Phelan. First, the recursive formulation has a different structure. While in our setup utility promises are conditional on the realized endowment, in Fernandes and Phelan the promises are conditional on the unobserved action in the preceding period. Both methods should lead to the same results if the informational environment is the same, but they differ in computational efficiency. Even if incomes were observed, our method would be preferable in environments with many possible actions, but relatively few endowment values. Indeed, Kocherlakota (2002) has made the point that with a continuum of possible values for savings, the approach of Fernandes and Phelan is not computationally practical. A second more fundamental difference is that Fernandes and Phelan do not consider environments in which both actions and states are unobservable to the planner, the main source of complexity in our setup (they consider each problem separately). Unobserved incomes or endowments lead naturally to the use of state contingent utility promises, to deal with the truth-telling constraints at the time endowments are realized. Our methods are therefore applicable to a much wider range of problems, including those where the planner (or larger community) knows virtually nothing, only what has been transferred to the agent. Finally, our analysis is different from Fernandes and Phelan in that we allow for randomization and unrestricted communication, and we show from first principles that various recursive formulations are equivalent to the general formulation of the planning problem.

Our emphasis on first principles, i.e., proof of the revelation principle, is not for elegance and completeness alone. Rather, it is an essential part of what we do. In our "standard" formulation, there is a curse of dimensionality which arises from the interaction of hidden endowment and unobserved actions. We overcome the curse by guessing at various alternative formulations and then establishing that these are all equivalent to one another, that is, all are derived via the revelation principle from the common unrestricted general structure. The formulations differ, however, with respect to the dimensionality of variables, constraints, and the number of sub-programs. In one formulation, agents announce unobserved income twice, with transfers from the planner in between reports. In a second, we create utility bounds for all deviation strategies (lying and disobedience). The idea of using bounds comes from Prescott (1997). In effect, we bring together various pieces of the literature, point out what they have in common, show what is essential, and indicate the various tradeoffs. Our goal is to allow researchers to attack heretofore intractable problems.

Another possible approach for the class of problems considered here is the recursive first-order method used by Werning (2001) to analyze a dynamic moral hazard problem with storage. Kocherlakota (2002) casts doubts on the wider applicability of first-order methods, because the decision problem is not necessarily concave when there are complementarities between savings and other decision variables. Werning considers a restricted setting in which output is observed, the return on storage does not exceed the credit-market return, and storage is not subject to random shocks. To the extent that first-order methods are justified in a given application, they are computationally more efficient than our approach. However, once again the range of possible applications is much smaller.

In the following section, we provide an overview of the methods developed in this paper. In Section 3 we introduce the economic environment that underlies our mechanism design problem, and formulate a general planning problem with unrestricted communication and full history dependence. In Section 4, we invoke and reprove the revelation principle to reformulate the planning problem, using direct message spaces and enforcing truth-telling and obedience. We then provide a recursive formulation of this problem with a vector of utility promises as the state variable (Program 1). In Section 5, we develop alternative formulations which allow the planner to specify behavior and utility promises off the equilibrium path, either by allowing the agent to report the endowment twice (Program 2), or by imposing off-path utility bounds (Program 3). These methods lead to a dramatic reduction in the number of incentive constraints that need to be imposed when computing the optimal contract. In Section 6 we use our methods to compute optimal contracts in an environment characterized by hidden storage with a fixed return. Section 7 concludes, and all proofs are contained in the mathematical appendix.

2 Outline of Methods

We consider a dynamic mechanism design problem with hidden income and hidden actions. The agent realizes an unobserved endowment at the beginning of the period, then the planner makes a transfer to the agent, and at the end of period the agent takes an action which influences the probability distribution over future income realizations. We formulate a general planning problem of providing optimal incentive-constrained insurance to the agent. Our ultimate aim is to derive a formulation of the planning problem which can be solved numerically, using linear programming techniques as in Phelan and Townsend (1991). In order to make the problem tractable, we employ dynamic programming techniques to convert the general planning problem to problems which are recursive and of relatively low dimension. Dynamic programming is applied at two different levels. First, building on Spear and Srivastava (1987), we use utility promises as a state variable to gain a recursive formulation of the planning problem. In addition, we apply similar techniques within the period as a method for reducing the dimensionality of the resulting programming problem.

A key feature of our approach, central to what we do, is that we start from a general setup which allows randomization, full history dependence, and unrestricted communication. We formulate a general planning problem in the unrestricted setup (Section 3.2), and show from first principles that our recursive formulations are equivalent to the original formulation. Rather than imposing truth-telling and obedience from the outset, we prove a version of the revelation principle for our environment (Proposition 2).[2] Truth-telling and obedience are thus derived as endogenous constraints capturing all the information problems inherent in our setup. This is done not only to ensure that the revelation principle applies in our dynamic setting (as is often just assumed), but also because we actually apply two different versions of the revelation principle, one with single and one with double reporting. We thus derive two different direct mechanisms from the same general planning problem. Even though both mechanisms deliver exactly the same equilibrium allocation, and in that sense double reporting is not necessary, deriving separate mechanisms is still useful, since it can lead to enormous computational gains. Which of the mechanisms is used determines to a large extent whether the resulting program is computable.

After proving the revelation principle, the next step is to work towards a recursive formulation of the planning problem. Given that in our model both endowments and actions are hidden, so that in fact the planner does not observe anything, standard methods need to be extended to be applicable to our environment. The main problem is that with hidden

[2]Initially, derivations of the revelation principle as in Harris and Townsend (1977), Harris and Townsend (1981), Myerson (1979), and Myerson (1982) were for the most part static formulations. Here we are more explicit about the deviation possibilities in dynamic games. Unlike Myerson (1986), we do not focus on zero-probability events, but instead concentrate on within-period maximization operators, either using dynamic programming to work backwards from the end of the period, or off-path utility bounds to summarize all possible deviation behavior.

endowments and actions a scalar utility promise is not a sufficient state variable, since agent and planner do not have common knowledge over probabilities of current and future states. However, our underlying model environment is Markov in the sense that the unobserved action only affects the probability of tomorrow's income. Once that income is realized, it completely describes the current state of affairs for the agent, apart from any history dependence generated by the mechanism itself. We thus show that the planning problem can still be formulated recursively by using a vector of endowment-specific utility promises as the state variable (see Proposition 4 and the ensuing discussion).

It is a crucial if well-understood result that the equilibrium of a mechanism generates utility realizations. That is, along the equilibrium path a utility vector is implicitly being assigned, a scalar number for each possible endowment (though a function of the realized history). If the planner were to take that vector of utility promises as given and reoptimize so as to potentially increase surplus, the planner could do no better and no worse than under the original mechanism. Thus, equivalently, we can assign utility promises explicitly, and allow the planner to reoptimize at the beginning of each date. Using a vector of utility promises as the state variable introduces an additional complication, since the set of feasible utility vectors is not known in advance. To this end, we show that the set of feasible utility vectors can be computed recursively as well, by applying a variant the methods developed in Abreu, Pearce, and Stacchetti (1990) (Propositions 7 and 8 in Appendix A.2).[3]

Starting from our recursive formulation, we discretize the state space to formulate a version of the planning problem which can be solved using linear programming and value-function iteration (Program 1 in Section 4.3 below). However, we now face the problem that in this "standard" recursive formulation a "curse of dimensionality" arises, in the sense that the number of constraints that need to be imposed when computing the optimal mechanism becomes very large. The problem is caused by the truth-telling constraints, which describe that reporting the true endowment at the beginning of the period is optimal for the agent. In these constraints, the utility resulting from truthful reporting has to be compared to all possible deviations. When both endowments and actions are unobserved, the number of such deviations is large. A deviation consists of lying about the endowment, combined with an "action plan" which specifies which action the agent will take, given any transfer and recommendation he may receive from the planner before taking the action. The number of such action plans is equal to the number of possible actions taken to the power of the product of the number of actions and the number of transfers (recall that there are finite grids for all choice variables to allow linear programming). The number of constraints therefore grows exponentially in the number of transfers and the number of actions. Thus even for moderate sizes for these grids, the number of constraints becomes too large to be handled on any computer.

To deal with this problem, we show that the number of constraints can be reduced dramatically by allowing the planner to specify outcomes off the equilibrium path. The in-

[3]See also Cole and Kocherlakota (2001a) in an application to dynamic games with hidden states and actions.

tuition is that we use a maximization operator which makes it unnecessary to check all possible deviations. This can be done either by having the agent report the endowment twice, or by imposing off-path utility bounds. The advantage of specifying behavior off the equilibrium path is that optimal behavior will be at least partly defined even if the agent misreports, so that not all possible deviations need to be checked. In the version with double reporting (Program 2 in Section 5.1 below), the agent reports his endowment a second time after receiving a transfer from the planner. Incentive constraints ensure that the second report is correct, regardless of the first report. In this double-reporting version, at the first report there is just one truth-telling constraint for each alternative endowment, since the agent already knows that even conditional on lying it will be optimal to tell the truth at the second report and follow the recommendations after that. There are additional truth-telling constraints for the second report, but these are simplified since the report is made after a specific transfer has been realized. In particular, the number of possible transfers no longer enters the number of deviation action plans. The result is that with double reporting the number of constraints is approximately linear in the number of possible transfers, and grows exponentially only in the number of possible actions. In environments with a fine grid for the transfer, the number of constraints in Program 2 is dramatically lower than in Program 1. Program 2 is derived from the general planning problem with the same methods as Program 1, and the two programs are therefore equivalent.

Notice that the advantages of using double reporting are similar to the advantages of using a recursive formulation in the first place. One of the key advantages of a recursive formulation with utility promises as a state variable is that only one-shot deviations need to be considered. The agent knows that his future utility promise will be delivered, and therefore does not need to consider deviations that extend over multiple periods. Double reporting applies a similar intuition to the incentive constraints within a period. The agent knows that it will be in his interest to return to the equilibrium path in the second part of the period (from the second report on), which simplifies the incentive constraints in the first part of the period.

An alternative method for reducing the number of truth-telling constraints is to let the planner choose off-path utility bounds directly (see Program 3 in Section 5.2 below). This technique derives from Prescott (1997), who uses the same approach in a static moral-hazard framework. The planner now specifies upper bounds to the utility an agent can get by lying and receiving a specific transfer and recommendation afterwards. The truth-telling constraints can then be formulated in a particularly simple way by summing over the utility bounds. Additional constraints ensure that the utility bounds hold, i.e., the actual utility of deviating must not exceed the utility bound regardless what action the agent takes. The number of such constraints is equal to the product of the number of transfers and the square of the number of actions. The total number of constraints in Program 3 is approximately linear in the number transfers, and quadratic in the number of actions. Once again, putting structure off the equilibrium path leads to a reduction in the number of constraints. In Proposition 6, we show that Program 1 and Program 3 are equivalent. With Program 3, the planning problem can be solved with fine grids for all choice variables. In Appendix A.3, we describe how the dimensionality of this Program can be reduced even further by introducing subperiods as in Program 2. However, instead of reporting the endowment a second time, the period is subdivided completely, and utility promises in the middle of the period are assigned as an additional state variable. The reduction in the dimensionality of the programs comes at the expense of an increase in the number of programs that need to be computed.

3 The Model

In the following sections we develop a number of recursive formulations for a general mechanism design problem. For maximum generality, when deriving the different recursive formulations we concentrate on the case of infinitely many periods with unobserved endowments and actions in every period. With little change in notation, the formulations can be adapted to models with finitely many periods and/or partially observable endowments and actions.

3.1 Physical Setup

The physical setup is identical for all programs that we consider. At the beginning of each period the agent receives an income or endowment e from a finite set E. The income cannot be observed by the planner. Then the planner gives a transfer τ from a finite set T to the agent. A positive transfer can be interpreted as an indemnity and negative transfer as a premium. At the end of the period, the agent takes an action a from a finite set A. Again, the action is unobservable for the planner. In most examples below we will concentrate on positive a and interpret that as storage or investment, but without any changes in the setup we could also allow for a to be negative, which can be interpreted as borrowing. The interpretation is the usual small-economy one, with unrestricted access to outside credit markets. Indeed, we can think of approximating a continuum of possible values of actions A by a finite set and similarly for the set of realized incomes E.[4] The agent consumes the amount $e+\tau-a$ and enjoys period utility $u(e+\tau-a)$, where the utility function is defined on $E \times T \times A$. Our methods do not require any specific assumptions on the utility function $u(\cdot)$, apart from it being real-valued. Indeed, we could separate consumption $e+\tau$ from the action a and write $U(e+\tau, a)$, though we would lose the credit market or investment interpretation.

[4]With non-stochastic saving, next period's beginning-of-period "endowment" would be random income plus the return on saving. If income has finite support, then for a given saving decision, certain income realizations would be impossible. This complicates but does not invalidate our methods—we could impose zero consumption for detectable off-path behavior, and continue as below.

The action a influences the probability distribution over the income or endowment in the next period. Probability $\mu(e|a)$ denotes the probability of endowment e if the agent took action a in the previous period. The word "endowment" is thus a misnomer as income next period is endogenous, a function of investment or unobserved credit-market activity, through the return on the risky investment. It is only in the initial period that the probability $\mu(e)$ of endowment e does not depend on any prior actions. For tractability, and consistent with the classic formulation of a moral-hazard problem, we assume that all states occur with positive probability, regardless of the action:

Assumption 1 *The probability distribution over the endowment e satisfies $\mu(e|a) > 0$ for all $e \in E$, all $a \in A$.*

Otherwise, we place no restrictions on the investment technology. Note that since yesterday's action affects probabilities over endowments only in the current period[5], the income realization completely describes the current environment as far as the agent is concerned. Likewise, with time-separable utility, previous actions and endowments do not enter current utility as a state variables. Apart from physical transactions, there is also communication taking place between the agent and the planner. We do not place any prior restrictions on this communication, in order not to limit the attainable outcomes. At a minimum, the agent has to be able send a signal about his beginning-of-period endowment, and the planner has to be able to send a recommendation for the investment or unobserved action.

In what follows Q is the discount factor of the planner, and β is the discount factor of the agent. The planner is risk-neutral and minimizes the expected discounted transfer, while the agent maximizes expected discounted utility. The discount factor Q is given by $Q = \frac{1}{1+r}$, where r is taken to be the outside credit-market interest rate for borrowers and lenders in this small open economy. We assume that both discount factors are less than one so that utility is finite and our problem is well defined.

Assumption 2 *The discount factors Q and β of the planner and the agent satisfy $0 < Q < 1$ and $0 < \beta < 1$.*

When there are only finitely many periods, we only require that both discount factors be bigger than zero, because utility will still be well defined.

While we formulate the model in terms of a single agent, another powerful interpretation is that there is a continuum of agents with mass equal to unity. In that case, the probability of an event represents the fractions in the population experiencing that event. Here the planner is merely a programming device to compute an optimal allocation: when the discounted surplus of the continuum of agents is zero, then we have attained a Pareto optimum.

[5]This assumption could be weakened as in Fernandes and Phelan (2000). If the action had effects over multiple periods, additional state variables need to be introduced to recover a recursive formulation.

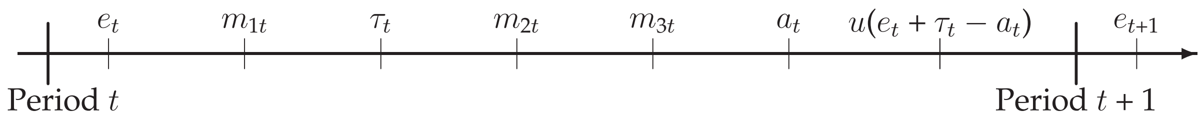

Figure 1: The Sequence of Events and Messages in Period t

3.2 The Planning Problem

We now want to formulate the Pareto problem of the planner maximizing surplus subject to providing reservation utility to the agent. Since the planner does not have any information on endowments and actions of the agent, we need to take a stand on what kind of communication is possible between planner and agent. In order not to impose any constraints from the outset, we start with a general communication game with arbitrary message spaces and full history-dependence. At the beginning of each period the agent realizes an endowment e. Then the agent sends a message or report m_1 to the planner, where m_1 is in a finite set M_1. Given the message, the planner assigns a transfer $\tau \in T$, possibly at random. Then the agent sends a second message m_2, where m_2 is in some finite set M_2. The planner responds by sending a message or recommendation $m_3 \in M_3$ to the agent, and M_3 is finite as well. Finally, the agent takes an action $a \in A$. In the direct mechanisms that we will introduce later, m_1 and m_2 will be reports on the endowment e, while m_3 will be a recommendation for the action a.[6]

We will use h_t to denote the realized endowment and all choices by planner and agent within period t:

$$h_t \equiv \{e_t, m_{1t}, \tau_t, m_{2t}, m_{3t}, a_t\}.$$

We denote the space of all possible h_t by H_t. The history up through time t will be denoted by h^t:

$$h^t \equiv \{h_{-1}, h_0, h_1, \ldots, h_t\}.$$

Here $t = 0$ is the initial date. The set of all possible histories up through time t is denoted by H^t and is thus given by:

$$H^t \equiv H_{-1} \times H_0 \times H_1 \times \ldots \times H_t.$$

At any time t, the agent knows the entire history up through time $t-1$. On the other hand, the planner never sees the true endowment or the true action. We will use s_t and s^t

[6]It is customary in the literature to start with a direct mechanism from the outset, assuming that the revelation principle holds. We start with the more general setup, since we are going to derive two different direct mechanisms and need to show that they are equivalent to each other and to the more general setup. Specifically, Program 2 relies on the presence of the second report m_2.

to denote the part of the history known to the planner. We therefore have within period t

$$s_t \equiv \{m_{1t}, \tau_t, m_{2t}, m_{3t}\},$$

where the planner's history of the game up through time t will be denoted by s^t, and the set S^t of all histories up through time t is defined analogously to the set H^t above. Since the planner sees a subset of what the agent sees, the history of the planner is uniquely determined by the history of the agent. We will therefore write the history of the planner as a function $s^t(h^t)$ of the history h^t of the agent. There is no information present at the beginning of time, and consequently we define $h_{-1} \equiv s_{-1} \equiv \emptyset$.

The choices by the planner are described by a pair of outcome functions $\pi(\tau_t|m_{1t}, s^{t-1})$ and $\pi(m_{3t}|m_{1t}, \tau_t, m_{2t}, s^{t-1})$ which map the history up to the last period as known by the planner and events (messages and transfers) that already occurred in the current period into a probability distribution over transfer τ_t and a report m_{3t}. The choices of the agent are described by a strategy. A strategy consists of a function $\sigma(m_{1t}|e_t, h^{t-1})$ which maps the history up to the last period as known by the agent and the current endowment into a probability distribution over the first report m_{1t}, a function $\sigma(m_{2t}|e_t, m_{1t}, \tau_t, h^{t-1})$ which determines a probability distribution over the second report m_{2t}, and a function $\sigma(a_t|e_t, m_{1t}, \tau_t, m_{2t}, m_{3t}, h^{t-1})$ which determines the action.

We use $p(h^t|\pi, \sigma)$ to denote the probability of history h^t under a given outcome function π and strategy σ. The probabilities over histories are defined recursively, given history h^{t-1} and action $a_{t-1}(h^{t-1})$, by:

$$\begin{aligned} p(h^t|\pi, \sigma) = p(h^{t-1}|\pi, \sigma)\; \mu(e_t|a_{t-1}(h^{t-1}))\; \sigma(m_{1t}|e_t, h^{t-1})\; \pi(\tau_t|m_{1t}, s^{t-1}(h^{t-1})) \\ \sigma(m_{2t}|e_t, m_{1t}, \tau_t, h^{t-1})\; \pi(m_{3t}|m_{1t}, \tau_t, m_{2t}, s^{t-1}(h^{t-1}))\; \sigma(a_t|e_t, m_{1t}, \tau_t, m_{2t}, m_{3t}, h^{t-1}). \end{aligned}$$

Also, $p(h^t|\pi, \sigma, h^k)$ is the conditional probability of history h^t given that history h^k occurred with $k \leq t$, and conditional probabilities are defined analogously. In the initial period, probabilities are given by:

$$\begin{aligned} p(h^0|\pi, \sigma) = \mu(e_0)\; \sigma(m_{10}|e_0)\; \pi(\tau_0|m_{10}) \\ \sigma(m_{20}|e_0, m_{10}, \tau_0)\; \pi(m_{30}|m_{10}, \tau_0, m_{20})\; \sigma(a_0|e_0, m_{10}, \tau_0, m_{20}, m_{30}). \end{aligned}$$

For a given outcome function π and strategy σ, the expected utility of the agent is:

$$U(\pi, \sigma) \equiv \sum_{t=0}^{\infty} \beta^t \left[\sum_{H^t} p(h^t|\pi, \sigma) u(e_t + \tau_t - a_t) \right]. \tag{1}$$

The expression above represents the utility of the agent as of time zero. We will also require that the agent use a maximizing strategy at all other nodes, even if they occur with probability zero. The utility of the agent given that history h^k has already been realized is given by:

$$U(\pi, \sigma | h^k) \equiv \sum_{t=k+1}^{\infty} \beta^{t-1-k} \left[\sum_{H^t} p(h^t | \pi, \sigma, h^k) u(e_t + \tau_t - a_t) \right]. \tag{2}$$

We now define an *optimal strategy* σ for a given outcome function π as a strategy that maximizes the utility of the agent at all nodes. The requirement that the strategy be utility maximizing can be described by a set of inequality constraints. Specifically, for a given outcome function π, for any alternative strategy $\hat{\sigma}$, and any history h^k, an optimal strategy σ has to satisfy:

$$\forall \hat{\sigma}, h^k: \quad U(\pi, \hat{\sigma} | h^k) \leq U(\pi, \sigma | h^k). \tag{3}$$

Inequality (3) thus imposes or describes optimization from any history h^k on.

In addition, we also require that the strategy be optimal at any node that starts after an arbitrary first report in a period is made, i.e., even if in any period $k+1$ the first report were generated by a strategy $\hat{\sigma}$, it is optimal to revert to σ from the second report in period $k+1$ on. For any alternative strategy $\hat{\sigma}$ and any history h^k, an optimal strategy σ therefore also has to satisfy:

$$\begin{aligned} \forall \hat{\sigma}, h^k: \quad & U(\pi, \hat{\sigma} | h^k) \\ \leq & \sum_{h_{k+1}} \mu(e_{k+1} | a_k(h^k)) \; \hat{\sigma}(m_{1k+1} | e_{k+1}, h^k) \; p(\tau_{k+1}, m_{2k+1}, m_{3k+1}, a_{k+1} | e_{k+1}, m_{1k+1}, \pi, \sigma, h^k) \\ & \left[u(e_{k+1} + \tau_{k+1} - a_{k+1}) + \beta U(\pi, \sigma | h^{k+1}) \right]. \end{aligned} \tag{4}$$

Notice that on the right-hand side the first report m_{1k+1} is generated by strategy $\hat{\sigma}$, the remaining messages, transfers, and actions are generated under π and σ, as captured by the second term $p(\cdot|\cdot)$, and the future is generated by π and σ as well. This condition is not restrictive, since by (3) even without this condition the agent chooses the second report optimally conditional on any first report that occurs with positive probability. The only additional effect of condition (4) is to impose or describe that the agent chooses the second report and action optimally even conditional on first reports that occur with zero probability under σ, but could be generated under a counterfactual strategy. Describing optimal behavior off the equilibrium path will help us later in deriving recursive formulations of the planning problem that can be computed efficiently.

We are now able to provide a formal definition of an optimal strategy:

Definition 1 *Given an outcome function π, an optimal strategy σ is a strategy such that inequalities* (3) *and* (4) *are satisfied for all k, all $h^k \in H^k$, and all alternative strategies $\hat{\sigma}$.*

Of course, for $h^k = h^{-1}$ this condition includes the maximization of expected utility (1) at time zero.

We imagine the planner as choosing an outcome function and a corresponding optimal strategy subject to the requirement that the agent realize at least reservation utility, W_0:

$$U(\pi, \sigma) \geq W_0. \tag{5}$$

Definition 2 *An equilibrium* $\{\pi, \sigma\}$ *is an outcome function* π *together with a corresponding optimal strategy* σ *such that* (5) *holds, i.e., the agent realizes at least his reservation utility. A feasible allocation is a probability distribution over endowments, transfers and actions that is generated by an equilibrium.*

The set of equilibria is characterized by the promise-keeping constraint (5), by the optimality conditions (3) and (4), and of course a number of adding-up constraints that ensure that both outcome function and strategy consist of probability measures. For brevity these latter constraints are not written explicitly.

The objective function of the planner is:

$$V(\pi, \sigma) \equiv \sum_{t=0}^{\infty} Q^t \left[\sum_{H^t} p(h^t|\pi, \sigma)(-\tau_t) \right] \tag{6}$$

When there is a continuum of agents, there is no aggregate uncertainty, and (6) is the actual surplus of the planner, or equivalently, the surplus of the community as a whole. Logically, that surplus should be set to zero. In the single-agent interpretation, there is uncertainty about the realization of transfers, and (6) is the expected surplus. In either case, the planner's problem is to choose an equilibrium that maximizes (6). By construction, this will be Pareto optimal (or at least necessary for an optimum). The Pareto frontier can be traced out by varying reservation utility W_0, and with a continuum of households, by picking the W_0 that generates zero surplus for the planner.

Definition 3 *An optimal equilibrium is an equilibrium that solves the planner's problem.*

Proposition 1 *There are reservation utilities* $W_0 \in R$ *such that an optimal equilibrium exists.*

4 Deriving a Recursive Formulation

4.1 The Revelation Principle

Our ultimate aim is to find a computable, recursive formulation of the planning problem. We begin by showing that without loss of generality we can restrict attention to a direct mechanism where there is just one message space each for the agent and the planner. The message space of the agent will be equal to the space of endowments E, and the agent will be induced to tell the truth. The message space for the planner will be equal to the space of actions A, and it will be in the interest of the agent to follow the recommended action. Since we fix the message spaces and require that truth-telling and obedience be optimal for the agent, instead of allowing any optimal strategy as before, it has to be the case that the set of feasible allocations in this setup is no larger than in the general setup with arbitrary message spaces. The purpose of this section is to show that the set of feasible allocations is in fact identical. Therefore there is no loss of generality in restricting attention to truth-telling and obedience from the outset.[7]

More formally, we consider the planning problem described above under the restriction that $M_1 = E$ and $M_3 = A$. M_2 is set to be a singleton, so that the agent does not have an actual choice over the second report. For simplicity, we will suppress m_2 in the notation below. We can then express the contemporary part of the history of the planner as:

$$s_t \equiv \{e_t, \tau_t, a_t\},$$

with history s^t up through time t defined as above. Notice that since we are considering the history of the planner, e_t is the *reported*, not necessary actual, endowment, and a_t is the *recommended* action, not necessarily the one actually taken. This will be different once we arrive at the mechanism with truth-telling and obedience, where reported and actual endowment and recommended and actual action always coincide.

As before, the planner chooses an outcome function consisting of probability distributions over transfers and reports. For notational convenience, we express the outcome function as the joint probability distribution over combinations of transfer and recommendation. This is equivalent to choosing marginal probabilities as above. The planner therefore chooses probabilities $\pi(\tau_t, a_t|e_t, s^{t-1})$ that determine the transfer τ_t and the recommended action a_t as a function of the reported endowment e_t and the history up to the last period s^{t-1}.

We now impose constraints on the outcome function π that ensure that the outcome function together with a specific strategy of the agent, namely truth-telling and obedience, are an equilibrium. First, the outcome function has to define probability measures. We require that $\pi(\tau_t, a_t|e_t, s^{t-1}) \geq 0$ for all transfers, actions, endowments and histories, and that:

$$\forall e_t, s^{t-1}: \quad \sum_{T,A} \pi(\tau_t, a_t|e_t, s^{t-1}) = 1. \tag{7}$$

Given an outcome function, we define probabilities $p(s^t|\pi)$ over histories in the obvious way, where the notation for σ is suppressed on the premise that the agent is honest and obedient. Given these probabilities, as in (5), the outcome function has to deliver reser-

[7] We still prefer to start from the general setup (as opposed to just assuming that the revelation principle holds) since we are deriving two different direct mechanisms from the same general setup, and we need to show that they are equivalent. More precisely, in this section we develop a formulation where the endowment is reported just once, whereas in Section 5.1 we derive an alternative formulation where the second report corresponds to reporting the endowment a second time.

vation utility W_0 to the agent, provided that the agent reports truthfully and takes the recommended actions:

$$\sum_{t=0}^{\infty} \beta^t \left[\sum_{S^t} p(s^t|\pi) u(e_t + \tau_t - a_t) \right] \geq W_0. \tag{8}$$

Finally, it has to be optimal for the agent to tell the truth and follow the recommended action, so that (3) holds for the outcome function and the maximizing strategy σ of the agent, which is to be truthful and obedient. In particular, the utility of honesty and obedience must weakly dominate the utility derived from any possible deviation strategy mapping any realized history, which may be generated by possible earlier lies and disobedient actions, into possible lies and alternative actions today, with plans for possible deviations in the future. We write a possible deviation strategy δ, which is allowed to be fully history-dependent, as a set of functions $\delta_e(h^{t-1}, e_t)$ that determine the reported endowment as a function of the actual history h^{t-1} and the true endowment e_t, and functions $\delta_a(h^{t-1}, e_t, \tau_t, a_t)$ that determine the actual action as a function of the history h^{t-1}, endowment e_t, transfer τ_t, and recommended action a_t. Since the actual action may be different from the recommendation, this deviation also changes the probability distribution over histories and states. The agent takes this change into account, and the changed probabilities are denoted as $p(h^t|\pi, \delta)$, with the inclusion of other conditioning elements where appropriate. In particular, we require that the actions of the agent be optimal from any history of the planner s^k on, and it will also be useful to write down separate constraints for each possible endowment e_{k+1} in period $k+1$. Then for every possible deviation (δ_e, δ_a), any history s^k, and any e_{k+1}, the outcome function has to satisfy:

$$\forall \delta, s^k, e_{k+1}: \quad \sum_{t=k+1}^{\infty} \beta^t \left[\sum_{H^t} p(h^t|\pi, \delta, s^k, e_{k+1}) u(e_t + \tau_t - \delta_a(h^{t-1}, e_t, \tau_t, a_t)) \right]$$
$$\leq \sum_{t=k+1}^{\infty} \beta^t \left[\sum_{S^t} p(s^t|\pi, s^k, e_{k+1}) u(e_t + \tau_t - a_t) \right]. \tag{9}$$

Here $p(h^t|\pi, \delta, s^k, e_{k+1})$ on the left-hand side is the probability of actual history h^t implied by outcome function π and deviation δ, conditional on the planner's history s^k and realized endowment e_{k+1}, and the $p(s^t|\pi, s^k, e_{k+1})$ on the right-hand side is the probability under truth-telling and obedience as above, but now conditioned on s^k and e_{k+1}. Condition (9) imposes or describes honesty and obedience on the equilibrium path, similar to (3).

It might seem at first sight that (9) is less restrictive than (3), because only a subset of possible deviations is considered. Specifically, deviations are nonrandom, and a constraint is imposed only at every s^t node, instead of every node h^k of the agent's history. However, none of these limitations are restrictive. Allowing for randomized deviations would lead to constraints which are linear combinations of the constraints already imposed. Imposing (9) is therefore sufficient to ensure that the agent cannot gain from randomized deviations. Also, notice that the conditioning history s^k enters (9) only by affecting probabilities over future states s^t through the history-dependent outcome function π. These probabilities are identical for all h^k that coincide in the s^k part once e_{k+1} is realized. The agent's private information on past endowments e and actions a affects the present only through the probabilities over different future endowments. Imposing a separate constraint for each h^k therefore would not put additional restrictions on π.

Definition 4 *An outcome function is an equilibrium outcome function under truth-telling and obedience if it satisfies the constraints* (7), (8) *and* (9) *above. A feasible allocation in the truth-telling mechanism is a probability distribution over endowments, transfers and actions that is implied by an equilibrium outcome function.*

Feasible allocations under truth-telling and obedience are a subset of the feasible allocations in the general setup, since (8) implies that (5) holds, (9) implies that (3) holds, and (4) does not constrain allocations and could be satisfied by specifying off-path behavior appropriately. In fact, we can show that the set of feasible allocations in the general and the restricted setup are in fact identical.

Proposition 2 (Revelation Principle) *For any message spaces M_1, M_2, and M_3, any allocation that is feasible in the general mechanism is also feasible in the truth-telling-and-obedience mechanism.*

The proof (outlined in the appendix) takes the usual approach of mapping an equilibrium of the general setup into an equilibrium outcome function in the restricted setup. Specifically, given an equilibrium (π, σ) in the general setup, the corresponding outcome function in the restricted setup is gained by prescribing the outcomes on the equilibrium path, while integrating out all the message spaces:

$$\pi^\star(\tau_t, a_t|e_t, s^{t-1}) \equiv \sum_{H^{t-1}(s^{t-1}),M_1,M_2,M_3} p(h^{t-1}|s^{t-1})\ \sigma(m_{1t}|e_t, h^{t-1})\ \pi(\tau_t|m_{1t}, s^{t-1}(h^{t-1}))$$
$$\sigma(m_{2t}|e_t, m_{1t}, \tau_t, h^{t-1})\ \pi(m_{3t}|m_{1t}, \tau_t, m_{2t}, s^{t-1}(h^{t-1}))\ \sigma(a_t|e_t, m_{1t}, \tau_t, m_{2t}, m_{3t}, h^{t-1}).$$

The proof then proceeds by showing that the outcome function $\pi^\star$ on the left-hand side satisfies all the required constraints. The essence of the matter is that lying or deviating under the new outcome function would be equivalent to using the optimizing strategy function under the original outcome function, but evaluated at a counterfactual realization. For example, an agent who has endowment e but reports $\hat{e}$ will face the same probability distribution over transfers and recommendations as an agent who under the original outcome function behaved "as if" the endowment were $\hat{e}$. The agent can never gain this way, since σ is an optimal strategy, and it is therefore preferable to receive the transfers and recommendations intended for endowment e instead of $\hat{e}$.

We are therefore justified in continuing with the restricted setup which imposes truth-telling and obedience. The objective function of the planner is now:

$$V(\pi) \equiv \sum_{t=0}^{\infty} Q^t \left[\sum_{S^t} p(s^t|\pi)(-\tau_t) \right], \tag{10}$$

and the original planning problem can be expressed as maximizing (10) subject to (7), (8), and (9) above.

4.2 Utility Vectors as State Variables

We now have a representation of the planning problem that requires truth-telling and obedience, and yet does not constitute any loss of generality. However, we still allow fully history-dependent outcome functions. The next step is to reduce the planning problem to a recursive version with a vector of promised utilities as the state variable.

We wish to work towards a problem in which the planner has to deliver a vector of promised utilities at the beginning of period k, with elements depending on the endowment e_k. It will be useful to consider an auxiliary problem in which the planner has to deliver a vector of reservation utilities $\mathbf{w}_0$, depending on the endowment in the initial period. The original planning problem can then be cast, as we shall see below, as choosing the vector of initial utility assignments $\mathbf{w}_0$ which yields the highest expected surplus for the planner, given the initial exogenous probability distribution over states $e \in E$ at time $t = 0$.

In the auxiliary planning problem, we impose the same probability constraints (7) and incentive constraints (9) as before. However, instead of a single promise-keeping constraint (8) there is now a separate promise-keeping constraint for each possible initial endowment. For all e_0, we require:

$$\forall e_0 :$$

$$\sum_{T,A} \pi(\tau_0, a_0|\mathbf{w}_0, e_0) \left[u(e_0 + \tau_0 - a_0) + \sum_{t=1}^{\infty} \beta^t \Big[\sum_{S^t} p(s^t|\pi, s_0) u(e_t + \tau_t - a_t) \Big] \right] = w_0(e_0). \tag{11}$$

Here the vector $\mathbf{w}_0$ of endowment-specific utility promises $w_0(e_0)$ is taken as given. Notice that we write the outcome function π as a function of the vector of initial utility promises $\mathbf{w}_0$. In period 0, there is no prior history, but in a subsequent period t the outcome function also depends on the history up to period $t-1$, so that the outcome function would be written as $\pi(\tau_t, a_t|\mathbf{w}_0, e_t, s^{t-1})$.

In principle, specifying a separate utility promise for each endowment is more restrictive than a requiring that a scalar utility promise be delivered in expected value across endowments. However, the original planning problem can be recovered by introducing an initial stage at which the initial utility vector is chosen by the planner. Indeed one might think that we should stick to using a scalar utility promise for the next period, not a vector over state-contingent e'. This would be closer to the timing of Fernandes and Phelan, but here with e' unobserved we would need to be explicit about incentives to announce e', leading naturally to state-contingent utilities and our formulation.

Since the vector of promised utilities $\mathbf{w}_0$ will serve as our state variable, it will be important to show that the set of all feasible utility vectors has nice properties.

Definition 5 *The set* $\mathbf{W}$ *is given by all vectors* $\mathbf{w}_0 \in R^{\#E}$ *that satisfy constraints* (7), (9), *and* (11) *for some outcome function* $\pi(\tau_t, a_t | e_t, s^{t-1})$.

Proposition 3 *The set* $\mathbf{W}$ *is nonempty and compact.*

Now we consider the problem of a planner who has promised utility vector $\mathbf{w}_0 \in \mathbf{W}$ and has received report e_0 from the agent. In the auxiliary planning problem, the maximized surplus of the planner is given by:

$$V(\mathbf{w}_0, e_0) = \max_{\pi} \sum_{T,A} \pi(\tau_0, a_0 | \mathbf{w}_0, e_0) \left[-\tau_0 + \sum_{t=1}^{\infty} Q^t \Big[\sum_{S^t} p(s^t | \pi, s_0)(-\tau_t) \Big] \right], \tag{12}$$

where the maximization over current and future π is subject to constraints (7), (9), and (11) above, for a given $\mathbf{w}_0 \in \mathbf{W}$ and $e_0 \in E$.

We want to show that this problem has a recursive structure. To do this, we need to define on-path future utilities that result from a given choice of π. For all s^{k-1}, e^k, let:

$$w(e_k, s^{k-1}, \pi) =$$
$$\sum_{T,A} \pi(\tau_k, a_k | \mathbf{w}_0, e_k, s^{k-1}) \left[u(e_k + \tau_k - a_k) + \sum_{t=k+1}^{\infty} \beta^{t-k} \Big[\sum_{S^t} p(s^t | \pi, s^k) u(e_t + \tau_t - a_t) \Big] \right], \tag{13}$$

and let $\mathbf{w}(s^{k-1}, \pi)$ be the vector of these utilities over all e_k. We can now show a version of the principle of optimality for our environment:

Proposition 4 *For all* $\mathbf{w}_0 \in \mathbf{W}$ *and* $e_0 \in E$*, and for any* s^{k-1} *and* e_k*, there is an optimal contract* $\pi^\star$ *such that the remaining contract from* s^{k-1} *and* e_k *on is an optimal contract for the auxiliary planning problem with* $e_0 = e_k$ *and* $\mathbf{w}_0 = \mathbf{w}(s^{k-1}, \pi^\star)$.

Thus the planner is able to reoptimize the contract at any future node. For Proposition 4 to go through, it is essential that we chose a vector of utility promises as the state variable, as opposed to the usual scalar utility promise which is realized in expected value across states. If the planner reoptimized given a scalar utility promise at a given date, the distribution of expected utilities across states might be different than in the original contract. Such a reallocation of utilities would change the incentives for lying and disobedience in the preceding period, so that incentive-compatibility of the complete contract would no longer be guaranteed. This problem is avoided by specifying a separate utility promise for each possible endowment. Likewise, in implementing the utility promises it does not matter whether the agent lied or was disobedient in the past, since the agent has to report the realized endowment anyway, and once the endowment is realized past actions have no further effects.[8]

Given Proposition 4, we know that the maximized surplus of the planner can be written as:

$$V(\mathbf{w}_0, e_0) = \sum_{A,T} \pi^\star(\tau_0, a_0|\mathbf{w}_0, e_0) \left[-\tau_0 + Q \sum_E \mu(e_1|a_0(s^0)) V(\mathbf{w}(s^0, \pi^\star), e_1) \right]. \tag{14}$$

In light of (14), we can cast the auxiliary planning problem as choosing transfers and actions in the initial period, and choosing continuation utilities from the set $\mathbf{W}$, conditional on history $s^0 = \{e_0, \tau_0, a_0\}$.

We are now close to the recursive formulation of the planning problem that we are looking for. We will drop the time subscripts from here on, and write the choices of the planner as a function of the current state, namely the vector of promised utilities w that has to be delivered in the current period, and the reported endowment e. The choices are functions $\pi(\tau, a|\mathbf{w}, e)$ and $\mathbf{w}'(\mathbf{w}, e, \tau, a)$, where $\mathbf{w}'$ is the vector of utilities promised from tomorrow on, which is restricted to lie in $\mathbf{W}$. Assuming that the value function V is known (it needs to be computed in practice), the auxiliary planning problem can be solved by solving a static optimization problem for all vectors in $\mathbf{W}$. An optimal contract for the non-recursive auxiliary planning problem can be found by assembling the appropriate solutions of the static problem.

We still need to determine which constraints need to be placed on the static choices $\pi(\tau, a|\mathbf{w}, e)$ and $\mathbf{w}'(\mathbf{w}, e, \tau, a)$ in order to guarantee that the implied contract satisfies probability measure constraints (7), maximization (9), and promise keeping (11) above. In order to reproduce (7), we need to impose:

$$\sum_{T,A} \pi(\tau, a|\mathbf{w}, e) = 1. \tag{15}$$

The promise-keeping constraint (11) will be satisfied if we impose:

$$\sum_{T,A} \pi(\tau, a|\mathbf{w}, e) \left[u(e + \tau - a) + \beta \sum_E \mu(e'|a) w'(\mathbf{w}, e, \tau, a)(e') \right] = w(e) \tag{16}$$

[8] The state space would have to be extended further if the action affected outcomes for more than one period into the future.

where along the equilibrium path, honesty and obedience prevails in reports e and actions a. Note that $w'(\mathbf{w}, e, \tau, a)(e')$ is the appropriate scalar utility promise for e'. The incentive constraints are imposed in two parts. We first require that the agent cannot gain by following another action strategy $\delta_a(\tau, a)$, assuming that the reported endowment e was correct. Note that e enters the utility function as the actual value and as the conditioning element in π as the reported value.

$$\forall \delta_a :$$

$$\sum_{T,A} \pi(\tau, a|\mathbf{w}, e) \left[u(e + \tau - \delta_a(\tau, a)) + \beta \sum_E \mu(e'|\delta_a(\tau, a)) w'(\mathbf{w}, e, \tau, a)(e') \right] \leq w(e). \quad (17)$$

A similar constraint on disobedience is also required if the initial report was e, but the true state was $\hat{e}$, i.e., false reporting. Note that $\hat{e}$ enters the utility function as the actual value but e is the conditioning element in π on the left-hand side, and $w(\hat{e})$ is the on-path realized utility under honesty and obedience at $\hat{e}$.

$$\forall \hat{e} \neq e, \delta_a :$$

$$\sum_{T,A} \pi(\tau, a|\mathbf{w}, e) \left[u(\hat{e} + \tau - \delta_a(\tau, a)) + \beta \sum_E \mu(e'|\delta_a(\tau, a)) w'(\mathbf{w}, e, \tau, a)(e') \right] \leq w(\hat{e}). \quad (18)$$

Conditions (17) and (18) impose a sequence of period-by-period incentive constraints on the implied full contract. The constraints rule out that the agent can gain from disobedience or misreporting in any period, given that he goes back to truth-telling and obedience from the next period on. Equations (17) and (18) therefore imply that (9) holds for one-shot deviations. We still have to show that (17) and (18) are sufficient to prevent deviations in multiple periods, but the argument follows as in Phelan and Townsend (1991). That is, for a finite number of deviations, we can show that the original constraints are satisfied by backward induction. The agent clearly does not gain in the last period when he deviates, since this is just a one-time deviation and by (17) and (18) is not optimal. Going back one period, the agent has merely worsened his future expected utility by lying or being disobedient in the last period. Since one-shot deviations do not improve utility, the agent cannot make up for this. Going on this way, we can show by induction that any finite number of deviations does not improve utility. Lastly, consider an infinite number of deviations. Let us assume that there is a deviation that gives a gain of ϵ. Since $\beta < 1$, there is a period T such that at most $\epsilon/2$ utils can be gained from period T on. This implies that at least $\epsilon/2$ utils have to be gained until period T. But this contradicts our result that there cannot be any gain from deviations with a finite horizon.

Thus we are justified to pose the auxiliary planning problem as solving:

$$V(\mathbf{w}, e) = \max_{\pi \geq 0, \mathbf{w}'} \sum_{A,T} \pi(\tau, a|\mathbf{w}, e) \left[-\tau + Q \sum_E \mu(e'|a) V(\mathbf{w}'(\mathbf{w}, e, \tau, a), e') \right] \quad (19)$$

by choice of π and $\mathbf{w}'$, subject to constraints (15) to (18) above. Program 1 below is a version of this problem with a discrete grid for promised utilities as an approximation. We have assumed that the function $V(\mathbf{w}, e)$ is known. In practice, $V(\mathbf{w}, e)$ can be computed with standard dynamic programming techniques. Specifically, the right-hand side of (19) defines an operator T that maps functions $V(\mathbf{w}, e)$ into $TV(\mathbf{w}, e)$. It is easy to show, as in Phelan and Townsend (1991), that T maps bounded continuous functions into bounded continuous functions, and that T is a contraction. It then follows that T has a unique fixed point, and the fixed point can be computed by iterating on the operator T.

The preceding discussion was based on the assumption that the set $\mathbf{W}$ of feasible utility vectors is known in advance. In practice, $\mathbf{W}$ is not known and needs to be computed alongside the value function $V(\mathbf{w}, e)$. $\mathbf{W}$ can be computed with the dynamic-programming methods described in detail in Abreu, Pearce, and Stacchetti (1990). An outline of the method is given in Appendix A.2.

Finally, the entire discussion is easily specialized to the case of a finite horizon T. V_T would be the value function for period T, V_{T-1} for period $T-1$, $\mathbf{W_{T-1}}$ the set of feasible promised utilities at time $T-1$, and so on.

4.3 The Discretized Version

For numerical implementation of the recursive formulation of the planning problem, we require finite grids for all choice variables in order to employ linear programming techniques. $\#E$ is the number of grid points for the endowment, $\#T$ is the number of possible transfers, and $\#A$ is the number of actions. The vector of promised utilities is also assumed to be in a finite set $\mathbf{W}$, and the number of possible choices is $\#\mathbf{W}$. To stay in the linear programming framework, we let the planner choose a probability distribution over vectors of utility promises, instead of choosing a specific utility vector.[9] That is, τ, a, and $\mathbf{w}'$ are chosen jointly under π. Notice that while the finite grids for endowment, transfer, and action are features of the physical setup of the model, the finite grid for utility promises is merely a numerical approximation of the continuous set in our theoretical formulation.

With finite grids, the optimization problem of a planner who has promised vector $\mathbf{w}$ and *has received report* e is:

Program 1:

$$V(\mathbf{w}, e) = \max_{\pi \geq 0} \sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[-\tau + Q \sum\nolimits_E \mu(e'|a) V(\mathbf{w}', e')\right] \tag{20}$$

[9]This imposes no loss of generality. If the contract puts weight on more than one utility vector, the corresponding mixed contract is a feasible choice for the planner who wants to implement the implied expected utility vector. The planner therefore cannot do better by choosing lotteries. It is also not possible to do worse, since the planner is free to place all weight on just one utility vector.

subject to the constraints (21) to (24) below. The first constraint is that the $\pi(\cdot)$ sum to one to form a probability measure, as in (15):

$$\sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) = 1. \tag{21}$$

Second, the contract has to deliver the utility that was promised for state e, as in (16):

$$\sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(e+\tau-a) + \beta \sum\nolimits_E \mu(e'|a)w'(e')\right] = w(e). \tag{22}$$

Third, the agent needs incentives to be obedient. Corresponding to (17), for each transfer τ and recommended action a, the agent has to prefer to take action a over any other action $\hat{a} \neq a$:

$$\begin{aligned} \forall \tau, a, \hat{a} \neq a: \quad & \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(e+\tau-\hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a})w'(e')\right] \\ & \leq \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(e+\tau-a) + \beta \sum\nolimits_E \mu(e'|a)w'(e')\right]. \end{aligned} \tag{23}$$

Finally, the agent needs incentives to tell the truth, so that no agent with endowment $\hat{e} \neq e$ would find this branch attractive. Under the promised utility vector $\mathbf{w}$, agents at $\hat{e}$ should get $w(\hat{e})$. Thus, an agent who actually has endowment $\hat{e}$ but says e nevertheless must not get more utility than was promised for state $\hat{e}$. This has to be the case regardless whether the agent follows the recommendations for the action or not. Thus, for all states $\hat{e} \neq e$ and all functions $\delta : T \times A \rightarrow A$ mapping transfer τ and recommended action a into an action $\delta(\tau, a)$ actually taken, we require as in (18):

$$\begin{aligned} & \forall \hat{e} \neq e, \delta: \\ & \sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(\hat{e}+\tau-\delta(\tau,a)) + \beta \sum\nolimits_E \mu(e'|\delta(\tau,a))w'(e')\right] \leq w(\hat{e}). \end{aligned} \tag{24}$$

Note that similar constraints are written for the $\hat{e}$ problem, so that agents with $\hat{e}$ receive $w(\hat{e})$ from a constraint like (22). For a given vector of utility promises, there are $\#E$ Program 1's to solve.

Program 1 allows us to numerically solve the auxiliary planning problem for a given vector of utility promises, by using linear programming and iteration on the value function. To recover the original planning problem with a scalar utility promise W_0, we let the planner offer a lottery $\pi(\mathbf{w}|W_0)$ over utility vectors $\mathbf{w}$ before the first period starts and before e is known. The problem of the planner at this initial stage is:

$$V(W_0) = \max_{\pi \geq 0} \sum_{\mathbf{W}} \pi(\mathbf{w}|W_0) \left[\sum\nolimits_E \mu(e)V(e, \mathbf{w})\right] \tag{25}$$

subject to a probability and a promise-keeping constraint:

$$\sum_{\mathbf{w}} \pi(\mathbf{w}|W_0) = 1. \tag{26}$$

$$\sum_{\mathbf{w}} \pi(\mathbf{w}|W_0) \left[\sum\nolimits_E \mu(e)\ w(e)\right] \geq W_0. \tag{27}$$

The same methods can be used for computing models with finitely many periods. With finitely many periods, the value functions carry time subscripts. The last period T would be computed first, by solving Program 1 with all terms involving utility promises omitted. The computed value function $V_T(\mathbf{w}, e)$ for period T is then an input in the computation of the value function for period $T-1$. Moving backward in time, the value function for the initial period is computed last.

An important practical limitation of the approach outlined thus far is that the number of truth-telling constraints in Program 1 is very large, which makes computation practically infeasible even for problem with relatively small grids. For each state $\hat{e}$ there is a constraint for each function $\delta : T \times A \rightarrow A$, and there are $(\#A)^{(\#T\times\#A)}$ such functions. Unless the grids for τ and a are rather sparse, memory problems make the computation of this program infeasible. The total number of variables in this formulation, the number of objects under $\pi(\cdot)$, is $\#T \times \#A \times \#\mathbf{W}$. There is one probability constraint (21) and one promise-keeping constraint (22). The number of obedience constraints (23) is $\#T \times \#A \times (\#A - 1)$, and the number of truth-telling constraints (24) is $(\#E - 1) \times (\#A)^{(\#T\times\#A)}$. Thus, the number of constraints grows exponentially with the product of the grid sizes for actions and transfers.

As an example, consider a program with two states e, ten transfers τ, two actions a, and ten utility vectors $\mathbf{w}'$. With only ten points, the grids for transfers and utility promises are rather sparse. Still, for this example and a given vector of utility promises $\mathbf{w}$ and realized state e Program 1 is a linear program with 200 variables and 1,048,598 constraints. If we increase the number of possible actions a to ten, the number of truth-telling constraints alone is 10^{100}. Clearly, such programs will not be computable now or any time in the future. It is especially harmful that the grid size for the transfer causes computational problems, as it does here because of the dimensionality of $\delta(\tau, a)$. One can imagine economic environments in which there are only a small number of options for actions available, but it is much harder to come up with a reason why the planner should be restricted to a small set of transfers. In the next section we present alternative formulations, equivalent to the one developed here, that can be used for computing solutions in environments with many transfers and many actions.

5 Computationally Efficient Formulations

In this section we develop a series of alternative recursive formulations of the planning problem which are equivalent to Program 1, but require a much smaller number of constraints. The key method for reducing the number of constraints in the program is to allow the planner to specify behavior and utility promises off the equilibrium path. This can be achieved by multiple reporting, or by incorporating off-path utility bounds as choice variables for the planner. Further efficiency gains are possible if the period is divided into two subperiods with separate planning problems. We will address each method in turn.

5.1 A Version with Double Reporting

The basic idea of this section is to let the agent report the endowment a second time after the transfer is received, but before a recommendation for the action is received. On the equilibrium path, the agent will make the correct report twice and follow the recommended action, and the optimal allocation will be the same as in the first formulation. At first sight, it might therefore appear that the second report is not necessary, since in equilibrium it will always coincide with the first report. The advantage of double reporting is that it allows the planner to specify behavior off the equilibrium path, because outcomes are determined even if the two reports differ. We will see that this possibility leads to a significant reduction in the number of incentive constraints.[10]

To see how the version with double reporting works, it is instructive to retrace some of the steps which led from the general setup to Program 1. We started by applying the revelation principle to argue that without loss of generality we can restrict attention to direct mechanisms with truth-telling and obedience. The proof of the revelation principle proceeds by showing that any equilibrium outcome in the general setup can be mapped into an equivalent outcome function in the restricted setup. The outcome function π in the restricted setup is derived from an equilibrium outcome function and strategy (π, σ) in the general setup by integrating out the message spaces M_1, M_2, and M_3, and prescribing the outcomes that occur on the equilibrium path (see equation (45) in the Appendix):

$$\pi^\star(\tau_t, a_t|e_t, s^{t-1}) \equiv \sum_{H^{t-1}(s^{t-1}),M_1,M_2,M_3} p(h^{t-1}|s^{t-1})\ \sigma(m_{1t}|e_t, h^{t-1})\ \pi(\tau_t|m_{1t}, s^{t-1}(h^{t-1}))$$
$$\sigma(m_{2t}|e_t, m_{1t}, \tau_t, h^{t-1})\ \pi(m_{3t}|m_{1t}, \tau_t, m_{2t}, s^{t-1}(h^{t-1}))\ \sigma(a_t|e_t, m_{1t}, \tau_t, m_{2t}, m_{3t}, h^{t-1}). \quad (28)$$

[10] This idea comes naturally from the revelation principle. The agent is about to take an action conditioned on his immediate state, which is the actual endowment and realized transfer, plus any history dependence resulting from previous beginning-of-period reports of income. This bears a resemblance to Fernandes and Phelan, where previous announcement of income realizations are the natural state variable (and the utility is assigned accordingly).

While outcomes on the equilibrium path are unchanged, the original equilibrium (π, σ) contains information on outcomes off the equilibrium path which are lost in the switch to the truth-telling-and-obedience outcome function $\pi^\star$. Specifically, the original equilibrium strategy prescribes optimal behavior following any initial report m_{1t}, even after those reports which never actually occur in equilibrium. This would include reports that are generated under σ, but at a counterfactual e_t.

In the restricted setup with double reporting the two message spaces of the agent are given by the space of endowments, $M_1 = M_2 = E$, and the message space of the planner is the space of possible actions, $M_3 = A$. The planner chooses an outcome function $\pi^\star(\tau_t, a_t | e_{1t}, e_{2t}, s^{t-1})$ which determines the transfer and the action as a function of the two reported endowments by the agent and the history up to period $t-1$. Notice that since this function is also specified for the case that the two reports differ, the planner in effect specifies behavior off the equilibrium path. Exactly what happens if the two reports differ is tightly linked to the prescriptions for off-path behavior of the equilibrium (π, σ) in the general setup. Specifically, $\pi^\star(\tau_t, a_t | e_{1t}, e_{2t}, s^{t-1})$ describes the outcome if the actual endowment is e_{2t}, but the first report of the agent was governed by e_{1t}, as if the agent acted as if the true endowment were e_{1t} when making the first report. Formally, an equilibrium (π, σ) of the original game is transformed into a new outcome function $\pi^\star$ under double reporting in the following way:

$$\pi^\star(\tau_t, a_t | e_{1t}, e_{2t}, s^{t-1}) \equiv \sum_{H^{t-1}(s^{t-1}) M_1, M_2, M_3} p(h^{t-1}|s^{t-1})\ \sigma(m_{1t}|e_{1t}, h^{t-1})\ \pi(\tau_t | m_{1t}, s^{t-1}(h^{t-1}))$$
$$\sigma(m_{2t}|e_{2t}, m_{1t}, \tau_t, h^{t-1})\ \pi(m_{3t}|m_{1t}, \tau_t, m_{2t}, s^{t-1}(h^{t-1}))\ \sigma(a_t|e_{2t}, m_{1t}, \tau_t, m_{2t}, m_{3t}, h^{t-1}). \quad (29)$$

Notice that the first report m_{1t} is governed by e_{1t}, while the second report m_{2t} and the action a_t are governed by e_{2t}. If e_{1t} and e_{2t} coincide, the prescriptions are the usual on-path equilibrium outcomes, and (28) and (29) are the same. If e_{1t} and e_{2t} are different, the outcomes are as if the agent mistakenly assumed that the endowment was e_{1t} when making the first report, a non-maximizing strategy, but then realized that the endowment was actually e_{2t} when making the second report and choosing the action. In the general setup, such off-path behavior is always well defined, since the agent needs to specify a strategy for all possible initial reports, even those that occur with zero probability.

The advantage of using this information in a version with truth-telling and obedience is that the resulting outcome function satisfies a version of constraint (4), which states that the actions of the agent from the second report on have to be optimal *regardless* of what the first report was. Translated into the version with truth-telling and obedience, the constraint requires on the right-hand side of (30) below that after any history h^k, even if the first report in period $k+1$ were wrong and generated by $\delta_{e1}(h^k, e_{k+1})$, it would be optimal for the agent to tell the truth e_{k+1} at the second report in $k+1$, follow the recommended action a_{k+1}, and be honest and obedient in the future, instead of following some deviation

δ in the present and future as on the left-hand side:

$$\forall \delta, s^k, e_{k+1}: \quad \sum_{t=k+1}^{\infty} \beta^t \left[\sum_{H^t} p(h^t|\pi^\star, \delta, s^k, e_{k+1}) u(e_t + \tau_t - \delta_a(h^{t-1}, e_t, \tau_t, a_t)) \right]$$
$$\leq \beta^{k+1} \left[\sum_{T,A} \pi^\star \left(\tau_{k+1}, a_{k+1} | \delta_{e1}(h^k, e_{k+1}), e_{k+1}, s^k \right) u(e_{k+1} + \tau_{k+1} - a_{k+1}) \right]$$
$$+ \sum_{t=k+2}^{\infty} \beta^t \left[\sum_{S^t} p(s^t|\pi^\star, s^k, e_{k+1}) u(e_t + \tau_t - a_t) \right]. \quad (30)$$

This constraint is imposed in addition to a version of the usual, beginning-of-the-period incentive constraint (9). Imposing (30) simplifies the incentive constraints for being truthful at the first report, since the agent already knows that he will be truthful and obedient from the second report on, regardless of what he says at the first report. This leads to a dramatic reduction in the number of incentive constraints.

The derivation of the version with double reporting follows the same outline taken above for Program 1. Since the steps and proofs are virtually identical, we omit them here. The point is that both programs are derived from first principles. The apparently more general double-reporting version is however easily reduced to the single-reporting version along the equilibrium path. Thus both are equivalent.[11]

We go directly to the discretized recursive version of the planning problem with double reporting. We will concentrate on the differences between single and double reporting, and the efficiency gains resulting from specifying behavior off the equilibrium path. As in Program 1, the agent comes into the period with a vector of promised utilities $\mathbf{w}$. At the beginning of the period, the agent observes the state e and makes a first report to the planner. Then the planner delivers the transfer τ, and afterwards the agent reports the endowment e again. Incentive-compatibility constraints, based on (30), ensure that this second report will be correct, even if the first report were false. Because now the planner receives a report after the transfer, the number of possible transfers does not affect the number of truth-telling constraints, as it did in (24). This is the main source of the efficiency gain of double reporting.

Program 2:
The optimization problem of a planner who promised utility vector $\mathbf{w}$ and *has already received first report* e is:

$$V(\mathbf{w}, e) = \max_{\pi \geq 0} \sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, e) \left[-\tau + Q \sum\nolimits_E \mu(e'|a) V(\mathbf{w}', e') \right] \quad (31)$$

subject to constraints (32)-(37) below. Notice that the contract $\pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, e)$ is condi-

[11] The full derivations are carried out in Doepke and Townsend (2002).

tioned on two reports e, unlike in (20). The first constraint, much like (21), is that the $\pi(\cdot)$ form a probability measure for any second report $\hat{e}$. Note again that in the program endowment e is a fixed state or parameter.

$$\forall \hat{e}: \quad \sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \hat{e}) = 1. \tag{32}$$

Since the second report is made *after* the transfer, we have to enforce that the transfer does not depend on the second report. For all $\hat{e} \neq \tilde{e}$ and all τ, we require:

$$\forall \hat{e} \neq \tilde{e}, \tau: \quad \sum_{A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \hat{e}) = \sum_{A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \tilde{e}). \tag{33}$$

Given that the agent told the truth twice, the contract has to deliver the promised utility $w(e)$ for state e from vector $\mathbf{w}$. That is, much like (22) above:

$$\sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, e) \left[u(e + \tau - a) + \beta \sum\nolimits_E \mu(e'|a) w'(e')\right] = w(e). \tag{34}$$

Next, the agent needs to be obedient. Given that the second report is true, it has to be optimal for the agent to follow the recommended action a. For each true state $\hat{e}$, transfer τ, recommended action a, and alternative action $\hat{a} \neq a$ we require much like (23):

$$\forall \hat{e}, \tau, a, \hat{a} \neq a: \quad \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \hat{e}) \left[u(\hat{e} + \tau - \hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a}) w'(e')\right]$$
$$\leq \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \hat{e}) \left[u(\hat{e} + \tau - a) + \beta \sum\nolimits_E \mu(e'|a) w'(e')\right]. \tag{35}$$

We also have to ensure that the agent prefers to tell the truth at the second report, no matter what he reported the first time around. *Since the transfer is already known at the time the second report is made, the number of deviations from the recommended actions that we have to consider does not depend on the number of possible transfers*. For each actual $\hat{e}$, transfer τ, second report $\hat{\hat{e}} \neq \hat{e}$, and action strategy $\delta: A \to A$, we require:

$$\forall \hat{e}, \tau, \hat{\hat{e}} \neq \hat{e}, \delta: \quad \sum_{A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \hat{\hat{e}}) \left[u(\hat{e} + \tau - \delta(a)) + \beta \sum\nolimits_E \mu(e'|\delta(a)) w'(e')\right]$$
$$\leq \sum_{A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \hat{e}) \left[u(\hat{e} + \tau - a) + \beta \sum\nolimits_E \mu(e'|a) w'(e')\right]. \tag{36}$$

This last constraint is derived from constraint (30) above, and is specific to the version with double reporting. Finally, we also have to ensure that the first report e be correct. That is, an agent who is truly at state $\hat{e}$ and should get $w(\hat{e})$, but made a counterfactual first report e, cannot get more utility than was promised for state $\hat{e}$. For all $\hat{e} \neq e$ we require:

$$\forall \hat{e} \neq e: \quad \sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e, \hat{e}) \left[u(\hat{e} + \tau - a) + \beta \sum\nolimits_{E} \mu(e'|a) w'(e') \right] \leq w(\hat{e}). \tag{37}$$

Notice that these latter truth-telling constraints do not involve deviations in the action a. At the time of the first report the agent knows that the second report will be correct and that he will take the recommended action, because constraints (35) and (36) hold. In Program 1 the agent had to consider a much more complicated set of deviations when deciding on the the first report, resulting in a much higher number of constraints. Despite the differences in the sets of constraints, Program 1 and Program 2 are equivalent.

Proposition 5 *Program 1 and Program 2 are equivalent.*

The proof for this proposition proceeds by showing that Program 2 is equivalent to the general planning problem, using the same arguments as in the derivation of Program 1 above.[12] Since both Program 1 and Program 2 are equivalent to the general planning problem, they are also equivalent to each other.

The number of variables in this formulation is $\#E \times \#T \times \#A \times \#\mathbf{W}$. Thus, the number of variables increased relative to Program 1, since π now also depends on the second report $\hat{e}$. The number of constraints, however, is much lower than in Program 1. There are $\#E$ probability constraints (32), an increase, $(\#E - 1) \times \#T$ independence constraints (33), entirely new, and there is one promise-keeping constraint (34), as before. The total number of obedience constraints (35) is $\#E \times \#T \times \#A \times (\#A - 1)$, an increase. The number of truth-telling constraints for the second report (36) is $\#E \times \#T \times (\#E - 1) \times (\#A)^{(\#A)}$, and the number of truth-telling constraints for the first report (37) is $\#E - 1$. This is where we obtain a huge reduction in the number of constraints.

Going back to our example, consider a program with two states e, ten transfers τ, two actions a, and ten utility vectors $\mathbf{w}'$. For this example Program 2 has 400 variables and 134 constraints. Compared to Program 1, the number of variables increases by 200, but the number of constraints decreases by more than one million. This makes it possible to solve Program 2 on a standard personal computer. Program 2 does less well if the number of actions is large. If we increase the number of actions a to ten, Program 2 has 2000 variables and more than 10^{11} truth-telling constraints. This is many orders of magnitude smaller than Program 1, but still too big to be handled by standard computer hardware. The key advantage of Program 2 relative to Program 1 is that the number of constraints does not increase exponentially with the number of possible transfers τ. As long as the number of possible actions a is small, this formulation allows computation with fine grids for the other variables. However, the number of constraints still increases exponentially with the number of actions. In the next section we will present yet another formulation of our original program which solves this problem, once again by specifying outcomes off the equilibrium path.

[12] The proof is carried out in detail in Doepke and Townsend (2002).

5.2 A Version With Off-Path Utility Bounds

We saw already in the last section that specifying behavior off the equilibrium path can lead to a reduction in the number of incentive-compatibility constraints. We will now exploit this idea in a way similar to Prescott (1997) in order to reduce the number of truth-telling constraints. The choice variables in the new formulation include utility bounds $v(\cdot)$ that specify the maximum utility (that is, current utility plus expected future utility) an agent can get when lying about the endowment and receiving a certain recommendation. Specifically, for a given reported endowment e, $v(\hat{e}, e, \tau, a)$ is an upper bound for the utility of an agent who actually has endowment $\hat{e} \neq e$, reported endowment e nevertheless, and received transfer τ and recommendation a. An intermediate step would be to assign exact utilities for $\hat{e}, e, \tau, a$, in which case we have something like the threat keeping in Fernandes and Phelan (2000). But as in Prescott (1997), we do better by only imposing utility bounds. This utility bound is already weighted by the probability of receiving transfer τ and recommendation a. Thus, in order to compute the total expected utility that can be achieved by reporting e when the true state is $\hat{e}$, we simply have to sum the $v(\hat{e}, e, \tau, a)$ over all possible transfers τ and recommendations a. The truth-telling constraint is then that the utility of saying e when being at state $\hat{e}$ is no larger than the utility promise $w(\hat{e})$ for $\hat{e}$.

Program 3:
The optimization problem of the planner in this formulation given report e and promised utility vector $\mathbf{w}$ is:

$$V(\mathbf{w}, e) = \max_{\pi \geq 0, v} \sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[-\tau + Q \sum\nolimits_E \mu(e'|a) V(\mathbf{w}', e') \right] \tag{38}$$

subject to the constraints (39)-(43) below. Apart from the addition of the utility bounds $v(\cdot)$ the objective function (38) is identical to (20). The first constraint is the probability measure constraint, identical with (21):

$$\sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) = 1. \tag{39}$$

The second constraint is the promise-keeping constraint, identical with (22):

$$\sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(e + \tau - a) + \beta \sum\nolimits_E \mu(e'|a) w'(e') \right] = w(e). \tag{40}$$

We have to ensure that the agent is obedient and follows the recommendations of the planner, given that the report is true. For each transfer τ, recommended action a, and alternative action $\hat{a} \neq a$, we require as in (23):

$$\forall \tau, a, \hat{a} \neq a: \quad \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(e + \tau - \hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a}) w'(e') \right]$$
$$\leq \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(e + \tau - a) + \beta \sum\nolimits_E \mu(e'|a) w'(e') \right]. \qquad (41)$$

Next, the utility bounds have to be observed. An agent who reported state e, is in fact at state $\hat{e}$, received transfer τ, and got the recommendation a, cannot receive more utility than $v(\hat{e}, e, \tau, a)$, where again $v(\hat{e}, e, \tau, a)$ incorporates the probabilities of transfer τ and recommendation a. For each state $\hat{e} \neq e$, transfer τ, recommendation a, and all possible actions $\hat{a}$ we require:

$$\forall \hat{e} \neq e, \tau, a, \hat{a}: \quad \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(\hat{e} + \tau - \hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a}) w'(e') \right] \leq v(\hat{e}, e, \tau, a). \qquad (42)$$

Finally, the truth-telling constraints are that the utility of an agent who is at state $\hat{e}$ but reports e cannot be larger than the utility promise for $\hat{e}$. For each $\hat{e} \neq e$ we require:

$$\forall \hat{e} \neq e: \quad \sum_{T,A} v(\hat{e}, e, \tau, a) \leq w(\hat{e}). \qquad (43)$$

The number of variables in this problem is $\#T \times \#A \times \#\mathbf{W}$ under $\pi(\cdot)$ plus $(\#E-1) \times \#T \times \#A$, where the latter terms reflect the utility bounds $v(\cdot)$ that are now choice variables. There is one probability constraint (39) and one promise-keeping constraint (40). The number of obedience constraints (41) is $\#T \times \#A \times (\#A-1)$. There are $(\#E-1) \times \#T \times (\#A)^2$ constraints (42) to implement the utility bounds, and $(\#E - 1)$ truth-telling constraints (43). Notice that the number of constraints does not increase exponentially in any of the grid sizes. The number of constraints is approximately quadratic in $\#A$ and approximately linear in all other grid sizes. This makes it possible to compute models with a large number of actions. In Program 3, our example with two states e, ten transfers τ, two actions a, and ten utility vectors $\mathbf{w}'$ is a linear program with 220 variables and 63 constraints, even smaller than Program 2. For the first time, the program is still computable if we increase the number of actions a to ten. In that case, Program 3 has 1100 variables and 1903 constraints, which is still sufficiently small to be solved on a personal computer.

We now want to show that Program 3 is equivalent to Program 1. In both programs, the planner chooses lotteries over transfer, action, and promised utilities. Even though in Program 3 the planner also chooses utility bounds, in both programs the planner's utility depends only on the lotteries, and not on the bounds. The objective functions are identical. In order to demonstrate that the two programs are equivalent, it is therefore sufficient to show that the set of feasible lotteries is identical. We therefore have to compare the set of constraints in the two programs.

Proposition 6 *Program 1 and Program 3 are equivalent.*

The proof (contained in the Appendix) consists of showing that constraints (39)-(43) in Program 3 place the same restrictions on the outcome function $\pi(\cdot)$ as the constraints (21)-(24) of Program 1. The probability constraints (21) and (39), the promise-keeping constraints (22) and (40), and the obedience constraints (23) and (41) are in fact identical. Therefore one only needs to show that for any $\pi(\cdot)$ that satisfies the incentive constraints (24) in Program 1, one can find utility bounds such that the same outcome function satisfies (42) and (43) in Program 3 (and vice versa). Since the objective function is identical, it then follows that the programs are equivalent.

In Program 3, the number of constraints gets large if both the grids for transfer τ and action a are made very fine. In practice, this may lead to memory problems when computing. Further reductions in the number of constraints are possible in a formulation in which *the transfer and the recommendation are assigned at two different stages.* In other words, we are subdividing the period into two parts as in Program 2. However, instead of having a second report of the endowment, we subdivide the period completely and assign interim utility promises as an additional state variable. This procedure yields even smaller programs. However, the reduction comes at the expense of an increase in the number of programs that needs to be computed. The programs for both stages are described in Appendix A.3.

6 A Hidden Storage Application

In this section, we use our methods to compute optimal allocations in an environment where the hidden action is designed to resemble storage at a fixed return. The planner has access to an outside credit market at a fixed interest rate, and wants to provide insurance to an agent who has hidden income shocks. The agent can hide part of the income, and invest in a technology which raises the expected endowment in the following period. As discussed earlier, two special cases of this environment have been analyzed in the theoretical literature. Allen (1985) shows that if the hidden technology amounts to unobserved access to a perfect credit market at the same interest rate that the planner has access to, the principal cannot offer any insurance beyond self-insurance in the credit market. Cole and Kocherlakota (2001b) and Ljungqvist and Sargent (2003) extend this result to the case where the agent can only store the endowment, but not borrow. As long as the return on storage is identical to the planner's credit market return, the optimal outcome is once again equivalent to credit market access for the agent.

If the results of Allen (1985), Cole and Kocherlakota (2001b), and Ljungqvist and Sargent (2003) held more generally, the implementation of optimal policies would be straightforward: the planner only would have to provide credit market access to the agent, which is much simpler than the history-contingent transfer schemes that arise in environments without hidden actions. However, all of the literature relies on the assumption that planner and agent face the same return. Our recursive methods make it possible to compute outcomes for a variety of returns of the hidden storage technology, both below and above the credit-market interest rate. We find in these computations that once the hidden storage technology has a return sufficiently different from the credit market interest rate, the equivalence of optimal insurance and credit market access breaks down. This is true regardless of whether the return on storage is above or below the interest rate; it is only required that the deviation be sufficiently large.

We illustrate this result in an environment with three periods. Planner and agent have the same discount factor $\beta = Q = 1.05^{-1}$, corresponding to a risk-free interest rate of five percent for the planner. The agent has logarithmic utility. In each period there are two possible endowments e (2.1 or 3.1) and three storage levels a (0, 0.3, or 0.6). In periods one and two, the planner can use a grid of 50 transfers τ (from -1.4 to 1.4). In period three outcomes are easier to compute since there are no further actions utility promises, which allows us to use a grid with 4,000 grid points for transfers. In each period, the utility grids have 100 elements per endowment, so that (given two endowments) there are 10,000 possible utility vectors. In the initial period, each endowment occurs with probability 0.5. In all other periods, the probability distribution over endowments depends on the storage level in the previous period. If storage is zero, the high endowment occurs with probability 0.025. The probability of the high endowment increases linearly with storage. The expected return on storage R determines how fast the probability of the high endowment increases.[13] For example, for $R = 1$, an increase in storage of 0.1 results in an increase in the expected endowment of 0.1. Since the difference between the two endowments equals one ($3.1 - 2.1$), with $R = 1$ increasing the storage level a by 0.1 results in an increase in the probability of the high endowment of 3.1 by 0.1, and an equal decrease in the probability of the low endowment of 2.1.

We computed optimal policy functions for a variety of returns R as a function of the (scalar) initial utility promise W_0. Program 3 was used for all computations.[14] Figure

[13]Notice that our storage technology is non-standard in the sense that it is stochastic. Storage increases the probability of a high income realization within a given distribution, as opposed to shifting the entire distribution upwards. However, comparing our computations to existing theoretical results indicates that this choice does not have a major effect on the results (see below).

[14]The computations were carried out on a PC with an Intel P4 2.1 GHz CPU, 512 MB of RAM, and running Windows XP. The matrices were set up in MATLAB, and the optimization steps were carried out using the ILOG CPLEX 7.5 linear program solver. In the last period (period 3), each linear program had 4,000 variables and 3 constraints. Each of these programs took about 0.1 seconds to compute. In period 2, the programs had 34,800 variables and 753 constraints, resulting in a computing time of 0.25 seconds per program, and in period 1 the programs had 200,100 variables and 753 constraints, with a computation time of 1.8 seconds per program. The variation in the number of variables between periods one and two stems from differences in the number of feasible utility vectors. At each stage, a separate program needs to be computed for each endowment and each feasible utility vector. The actual number of programs to be computed varies, since some infeasible utility vectors are disregarded from the outset. On average, 1500 programs had to be computed for period 3, and 3000 programs for periods 1 and 2. The total computation time averaged 3 minutes for period 3, 45 minutes for period 2, and 280 minutes for period 1. In all computed

2 shows the policy functions for $R = 1.1$ as a function of the initial utility promise W_0 and the realized endowment in the first period (low or high, represented by solid and dashed lines in the graph). In all graphs a vertical line marks the initial utility level at which the expected discounted surplus of the planner is zero. At $R = 1.1$, the net expected return on storage is 10 percent, which exceeds the credit market interest rate of 5 percent. Consequently, the storage technology is used to the maximum of 0.6 regardless of the realized endowment. The planner uses transfers and utility promises to provide incentive-compatible insurance to the agent with the low income realization. The transfer for the low endowment is higher than for the high endowment, and at zero expected surplus high-income agents today pay premia, while low-income agents receive indemnities. The difference in consumption between low- and high-endowment agents is only about 0.3, even though the endowment differs by 1. In other words, the planner is able to insure about seventy percent the income risk in the first period. In order to ensure incentive compatibility (i.e., induce the high-endowment agent to report truthfully), the planner promises a schedule of higher future utility from tomorrow on to agents with the high endowment. Note that separate utilities are promised for each of the two possible incomes tomorrow (the promises for the low income are on the left, and those for the high income on the right). In the first period, naturally, both consumption and utility promises increase with initial promised utility. The dynamics are implicit in the utility promise transitions. For example, agents with high income today are pushed towards higher expected utility tomorrow.

The optimal insurance scheme for returns R below 1.05 (the market rate) is similar to the outcome for $R = 1.1$, with the key difference that the private storage technology is not used. Interestingly, the planner still does not recommend the use of private storage if the return on storage is slightly above the credit market return. Even though using storage would increase the expected endowment, there would also be added uncertainty since the storage technology is stochastic. Using private storage pays off only if the return is sufficiently far above the credit-market return to compensate for the added risk.

Figure 3 shows how the expected utility of the agent varies with the return on storage, subject to the requirement that the discounted surplus of the planner be zero. The utility of the agent is shown for values of R between 0.6 and 1.1 (below $R = 0.6$, utilities no longer change with R). The top line is the utility the agent gets under full information, that is, when both endowment and storage are observable by the planner. In this case the planner provides full insurance, and consumption and utility promises are independent of the endowment. It turns out that the utility of the agent does not vary with the return on storage as long as $R \leq 1.05$. This is not surprising, since 1.05 is the risk-free interest rate. As long as the return on storage does not exceed the credit-market return, raising the return on storage does not extend the unconstrained Pareto frontier. Storage is never

solutions to the initial planning problem (25), the surplus of the planner $V(W_0)$ is a strictly decreasing function of the utility promise W_0, implying that the promise-keeping constraint (27) is always binding. This also means that in future periods it is not in the interest of both the agent and the principal to tear up the contract and start over in such a way as to make both better off.

used, and the utility of the agent does not depend on the return on storage. When R exceeds 1.05, the Pareto frontier expands and the utility of the agent increases dramatically. In effect, when $R > 1.05$ an arbitrage opportunity arises which is only limited by the upper bound on storage.

The lower line shows the utility of the agent under autarky. When the return on storage is sufficiently high (above about 0.91, still a relatively dismal return) the agent uses storage to self-insure against the income shocks. Since under autarky storage is the only insurance for the agent, utility increases with the return on storage.

The solid line shows the utility of the agent under optimal incentive-constrained insurance with hidden endowments and hidden actions. Once the return on storage exceeds some critical value (about 0.7), the utility of the agent decreases with the return on storage, instead of increasing as it does under autarky. As long as the planner never recommends positive private storage, a higher return on storage has no positive effects. On the contrary, with a high return on storage it becomes harder to satisfy the obedience constraints which require that the agent does not prefer a positive storage level when the planner recommends zero storage. Raising the return on storage shrinks the set of feasible allocations, and the utility of the agent falls. Conversely, lowering the return on private storage from about R = 1.05 to R = 0.7 steadily increases the expected utility of the agent. Hence, the properties of the storage technology influence the optimal insurance contract even if private storage is never used in equilibrium. For virtually all returns R less than the outside financial market return, the agent is strictly better off than he would be if given unrestricted access to borrowing and lending at the outside financial market rate of return. When R exceeds the credit-market return by a sufficient margin, private storage is used in the constrained solution, and consequently further increases in R raise utility. Unlike under full information, $R > 1.05$ does not imply an arbitrage opportunity, since insurance is constrained by incentives.

The dotted line represents the utility of an agent who has access to a perfect credit market under the same conditions as the planner, and can also use the private storage technology. This utility is computed by solving for the optimal saving and borrowing decisions explicitly, i.e., no finite grids on consumption or assets are imposed. As long as the return on storage does not exceed the credit market interest rate, once again zero storage is optimal, and the utility is independent of the return on storage. When the return on storage exceeds the interest rate, the agent may gain by using the storage technology in addition to smoothing income in the credit market.

We know that, in principle, the information-constrained solution can do no worse that what the agent would do on his own when given access to an outside credit market. Any borrowing-and-lending plan is incentive compatible, and has no influence on the principal. The optimal insurance scheme can therefore reproduce anything that can be achieved in the credit market. Hence, the credit-market solution provides a lower bound on the information-constrained solution; the latter cannot dip below the former. In Figure 3, the optimal constrained solution does dip below the credit-market solution for a small range of R, which can be attributed to the finite grids used to compute the information-constrained regime. Even so, the figure bears witness to the relative accuracy of our code: despite the grids, the difference between the solution and the lower bound is small.

If we conjecture that the entire utility schedule would be shifted upwards if we were able to compute with arbitrary fine grids, then the information-constrained solution would be slightly above the credit market solution at $R = 1.05$ and hit the lower bound (the credit-market utility) at a small range of returns to the immediate right of $R = 1.05$, near the point at which the credit-access-only agent begins to use the stochastic higher yield internal return. Thus our computations bear out the prediction by Cole and Kocherlakota (2001b) that the credit-market utility would equal the optimal-insurance utility at the point where the interest rate and return on storage are equal, but the prediction is slightly adjusted here to account for our stochastic storage technology. For yet higher internal returns, the utility of the agent increases under both regimes, but the information-constrained solution begins to dominate once again.

In sum, Figure 3 shows that if the return on private storage is sufficiently different from the outside interest rate, the agent does significantly better with optimal insurance than with borrowing and lending. What matters here is the ability of the agent to circumvent the insurance scheme offered by the planner. The planner provides incentives for truthful reporting by exploiting differences in the intertemporal rate of substitution between agents with high and low income realizations. When the return on storage is close to the credit-market interest rate, agents have the possibility of smoothing income on their own, so that differences in rates of substitution are small. Consequently, little additional insurance can be offered. When there is a large difference between return on storage and interest rate, however, it is costly for agents to deviate from the recommended storage scheme (no storage at low returns, maximum storage at high returns). The planner can use the resulting differences in rates of substitution to provide incentives for truthful reporting, which results in more insurance.[15]

The optimal allocations in this application are computed subject to two types of incentive constraints: obedience constraints force the agent to take the right action, and truth-telling constraints ensure that the agent reports the actual endowment. What happens if only one type of constraint is introduced? If we remove the truth-telling constraints (i.e., we assume that the actual endowment can be observed by the planner) and assume that the return on storage does not exceed the credit-market return, the situation is simple:

[15]The exact gain from optimal insurance depends on the return on storage. At low returns, the quantitative gains relative to credit-market access are small. The maximum utility differential is equivalent to a permanent increase in consumption of slightly above one-twentieth of one percent. If the return to storage is so low that the storage technology is never used, self-insurance in a credit market achieves 95 percent of the utility gain of optimal insurance relative to autarky. The relative gains from optimal insurance are higher when storage is actually used. At the maximum return of $R = 1.1$, the credit-market solution yields only 65 percent of the utility gain relative to autarky that can be attained by optimal insurance. These numbers, of course, are particular to our example. Generally, the absolute gains achieved by optimal insurance increase with the amount of risk inherent in income realizations and the agent's degree of risk aversion.

The planner can provide full insurance against income shocks. In this case the obedience constraints are not binding, since the optimal storage level is zero, which is also what the agent prefers if income shocks are fully insured. Conversely, if only the obedience constraints are removed, as if storage could be observed, full insurance cannot be achieved. Since the planner will require zero private storage, the outcome is the same as in the situation where the storage technology has zero gross return. The utilities are as in Figure 3 for $R = 0.6$. Without the obedience constraints, more insurance is possible, but it still falls well short of full insurance.

We have not computed a version of Phelan and Townsend (1991) or Atkeson and Lucas, Jr. (1992) with unobserved storage as these models are not equivalent to the one analyzed here. We conjecture however that the welfare consequences of Figure 3 would carry over in the sense that there would be a welfare loss but still nontrivial insurance relative to autarky.

7 Conclusions

In this paper we show how a general dynamic mechanism design problem with hidden income and hidden actions can be formulated in a way that is suitable for computation. We start from a general planning problem which allows arbitrary message spaces, lotteries, and history dependence. The planning problem is reduced to a recursive version which imposes truth-telling and obedience, and uses vectors of utility promises as state variables. We also develop methods to reduce the number of constraints that need to be imposed when computing the optimal contract. The main theme of these methods is to allow the planner to specify utility and behavior off the equilibrium path. In Program 2 this occurs because the agent reports his endowment more than once, so that the planner has to specify what happens if there are conflicting reports. In Program 3 the planner chooses bounds that limit the utility the agent can get on certain branches off the equilibrium path.

In an application to a model with hidden storage, we use our methods to examine how the optimal insurance contract varies with the return of the private storage technology. Specifically, we observe that in certain cases the utility of the agent can actually decrease as the return of the private technology rises. This occurs when the return on public storage is sufficiently high so that in the constrained optimum only public storage is used. If now the return to the private technology rises, it becomes more difficult for the planner to provide incentives for truth-telling and obedience, and consequently less insurance is possible. Thus the effect of a rising return of private investment on the agent's utility in the constrained optimum is exactly the opposite of the effect that prevails under autarky, where a higher return raises utility.

While in this paper we have presented our recursive formulations with double reporting and utility bounds as tools to allow to compute the optimal contract numerically, we also believe that the results are suggestive for the design of actual games and mechanisms. In dealing with the government or private businesses, we often find ourselves reporting the same information over and over again. While at first sight this may appear as unnecessary and inefficient, our results suggest that multiple reporting may be a useful feature of real-life mechanisms, because implementation is simplified. We plan to explore this perspective on the characteristics of actual mechanisms in future research.

A Mathematical Appendix

A.1 Proofs for all Propositions

Proposition 1 *There are reservation utilities $W_0 \in R$ such that an optimal equilibrium exists.*

Proof of Proposition 1 We need to show that for some W_0 the set of equilibria is nonempty and compact, and the objective function is continuous. To see that the set of equilibria is nonempty, notice that the planner can assign a zero transfer in all periods, and always send the same message. If the strategy of the agent is to choose the actions that are optimal under autarky, clearly all constraints are trivially satisfied for the corresponding initial utility W_0. The set of all contracts that satisfy the probability-measure constraints is compact in the product topology, since π and σ are probability measures on finite support. Since only equalities and weak inequalities are involved, it can be shown that the constraints (3), (4), and (5) define a closed subset of this set. Since closed subsets of compact sets are compact, the set of all feasible contracts is compact. We still need to show that the objective function of the planner is continuous. Notice that the product topology corresponds to pointwise convergence, i.e., we need to show that for a sequence of contracts that converges pointwise, the surplus of planner converges. This is easy to show since we assume that the discount factor of the planner is smaller than one, and that the set of transfers is bounded. Let π_n, σ_n, be a sequence of contracts that converges pointwise to π, σ, and choose $\epsilon > 0$. We have to show that there is an N such that $|V(\pi_n, \sigma_n) - V(\pi, \sigma)| < \epsilon$. Since the transfer τ is bounded and $Q < 1$, there is an T such that the discounted surplus of the planner from time T on is smaller than $\epsilon/2$. Thus we only have to make the difference for the first T periods smaller than $\epsilon/2$, which is the usual Euclidian finite-dimensional case. □

Proposition 2 (Revelation Principle) *For any message spaces M_1, M_2, and M_3, any allocation that is feasible in the general mechanism is also feasible in the truth-telling mechanism.*

Proof of Proposition 2 (Outline) Corresponding to any feasible allocation in the general setup there is a feasible contract that implements this allocation. Fix a feasible allocation and the corresponding contract $\{\pi, \sigma\}$. We will now define an outcome function for the truth-telling mechanism that implements the same allocation. To complete the proof, we then have to show that this outcome function satisfies constraints (7) to (9).

We will define the outcome function such that the allocation is the one implemented by (π, σ) along the equilibrium path. To do that, let $H^t(s^t)$ be the set of histories h^t in the general game such that the sequence of endowments, transfers, and actions in h^t coincides with the sequence of reported endowments, transfers, and recommended actions in history s^t in the restricted game. Likewise, define $p(h^t|s^t)$ as the probability of history h^t conditional on s^t:

$$p(h^t|s^t) \equiv \frac{p(h^t)}{\sum_{H^t(s^t)} p(h^t)} \tag{44}$$

If s^t has zero probability (that is, if the sequence s^t of endowments, transfers, and actions occurs with probability zero in the allocation implemented by $\{\pi, \sigma\}$), the definition of $p(h^t|s^t)$ is irrelevant, and is therefore left unspecified. Now we define an outcome function for the truth-telling mechanism by:

$$\begin{aligned}\pi^\star(\tau_t, a_t|e_t, s^{t-1}) \equiv \sum_{H^{t-1}(s^{t-1}), M_1, M_2, M_3} & p(h^{t-1}|s^{t-1})\, \sigma(m_{1t}|e_t, h^{t-1})\, \pi(\tau_t|m_{1t}, s^{t-1}(h^{t-1})) \\ & \sigma(m_{2t}|e_t, m_{1t}, \tau_t, h^{t-1})\, \pi(m_{3t}|m_{1t}, \tau_t, m_{2t}, s^{t-1}(h^{t-1}))\, \sigma(a_t|e_t, m_{1t}, \tau_t, m_{2t}, m_{3t}, h^{t-1}).\end{aligned} \tag{45}$$

Basically, the outcome function is gained by integrating out the message spaces M_1, M_2, and M_3 and prescribing the outcomes that occur on the equilibrium path.

We now have to verify that with this choice of an outcome function conditions (7) to (9) above are satisfied. In showing this, we can make use of the fact that $\{\pi, \sigma\}$ are probability measures and satisfy (3), (4), and (5). The proof proceeds by substituting (45) into (3), (4), and (5), and showing that the resulting equations imply (7) to (9). This is carried out in detail in Doepke and Townsend (2002). □

Proposition 3 *The set* $\mathbf{W}$ *is nonempty and compact.*

Proof of Proposition 3 To see that $\mathbf{W}$ is nonempty, notice that the planner can always assign a zero transfer in every period, and recommend the optimal action that the agent would have chosen without the planner. For the $w_0(e)$ that equals the expected utility of the agent under autarky under state e, all constraints are satisfied. To see that $\mathbf{W}$ is bounded, notice that there are finite grids for the endowment, the transfer, and the action. This implies that in every period consumption and therefore utility is bounded from above and from below. Since the discount factor β is smaller than one, total expected utility is also bounded. Since each $w_0(e)$ has to satisfy a promise-keeping constraint with equality, the set $\mathbf{W}$ must be bounded. To see that $\mathbf{W}$ is closed, assume there exists a converging sequence $\mathbf{w}_n$ such that $\mathbf{w}_n \in \mathbf{W}$ for all n. Corresponding to each $\mathbf{w}_n$ there is a contract $\pi(\tau_t, a_t|e_t, s^{t-1})_n$ satisfying constraints (7), (8), and (9). Since the contracts are within a compact subset of R^∞ with respect to the product topology, there is a convergent subsequence with limit $\pi(\tau_t, a_t|e_t, s^{t-1})$. It then follows that $\mathbf{w}$ must satisfy (7), (8), and (9) when $\pi(\tau_t, a_t|e_t, s^{t-1})$ is the chosen contract. □

Proposition 4 *For all $\mathbf{w}_0 \in \mathbf{W}$ and $e_0 \in E$, and for any s^{k-1} and e_k, there is an optimal contract $\pi^\star$ such that the remaining contract from s^{k-1} and e_k on is an optimal contract for the auxiliary planning problem with $e_0 = e_k$ and $\mathbf{w}_0 = \mathbf{w}(s^{k-1}, \pi^\star)$.*

Proof of Proposition 4 We will first construct $\pi^\star$ from a contract which is optimal from time zero and another contract which is optimal starting at s^{k-1} and e_k. We will then show by a contradiction argument that $\pi^\star$ is an optimal contract from time zero as well.

We have shown earlier that an optimal contract exists. Let π be an optimal contract from time zero, and π_k an optimal contract for $e_0 = e_k$ and $\mathbf{w}_0 = \mathbf{w}(s^{k-1}, \pi)$, with the elements of vector $\mathbf{w}(s^{k-1}, \pi)$ defined in (13). Now construct a new contract $\pi^\star$ that is equal to π_k from (e_k, s^{k-1}) on, and equals π until time k and on all future branches other than e_k, s^{k-1}. First, notice that by the way $\pi^\star$ is constructed, we have $\mathbf{w}(s^{k-1}, \pi) = \mathbf{w}(s^{k-1}, \pi^\star)$, and since $\pi^\star$ equals π_k from (e_k, s^{k-1}), $\pi^\star$ fulfills the reoptimization requirement of the proposition. We now claim that $\pi^\star$ is also an optimal contract. To show this, we have to demonstrate that $\pi^\star$ satisfies constraints (7), (8), and (9), and that it maximizes the surplus of the planner subject to these constraints. To start, notice that the constraints that are imposed if we compute an optimal contract taking $e_0 = e_k$ and $\mathbf{w}_0 = \mathbf{w}(s^{k-1}, \pi)$ as the starting point also constrain the choices of the planner in the original program from (e_k, s^{k-1}) on. By reoptimizing at (e_k, s^{k-1}) in period k as if the game were restarted, the planner clearly cannot lower his surplus, since no additional constraints are imposed. Therefore the total surplus from contract $\pi^\star$ cannot be lower than the surplus from π. Since π is assumed to be an optimal contract, if $\pi^\star$ satisfies (7), (8), and (9), it must be optimal as well. Thus we only have to show that (7), (8), and (9) are satisfied, or in other words, that reoptimizing at e_k, s^{k-1} does not violate any constraints of the original problem.

The probability constraints (7) are satisfied by contract $\pi^\star$, since the reoptimized contract is subject to the same probability constraints as the original contract. The promise-keeping constraint (8) is satisfied since the new contract delivers the same on-path utilities by construction. We still have to show that the incentive constraints (9) are satisfied. We will do this by contradiction. Suppose that (9) is not satisfied by contract $\pi^\star$. Then there is a deviation δ such that for some s^l, e_{l+1}:

$$\sum_{t=l+1}^{\infty} \beta^t \left[\sum_{H^t} p(h^t|\pi^\star, \delta, s^l, e_{l+1}) u(e_t + \tau_t - \delta_a(h^{t-1}, e_t, \tau_t, a_t)) \right] > \sum_{t=l+1}^{\infty} \beta^t \left[\sum_{S^t} p(s^t|\pi^\star, s^l, e_{l+1}) u(e_t + \tau_t - a_t) \right]. \quad (46)$$

Consider first the case $l + 1 \geq k$. On any branch that starts at or after time k, contract $\pi^\star$ is entirely identical to either π or π_k. But then (46) implies that either π or π_k violates incentive-compatibility (9), a contradiction. Consider now the case $l+1 < k$. Here the contradiction is not immediate, since the remaining contract is a mixture of π and π_k. Using $w(e_k, s^{k-1}, \delta)$ to denote the continuation utility of the agent from time k on under the deviation strategy, we can rewrite (46) as:

$$\sum_{t=l+1}^{k-1} \beta^t \left[\sum_{H^t} p(h^t|\pi^\star, \delta, s^l, e_{l+1}) u(e_t + \tau_t - \delta_a(h^{t-1}, e_t, \tau_t, a_t)) \right] + \beta^k \sum_{E, s^{k-1}} p(e_k, s^{k-1}|\pi^\star, \delta, s^l, e_{l+1}) w(e_k, s^{k-1}, \delta)$$
$$> \sum_{t=l+1}^{k-1} \beta^t \left[\sum_{S^t} p(s^t|\pi^\star, s^l, e_{l+1}) u(e_t + \tau_t - a_t) \right] + \beta^k \sum_{E, s^{k-1}} p(e_k, s^{k-1}|\pi^\star, s^l, e_{l+1}) w(e_k, s^{k-1}). \tag{47}$$

Notice that for s^{k-1} that are reached with positive probability under the deviation we have:

$$w(e_k, s^{k-1}, \delta) \leq w(e_k, \delta(s^{k-1})), \tag{48}$$

where $\delta(s^k)$ is the history as seen by the planner (reported endowments, delivered transfers, and recommended actions) under the deviation strategy. Otherwise, either π or π_k would violate incentive constraints. To see why, assume that e_k, s^{k-1} is a branch after which $\pi^\star$ is identical to π. If we had $w(e_k, s^{k-1}, \delta) > w(e_k, \delta(s^{k-1}))$, an agent under contract π who reached history $\delta(s^{k-1})$ could gain by following the deviation strategy δ afterwards. This cannot be the case since π is assumed to be an optimal contract, and therefore deviations are never profitable. Using (48) in (47) gives:

$$\sum_{t=l+1}^{k-1} \beta^t \left[\sum_{H^t} p(h^t|\pi^\star, \delta, s^l, e_{l+1}) u(e_t + \tau_t - \delta_a(h^{t-1}, e_t, \tau_t, a_t)) \right] + \beta^k \sum_{E, s^{k-1}} p(e_k, s^{k-1}|\pi^\star, \delta, s^l, e_{l+1}) w(e_k, \delta(s^{k-1}))$$
$$> \sum_{t=l+1}^{k-1} \beta^t \left[\sum_{S^t} p(s^t|\pi^\star, s^l, e_{l+1}) u(e_t + \tau_t - a_t) \right] + \beta^k \sum_{E, s^{k-1}} p(e_k, s^{k-1}|\pi^\star, s^l, e_{l+1}) w(e_k, s^{k-1}). \tag{49}$$

The outcome function $\pi^\star$ enters (49) only up to time $k-1$. Since up to time $k-1$ the outcome function $\pi^\star$ is identical to π, and since by construction of $\pi^\star$ continuation utilities at time k are the same under $\pi^\star$ and π, we can rewrite (49) as:

$$\sum_{t=l+1}^{k-1} \beta^t \left[\sum_{H^t} p(h^t|\pi, \delta, s^l, e_{l+1}) u(e_t + \tau_t - \delta_a(h^{t-1}, e_t, \tau_t, a_t)) \right] + \beta^k \sum_{E, s^{k-1}} p(e_k, s^{k-1}|\pi, \delta, s^l, e_{l+1}) w(e_k, \delta(s^{k-1}))$$

$$> \sum_{t=l+1}^{k-1} \beta^t \left[\sum_{S^t} p(s^t|\pi, s^l, e_{l+1}) u(e_t + \tau_t - a_t) \right] + \beta^k \sum_{E, s^{k-1}} p(e_k, s^{k-1}|\pi, s^l, e_{l+1}) w(e_k, s^{k-1}). \quad (50)$$

But now the left-hand side of (50) is the utility that the agent gets under plan π from following the deviation strategy until time k, and following the recommendations of the planner afterwards. Thus (50) contradicts the incentive compatibility of π. We obtain a contradiction, $\pi^\star$ actually satisfies (9). This shows that plan $\pi^\star$ is within the constraints of the original problem. Since $\pi^\star$ yields at least as much surplus as π and π is an optimal contract, $\pi^\star$ must be optimal as well. □

Proposition 5 *Program 1 and Program 2 are equivalent.*

Proof of Proposition 5 (This proof is carried out in Doepke and Townsend (2002). We derive Program 2 from the general planning problem, taking the same steps as in the derivation of Program 1. Since both Program 1 and Program 2 are equivalent to the general planning problem, they are also equivalent to each other.)

Proposition 6 *Program 1 and Program 3 are equivalent.*

Proof of Proposition 6 We want to show that constraints (39)-(43) in Program 3 place the same restrictions on the outcome function $\pi(\cdot)$ as the constraints (21)-(24) of Program 1. The probability constraints (21) and (39), the promise-keeping constraints (22) and (40), and the obedience constraints (23) and (41) are identical. This leaves us with the truth-telling constraints. Let us first assume we have found a lottery $\pi(\tau, a, \mathbf{w}'|\mathbf{w}, e)$ that satisfies the truth telling constraint (24) of Program 1 for all $\hat{e}$ and $\delta : T \times S \to A$. We have to show that there exist utility bounds $v(\hat{e}, e, \tau, a)$ such that the same lottery satisfies (42) and (43) in Program 3. For each $\hat{e}$, τ, and a, define $v(\hat{e}, e, \tau, a)$ as the maximum of the left hand side of (42) over all $\hat{a}$:

$$v(\hat{e}, e, \tau, a) \equiv \max_{\hat{a}} \left\{ \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(\hat{e} + \tau - \hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a}) w'(e') \right] \right\}. \quad (51)$$

Then clearly (42) is satisfied, since the left-hand side of (42) runs over $\hat{a}$. Now for each τ and a, define $\hat{\delta}(\cdot)$ by setting $\hat{\delta}(\tau, a)$ equal to the $\hat{a}$ that maximizes the left-hand side of (42):

$$\hat{\delta}(\tau, a) \equiv \text{argmax}_{\hat{a}} \left\{ \sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \ \left[u(\hat{e} + \tau - \hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a}) w'(e') \right] \right\}. \tag{52}$$

Since $\pi(\tau, a, \mathbf{w}'|\mathbf{w}, e)$ satisfies (24) for any function $\delta(\cdot)$ by assumption, we have for our particular $\hat{\delta}(\cdot)$:

$$\sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(\hat{e} + \tau - \hat{\delta}(\tau, a)) + \beta \sum\nolimits_E \mu(e'|\hat{\delta}(\tau, a)) w'(e') \right] \leq w(\hat{e}). \tag{53}$$

By the way we chose $\hat{\delta}(\cdot)$ and the $v(\cdot)$, we have from (42):

$$\sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \ \left[u(\hat{e} + \tau - \hat{\delta}(\tau, a)) + \beta \sum\nolimits_E \mu(e'|\hat{\delta}(\tau, a)) w'(e') \right] = v(\hat{e}, e, \tau, a). \tag{54}$$

Substituting the left-hand side into (53), we get:

$$\sum_{T,A} v(\hat{e}, e, \tau, a) \leq w(\hat{e}). \tag{55}$$

which is (43).

Conversely, suppose we have found a lottery $\pi(\tau, a, \mathbf{w}'|\mathbf{w}, e)$ that satisfies (42) and (43) in Program 3 for some choice of $v(\hat{e}, e, \tau, a)$. By (42), we have then for any $\hat{e}$ and $\hat{a}$ and hence any $\delta : T \times S \to A$:

$$\sum_{\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(\hat{e} + \tau - \delta(\tau, a)) + \beta \sum\nolimits_E \mu(e'|\delta(\tau, a)) w'(e') \right] \leq v(\hat{e}, e, \tau, a). \tag{56}$$

Substituting the left-hand side of (56) into the assumed (43) for the $v(\hat{e}, e, \tau, a)$, we maintain the inequality:

$$\sum_{T,A,\mathbf{W}'} \pi(\tau, a, \mathbf{w}'|\mathbf{w}, e) \left[u(\hat{e} + \tau - \delta(\tau, a)) + \beta \sum\nolimits_E \mu(e'|\delta(\tau, a)) w'(e') \right] \leq w(\hat{e}). \tag{57}$$

But this is (24) in Program 1. Therefore the sets of constraints are equivalent, which proves that Program 1 and Program 3 are equivalent. □

A.2 Computing the Value Set

Our analysis was based on the assumption that the set $\mathbf{W}$ of feasible utility vectors is known in advance. In practice, $\mathbf{W}$ is not known and needs to be computed alongside the value function $V(\mathbf{w}, e)$. $\mathbf{W}$ can be computed with the dynamic-programming methods described in detail in Abreu, Pierce, and Stachetti (1990), henceforth APS. An outline of the method follows.

We start by defining an operator B that maps nonempty compact subsets of $\mathbf{R}^{\#E}$ into nonempty compact subsets of $\mathbf{R}^{\#E}$. Let $\mathbf{W}'$ be a nonempty compact subset of $\mathbf{R}^{\#E}$. Then $B(\mathbf{W}')$ is defined as follows:

Definition 6 *A utility vector* $\mathbf{w} \in B(\mathbf{W}')$ *if there exist probabilities* $\pi(\tau, a|\mathbf{w}, e)$ *and future utilities* $w'(\mathbf{w}, e, \tau, a) \in \mathbf{W}'$ *such that* (15) *to* (18) *hold.*

The key point is that utility promises are chosen from the set $\mathbf{W}'$ instead of the true value set $\mathbf{W}$. Intuitively, $B(\mathbf{W}')$ consists of all utility vectors $\mathbf{w}$ that are feasible today (observing all incentive constraints), given that utility vectors from tomorrow on are drawn from the set $\mathbf{W}'$. The fact that B maps compact set into compact sets follows from the fact that all constraints are linear and involve only weak inequalities. Clearly, the true set of feasible utility vectors $\mathbf{W}$ satisfies $\mathbf{W} = B(\mathbf{W})$, thus $\mathbf{W}$ is a fixed point of B. The computational approach described in APS consists of using B to define a shrinking sequence of sets that converges to $\mathbf{W}$.

To do this, we need to start with a set $\mathbf{W}_0$ that is known to be larger than $\mathbf{W}$ a priori. In our case, this is easy to do, since consumption is bounded and therefore lifetime utility is bounded above and below. We can choose $\mathbf{W}_0$ as an interval in $\mathbf{R}^{\#E}$ from a lower bound that is lower than the utility from receiving the lowest consumption forever to a number that exceeds utility from consuming the highest consumption forever. We can now define a sequence of sets $\mathbf{W}_n$ by defining $\mathbf{W}_{n+1}$ as $\mathbf{W}_{n+1} = B(\mathbf{W}_n)$. We have the following results:

Proposition 7

- *The sequence* $\mathbf{W}_n$ *is shrinking, i.e., for any* n, $\mathbf{W}_{n+1}$ *is a subset of* $\mathbf{W}_n$.
- *For all* n, $\mathbf{W}$ *is a subset of* $\mathbf{W}_n$.
- *The sequence* $\mathbf{W}_n$ *converges to a limit* $\bar{\mathbf{W}}$, *and* $\mathbf{W}$ *is a subset of* $\bar{\mathbf{W}}$.

Proof of Proposition 7 To see that $\mathbf{W}_n$ is shrinking, we only need to show that $\mathbf{W}_1$ is a subset of $\mathbf{W}_0$. Since $\mathbf{W}_0$ is an interval, it suffices to show that the upper bound of $\mathbf{W}_1$ is lower than the upper bound of $\mathbf{W}_0$, and that the lower bound of $\mathbf{W}_1$ is higher than the lower bound of $\mathbf{W}_0$. The upper bound of $\mathbf{W}_1$ is reached by assigning maximum consumption in the first period and the maximum utility vector in $\mathbf{W}_0$ from the second period on. But the maximum utility vector $\mathbf{W}_0$ by construction corresponds to consuming more than maximum consumption every period, and since utility is discounted, the highest utility vector in $\mathbf{W}_1$ therefore is smaller than the highest utility vector in $\mathbf{W}_0$.

To see that $\mathbf{W}$ is a subset of all $\mathbf{W}_n$, notice that by the definition of B, if $\mathbf{C}$ is a subset of $\mathbf{D}$, $B(\mathbf{C})$ is a subset of $B(\mathbf{D})$. Since $\mathbf{W}$ is a subset of $\mathbf{W}_0$ and $\mathbf{W} = B(\mathbf{W})$, we have that $\mathbf{W}$ is a subset of $\mathbf{W}_1 = B(\mathbf{W}_0)$, and correspondingly for all the other elements.

Finally, $\mathbf{W}_n$ has to converge to a nonempty limit since it is a decreasing sequence of compact sets, and the nonempty set $\mathbf{W}$ is a subset of all elements of the sequence. □

Up to this point, we know that $\mathbf{W}_n$ converges to $\bar{\mathbf{W}}$ and that $\mathbf{W}$ is a subset of $\bar{\mathbf{W}}$. What we want to show is that $\bar{\mathbf{W}}$ and $\mathbf{W}$ are actually identical. What we still need to show, therefore, is that $\bar{\mathbf{W}}$ is also a subset of $\mathbf{W}$.

Proposition 8 *The limit set $\bar{\mathbf{W}}$ is a subset of the true value set $\mathbf{W}$.*

Proof of Proposition 8 The outline of the proof is as follows. To show that an element $\mathbf{w}$ of $\bar{\mathbf{W}}$ is in $\mathbf{W}$, we have to find $\pi(\tau_t, a_t|e_t, s^{t-1})$ that satisfy constraints (7), (9), and (11) for $\mathbf{w}$. These $\pi(\tau_t, a_t|e_t, s^{t-1})$ can be constructed period by period from the π that are implicit in the definition of the operator B. Notice that in each period continuation utilities are drawn from the same set $\bar{\mathbf{W}}$, since $\bar{\mathbf{W}}$ as the limit of the sequence $\mathbf{W}_n$ satisfies $\bar{\mathbf{W}} = B(\bar{\mathbf{W}})$. By definition of B, the resulting $\pi(\tau_t, a_t|e_t, s^{t-1})$ satisfy the period-by-period constraints (15) to (18). We therefore need to show that satisfying the period-by-period constraints (with a given set of continuation utilities) is equivalent to satisfying the original constraints (7), (9), and (11), which we have done above in Section 4.2. □

A.3 A Program With Two Subperiods

The period is divided into two parts. In the first subperiod the agent reports the endowment, and the planner assigns the transfer, an interim utility when the agent is telling the truth, as well as a vector of utility bounds in case the agent was lying. In the second subperiod the planner assigns an action and a vector of promised utilities for the next period. The solution to the problem in the second subperiod is computed for each combination of endowment e, transfer τ, interim utility $w_m(e)$ (m for "middle" or interim) along the truth-telling path, and vector of utility bounds for lying, $\bar{\mathbf{w}}_m(\hat{e}, e)$. Here $\bar{w}_m(\hat{e}, e)$ is an upper bound on the utility *an agent can get who has endowment $\hat{e}$, but reported e nevertheless.* $\bar{\mathbf{w}}_m(\hat{e}, e)$ is the vector of $\bar{w}_m(\hat{e}, e)$ with components running over endowments $\hat{e} \neq e$. The choice variables in the second subperiod are lotteries over the action a and the vector of promised utilities $\mathbf{w}'$ for the next period. We use $V_m[e, \tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e)]$ to denote the utility of the planner if the true state is e, the transfer is τ, and $w_m(e)$ and $\bar{\mathbf{w}}_m(\hat{e}, e)$ are the assigned interim utility and utility bounds for lying. The function $V_m(\cdot)$ is determined in the second subperiod (Program 4b below).

In the first subperiod the planner assigns transfers, interim utilities, and utility bounds. We use $\mathcal{W}(e, \tau)$ to denote the set of feasible utility assignments for a given state e and transfer τ. The agent comes into the first subperiod with a vector of promised utilities $\mathbf{w}$, to be effected depending on the realized state e. The planner assigns a transfer τ, on-path interim utilities $w_m(e)$, and off-path utility bounds $\bar{\mathbf{w}}_m(\hat{e}, e)$ subject to promise-keeping and truth-telling constraints.

Program 4a:
The maximization problem of the planner given reported endowment e and promised utility vector $\mathbf{w}$ is:

$$V(\mathbf{w}, e) = \max_{\pi \geq 0} \sum_{T} \sum_{\mathcal{W}(e,\tau)} \pi(\tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e) | \mathbf{w}, e)\; V_m[e, \tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e)] \tag{58}$$

subject to the constraints (59)-(61) below:

$$\sum_{T} \sum_{\mathcal{W}(e,\tau)} \pi(\tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e) | \mathbf{w}, e) = 1, \tag{59}$$

$$\sum_{T} \sum_{\mathcal{W}(e,\tau)} \pi(\tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e) | \mathbf{w}, e)\; w_m(e) = w(e), \tag{60}$$

$$\forall \hat{e} \neq e: \quad \sum_{T} \sum_{\mathcal{W}(e,\tau)} \pi(\tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e) | \mathbf{w}, e)\; \bar{w}_m(\hat{e}, e) \leq w(\hat{e}). \tag{61}$$

The number of variables in this program is $\sum_T \#\mathcal{W}(e, \tau)$, and there are $1 + (\#E)$ constraints: one probability constraint (59), one promise-keeping constraint (60), and (#E-1) truth-telling constraints (61).

We now turn to the second subperiod. As in the last section, apart from the lotteries over actions and utilities, the program in the second subperiod also assigns utility bounds. $v(\hat{e}, e, \tau, a)$ is an upper bound on the utility an agent can get who has true endowment $\hat{e}$, reported endowment e, and receives transfer τ and recommendation a. These utility bounds are weighted by the probability of receiving recommendation a.

Program 4b:
The following program, given endowment e, transfer τ, and interim utilities and utility bounds, determines $V_m[\cdot]$:

$$V_m[e, \tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e)] = \max_{\pi \geq 0, v} \sum_{A, \mathbf{W}'} \pi(a, \mathbf{w}')\; [-\tau + Q \textstyle\sum_E \mu(e'|a) V(\mathbf{w}', e')] \tag{62}$$

subject to constraints (63)-(67) below:

$$\sum_{A, \mathbf{W}'} \pi(a, \mathbf{w}') = 1, \tag{63}$$

$$\sum_{A, \mathbf{W}'} \pi(a, \mathbf{w}')\; \left[u(e + \tau - a) + \beta \textstyle\sum_E \mu(e'|a) w'(e')\right] = w_m(e), \tag{64}$$

$$\forall a, \hat{a} \neq a: \quad \sum_{\mathbf{W}'} \pi(a, \mathbf{w}') \left[u(e + \tau - \hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a}) w'(e')\right]$$
$$\leq \sum_{\mathbf{W}'} \pi(a, \mathbf{w}') \left[u(e + \tau - a) + \beta \sum\nolimits_E \mu(e'|a) w'(e')\right], \tag{65}$$

$$\forall \hat{e} \neq e, a, \hat{a}: \quad \sum_{\mathbf{W}'} \pi(a, \mathbf{w}') \left[u(\hat{e} + \tau - \hat{a}) + \beta \sum\nolimits_E \mu(e'|\hat{a}) w'(e')\right] \leq v(\hat{e}, e, \tau, a), \tag{66}$$

$$\forall \hat{e} \neq e: \quad \sum_A v(\hat{e}, e, \tau, a) \leq \bar{w}_m(\hat{e}, e). \tag{67}$$

The number of variables in this program is $\#A \times \#\mathbf{W}$ under $\pi(\cdot)$ plus $(\#E - 1) \times \#A$ under $v(\cdot)$, where again e and τ are fixed at this stage of the problem. In our example with two endowments e, ten transfers τ, ten actions a, and ten utility vectors $\mathbf{w}'$, Program 4b has 110 variables and 193 constraints, while Program 4a has 3 constraints. The number of variables in Program 4a and the number of Programs 4b that need to be computed depends on the grid for interim utilities. For example, if the grid for interim utilities for each τ and e has ten values, Program 4a has 100 variables, and 1000 Programs 4b need to be computed. In practice, it is generally faster to use Program 3 as long as it is feasible to do so. Large programs can only be computed using Program 4a and 4b, since the individual programs are smaller and require less memory than Program 3.

We still have to define the set $\mathcal{W}(e, \tau)$ of feasible interim utility assignments in Program 4a. It is the set of all assignments for which Program 4b has a solution. With $\mathbf{W}_m$ being the utility grid that is used for $w_m(e)$ and $\bar{\mathbf{w}}_m(\hat{e}, e)$, define:

$$\mathcal{W}(e, \tau) = \left\{(w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e)) \in (\mathbf{W}_m)^{(\#E)} \mid V_m[e, \tau, w_m(e), \bar{\mathbf{w}}_m(\hat{e}, e)] \text{ is defined}\right\} \tag{68}$$

In other words, we vary utility assignments as parameters or states in Program 4b and rule out the ones for which there is no feasible solution.

The equivalence of Programs 3 and Program 4a/4b can be shown by directly comparing the sets of constraints. This is carried out in detail in Doepke and Townsend (2002).

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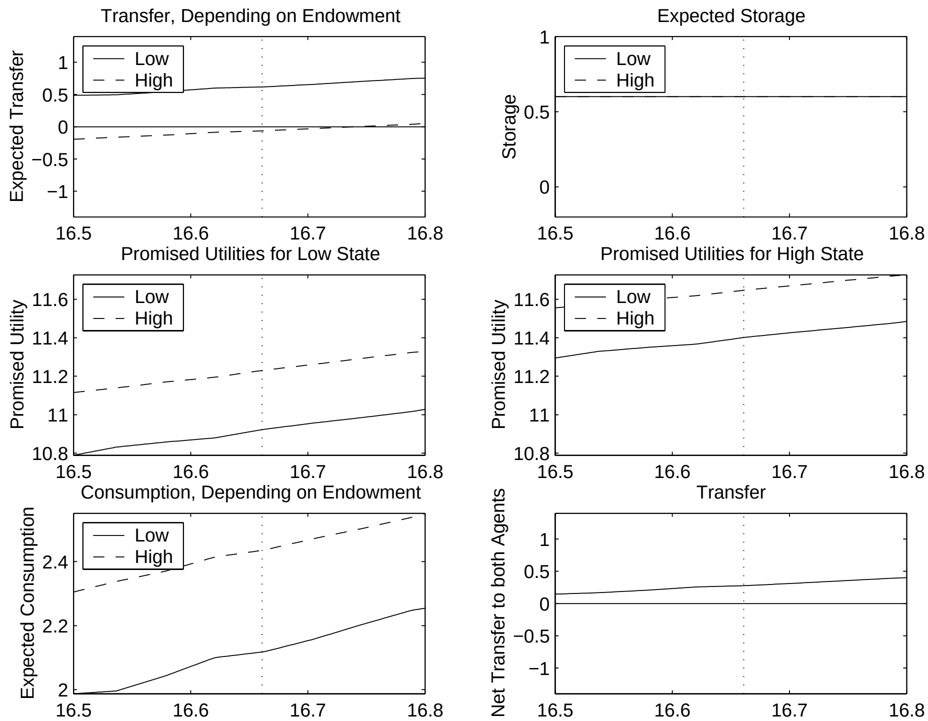

Figure 2: Policy Functions, R=1.1

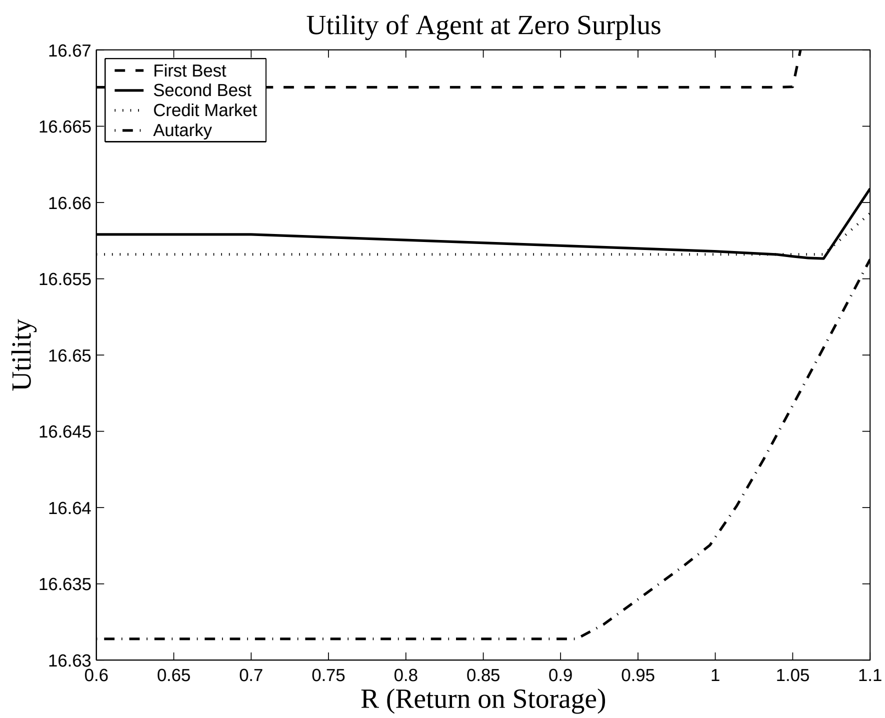

Figure 3: Utility of the Agent with Full Information, Private Information, Credit-Market Access, and Autarky