Chapter 2

Work Effort, Production, and Consumption

Robinson Crusoe is alone on an island, so he is an economy unto himself. He has preferences over consumption and leisure and can produce consumption goods by using labor and capital. We examine production first. Then we turn to preferences. Putting these two pieces together yields Crusoe's optimal choices of labor, leisure, and consumption.

2.1 Crusoe's Production Possibilities

Crusoe uses factors of production in order to make output y. We can think of this output as being coconuts. Two common factors of production, and those we consider here, are capital k and labor l. Capital might be coconut trees, and labor is the amount of time Crusoe works, measured as a fraction of a day. How much Crusoe produces with given resources depends on the type of technology A that he employs. We formalize this production process via a production function.

We often simplify our problems by assuming that the production function takes some particular functional form. As a first step, we often assume that it can be written: $y = Af(k, l)$, for some function $f(\cdot)$. This means that as technology A increases, Crusoe can get more output for any given inputs. It is reasonable to require the function $f(\cdot)$ to be increasing in each argument. This implies that increasing either input k or l will increase production. Another common assumption is that output is zero if either input is zero: $f(0, l) = 0$ and $f(k, 0) = 0$, for all k and l.

One functional form that has these properties is the Cobb–Douglas function, for example:

$y = Ak^{1-\alpha}l^{\alpha}$, for some α between zero and one. This particular Cobb-Douglas function exhibits constant returns to scale, since $(1-\alpha)+(\alpha)=1$. Figure 2.1 is a three-dimensional rendering of this function for particular values of A and α.

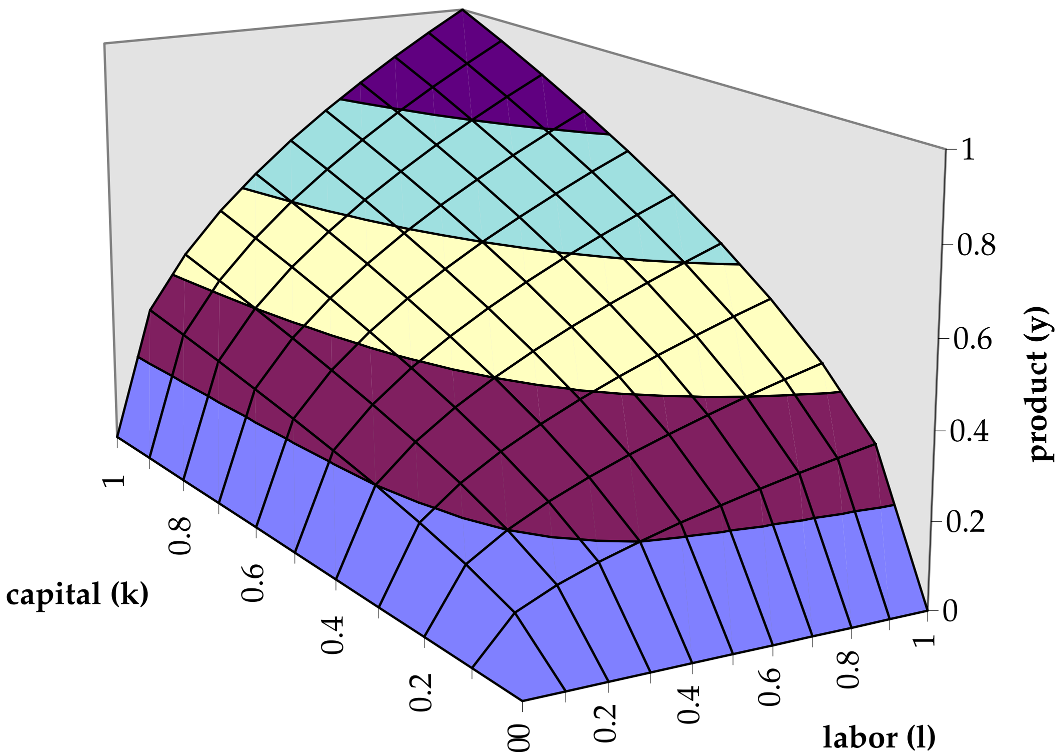

Figure 2.1: Cobb-Douglas Production

We will not be dealing with capital k until Chapter 9, so for now we assume that capital is fixed, say, at $k = 1$. This simplifies the production function. With a slight abuse of notation, we redefine $f(\cdot)$ and write production as $y = f(l)$. This is like what Barro uses in Chapter 2.

If the original production function was Cobb–Douglas, $y = Ak^{1-\alpha}l^{\alpha}$, then under $k = 1$ the production function becomes: $y = Al^{\alpha}$. The graph of this curve is just a slice through the surface depicted in Figure 2.1. It looks like Barro's Figure 2.1.

As you know, the marginal product of some factor of production (e.g., labor l) is the additional output, or "product", that results from increasing the input of that factor. Formally, the marginal product of an input is the derivative of the production function with respect to that input. For example, the marginal product of labor is: $dy/dl = f'(l)$.[1] Since the marginal product is the derivative of the the production function, and the derivative gives the slope, we can read the marginal product as the slope of the production function, as Barro does in his Figure 2.1.

In the particular case where production is Cobb–Douglas (and capital is fixed), the production function is: $y = Al^{\alpha}$, so the marginal product of labor is: $dy/dl = A\alpha l^{\alpha-1}$. This is always positive, as we require, and it decreases as we increase l. Accordingly, this production function exhibits diminishing marginal product: the first unit of labor is more productive than the tenth unit of labor. Graphing this marginal product equation gives us something like Barro's Figure 2.2.

Barro talks about improvements in technology and argues how both the production function and the marginal-product schedule shift as a result. The effects of such a change in technology are clearer when we examine a particular production function. For example, consider our production function: $y = Al^{\alpha}$. The improvement in technology means that A goes up. Accordingly, whatever production was before, it undergoes the same percentage increase as the increase in A. For example, if A doubles, then output at each l will be double what it used to be. Notably, when l is zero, output is zero just as before, since twice zero is still zero. The result is that the production function undergoes a kind of upward rotation, pivoting about the anchored origin, $l = 0$. That is precisely what Barro depicts in his Figure 2.3.

We can examine the marginal-product schedule as well. Under the particular functional form we are using, the marginal product of labor (MPL) is: $dy/dl = A\alpha l^{\alpha-1}$. Accordingly, the marginal product at each l undergoes the same percentage change as does A. Since the MPL is higher at low levels of l, the marginal-product curve shifts up more at those levels of l. Refer to Barro's Figure 2.4.

2.2 Crusoe's Preferences

Crusoe cares about his consumption c and his leisure. Since we are measuring labor l as the fraction of the day that Crusoe works, the remainder is leisure. Specifically, leisure is $1 - l$. We represent his preferences with a utility function $u(c, l)$. Take note, the second argument is not a "good" good, since Crusoe does not enjoy working. Accordingly, it might have been less confusing if Barro had written utility as $v(c, 1 - l)$, for some utility function $v(\cdot)$. We assume that Crusoe's preferences satisfy standard properties: they are increasing in each "good" good, they are convex, etc.

We will often simplify the analysis by assuming a particular functional form for Crusoe's

[1] Barro uses primes to denote shifted curves rather than derivatives. For example, when Barro shifts the $f(l)$ curve, he labels the new curve $f(l)'$. This is not a derivative. Barro's notation is unfortunate, but we are stuck with it.

preferences. For example, we might have: $u(c, l) = \ln(c) + \ln(1 - l)$. With such a function in hand, we can trace out indifference curves. To do so, we set $u(c, l)$ to some fixed number $\bar{u}$, and solve for c as a function of l. Under these preferences, we get:

$$c = \frac{e^{\bar{u}}}{1 - l}.$$

As we change $\bar{u}$, we get different indifference curves, and the set of those looks like Barro's Figure 2.6. These should look strange to you because they are increasing as we move to the right. This is because we are graphing a "bad" good (labor l) on the horizontal axis. If we graph leisure $(1 - l)$ instead, then we will get indifference curves that look like what you saw in your microeconomics courses.

2.3 Crusoe's Choices

When we put preferences and technology together, we get Crusoe's optimal choices of labor l, leisure $1 - l$, and consumption c. Formally, Crusoe's problem is:

$$\max_{c,l} \; u(c, l), \text{ such that:} \tag{2.1}$$

$$c \leq y, \text{ and:} \tag{2.2}$$

$$y = f(l). \tag{2.3}$$

There are two elements of equation (2.1). First, under the max, we indicate the variables that Crusoe gets to choose; in this case, he chooses c and l. Second, after the word "max" we place the maximand, which is the thing that Crusoe is trying to maximize; in this case, he cares about his utility.

Equation (2.2) says that Crusoe cannot consume more than he produces. We can use simple deduction to prove that we can replace the "$\leq$" symbol with "=". Suppose Crusoe chooses c and l such that $c < y$. This cannot be optimal because he could increase the maximand a little bit if he raised c, since $u(c, l)$ is increasing in c. Simply put: it will never be optimal for Crusoe to waste output y, so we know that $c = y$.

Finally, equation (2.3) simply codifies the production technology that is available to Crusoe.

With all this in mind, we can simplify the way we write Crusoe's problem as follows:

$$\max_{c,l} \; u(c, l), \text{ such that:}$$

$$c = f(l).$$

Here, we are making use of the fact that $c = y$, and we are substituting the second constraint into the first.

There are two principal ways to solve such a problem. The first is to substitute any constraints into the objective. The second is to use Lagrange multipliers. We consider these two methods in turn.

Substituting Constraints into the Objective

In the maximization problem we are considering, we have c in the objective, but we know that $c = f(l)$, so we can write the max problem as:

$$\max_{l} \ u[f(l), l].$$

We no longer have c in the maximand or in the constraints, so c is no longer a choice variable. Essentially, the $c = f(l)$ constraint tacks down c, so it is not a free choice. We exploit that fact when we substitute c out.

At this point, we have a problem of maximizing some function with respect to one variable, and we have no remaining constraints. To obtain the optimal choices, we take the derivative with respect to each choice variable, in this case l alone, and set that derivative equal to zero.[2] When we take a derivative and set it equal to zero, we call the resulting equation a *first-order condition*, which we often abbreviate as "FOC".

In our example, we get only one first-order condition:

$$\text{(FOC } l\text{)} \qquad \frac{d}{dl}\left\{u[f(l^\star), l^\star]\right\} = u_1[f(l^\star), l^\star]f'(l^\star) + u_2[f(l^\star), l^\star] = 0.$$

(See the Appendix for an explanation of the notation for calculus, and note how we had to use the chain rule for the first part.) We use $l^\star$ because the l that satisfies this equation will be Crusoe's optimal choice of labor.[3] We can then plug that choice back into $c = f(l)$ to get Crusoe's optimal consumption: $c^\star = f(l^\star)$. Obviously, his optimal choice of leisure will be $1 - l^\star$.

Under the particular functional forms for utility and consumption that we have been considering, we can get explicit answers for Crusoe's optimal choices. Recall, we have been using $u(c, l) = \ln(c) + \ln(1 - l)$ and $y = f(l) = Al^\alpha$. When we plug these functions into the first-order condition in equation (FOC l), we get:

$$\text{(2.4)} \qquad \left(\frac{1}{A(l^\star)^\alpha}\right)\left(A\alpha(l^\star)^{\alpha-1}\right) + \frac{-1}{1 - l^\star} = 0.$$

[2] The reason we set the derivative equal to zero is as follows. The maximand is some hump-shaped object. The derivative of the maximand gives the slope of that hump at each point. At the top of the hump, the slope will be zero, so we are solving for the point at which the slope is zero.

[3] Strictly speaking, we also need to check the second-order condition in order to make sure that we have solved for a maximum instead of a minimum. In this text we will ignore second-order conditions because they will always be satisfied in the sorts of problems we will be doing.

The first term in parentheses is from $u_1(c, l) = 1/c$, using the fact that $c = Al^\alpha$. The second term in parentheses is from the chain rule; it is the $f'(l)$ term. The final term is $u_2(c, l)$. We can cancel terms in equation (2.4) and rearrange to get:

$$\frac{\alpha}{l^\star} = \frac{1}{1 - l^\star}.$$

Cross multiplying and solving yields:

$$l^\star = \frac{\alpha}{1 + \alpha}.$$

When we plug this value of $l^\star$ into $c^\star = f(l^\star)$, we get:

$$c^\star = A\left(\frac{\alpha}{1 + \alpha}\right)^\alpha.$$

These are Crusoe's optimal choices of labor and consumption.

Using Lagrange Multipliers

In many problems, the technique of substituting the constraints into the objective is the quickest and easiest method of carrying out the constrained maximization. However, sometimes it is difficult to solve the constraints for a particular variable. For example, suppose you have a constraint like:

$$c + \ln(c) = 10 + l + \ln(1 - l).$$

You cannot solve for either c or l, so the solution method described above is not applicable.

Accordingly, we describe how to use Lagrange multipliers to tackle problems of constrained maximization when it is either difficult or impossible to solve the constraints for individual variables. At first we treat the method as a cook-book recipe. After we are done, we will try to develop intuition for why the technique works.

Recall, we are working with the following problem:

$$\max_{c,l} \; u(c, l), \text{ such that:}$$
$$c = f(l).$$

The first step in using Lagrange multipliers is to solve the constraint so that everything is on one side, leaving a zero on the other side. In that regard, we have either:

$$f(l) - c = 0, \text{ or:}$$
$$c - f(l) = 0.$$

Either of those two will work, but we want to choose the first one, for reasons that are described below. The general heuristic is to choose the one that has a minus sign in front of the variable that makes the maximand larger. In this case, more c makes utility higher, so we want the equation with $-c$.

The second step is to write down a function called the *Lagrangean*, defined as follows:

$$\mathcal{L}(c, l, \lambda) = u(c, l) + \lambda[f(l) - c].$$

As you can see, the Lagrangean is defined to be the original objective, $u(c, l)$, plus some variable λ times our constraint. The Lagrangean is a function; in this case its arguments are the three variables c, l, and λ. Sometimes we will write it simply as $\mathcal{L}$, suppressing the arguments. The variable λ is called the *Lagrange multiplier*; it is just some number that we will calculate. If there is more than one constraint, then each one is: (i) solved for zero; (ii) multiplied by its own Lagrange multiplier, e.g., λ_1, λ_2, etc.; and (iii) added to the Lagrangean. (See Chapter 3 for an example.)

Before we used calculus to maximize our objective directly. Now, we work instead with the Lagrangean. The standard approach is to set to zero the derivatives of the Lagrangian with respect to the choice variables and the Lagrange multiplier λ. The relevant first-order conditions are:

$$\text{(FOC } c\text{)} \qquad \frac{\partial}{\partial c}[\mathcal{L}(c^\star, l^\star, \lambda^\star)] = u_1(c^\star, l^\star) + \lambda^\star[-1] = 0;$$

$$\text{(FOC } l\text{)} \qquad \frac{\partial}{\partial l}[\mathcal{L}(c^\star, l^\star, \lambda^\star)] = u_2(c^\star, l^\star) + \lambda^\star[f'(l^\star)] = 0; \text{ and:}$$

$$\text{(FOC } \lambda\text{)} \qquad \frac{\partial}{\partial \lambda}[\mathcal{L}(c^\star, l^\star, \lambda^\star)] = f(l^\star) - c^\star = 0.$$

Again, we use starred variables in these first-order conditions to denote that it is only for the optimal values of c, l, and λ that these derivatives will be zero. Notice that differentiating the Lagrangian with respect to λ simply gives us back our budget equation. Now we have three equations in three unknowns ($c^\star$, $l^\star$, and $\lambda^\star$) and can solve for a solution. Typically, the first step is to use equations (FOC c) and (FOC l) to eliminate $\lambda^\star$. From (FOC c) we have:

$$u_1(c^\star, l^\star) = \lambda^\star,$$

and from (FOC l) we have:

$$-\frac{u_2(c^\star, l^\star)}{f'(l^\star)} = \lambda^\star.$$

Combining the two gives us:

$$u_1(c^\star, l^\star) = -\frac{u_2(c^\star, l^\star)}{f'(l^\star)}. \qquad (2.5)$$

When we are given particular functional forms for $u(\cdot)$ and $f(\cdot)$, then equation (2.5) gives us a relationship between $c^\star$ and $l^\star$ that we can plug into the budget equation and solve further. For example, under $u(c,l) = \ln(c) + \ln(1-l)$ and $f(l) = Al^\alpha$, equation (2.5) becomes:

$$\frac{1}{c^\star} = -\left(\frac{-1}{1-l^\star}\right)\left(\frac{1}{A\alpha(l^\star)^{\alpha-1}}\right),$$

or equivalently:

$$c^\star = A\alpha(1-l^\star)(l^\star)^{\alpha-1}.$$

Now we plug in the budget equation $c = Al^\alpha$ to get:

$$A(l^\star)^\alpha = A\alpha(1-l^\star)(l^\star)^{\alpha-1}.$$

After some canceling and algebraic manipulation, this reduces to:

$$l^\star = \frac{\alpha}{1+\alpha}.$$

Finally, we plug this answer for the optimal labor $l^\star$ back into the budget equation to get:

$$c^\star = A\left(\frac{\alpha}{1+\alpha}\right)^\alpha.$$

Notice that these are the same answers for $c^\star$ and $l^\star$ that we derived in the previous subsection, when we plugged the constraint into the objective instead of using a Lagrange multiplier.

Now let's try to figure out why the technique of Lagrange multipliers works. First, we want to understand better what the Lagrange multiplier λ is. Our first-order condition with respect to c gave us:

$$u_1(c^\star, l^\star) = \lambda^\star, \tag{2.6}$$

This tells us that, at the optimum, $\lambda^\star$ is the marginal utility of an extra unit of consumption, given by the left-hand side. It is this interpretation of λ that motivated our choice of $f(l) - c = 0$ rather than $c - f(l) = 0$ when we attached the constraint term to the Lagrangean. If we had used the latter version of the constraint, then the right-hand side of equation (2.6) would have been $-\lambda$, which would have been minus the marginal utility of income.

Now look at the terms in the Lagrangian:

$$\mathcal{L}(c,l,\lambda) = u(c,l) + \lambda[f(l) - c].$$

It contains our objective $u(\cdot)$ and then the Lagrange multiplier times the constraint. Remember, λ is the marginal utility of an additional unit of consumption. Notice that if the budget equation is satisfied, then $f(l) = c$, so the constraint term is zero, and the Lagrangian $\mathcal{L}$ and the objective $u(\cdot)$ are equal. Ceteris paribus, the Lagrangian will be big whenever the objective is.

Now, think about the contributions from the constraint term. Suppose Crusoe is at some choice of c and l such that the budget is exactly met. If he wants to decrease labor l by a little bit, then he will have to cut back on his consumption c. The constraint term in the Lagrangean is: $\lambda[f(l) - c]$. The Lagrangean, our new objective, goes down by the required cut in c times λ, which is the marginal utility of consumption. Essentially, the Lagrangean subtracts off the utility cost of reducing consumption to make up for shortfalls in budget balance. That way, the Lagrangean is an objective that incorporates costs from failing to meet the constraint.

2.4 Income and Substitution Effects

Barro uses graphs to examine how Crusoe's optimal choices of consumption and labor change when his production function shifts and rotates. He calls the changes in Crusoe's choices "wealth and substitution effects". That discussion is vaguely reminiscent of your study of income and substitution effects from microeconomics. In that context, you considered shifts and rotations of linear budget lines. Crusoe's "budget line" is his production function, which is not linear.

This difference turns out to make mathematical calculation of income and substitution effects impractical. Furthermore, the "wealth effects" that Barro considers violate our assumption that production is zero when labor l is zero. Such a wealth effect is depicted as an upward shift of the production function in Barro's Figure 2.8. This corresponds to adding a constant to Crusoe's production function, which means that production is not zero when l is.

Barro's Figure 2.10 depicts a pivot of the production about the origin. This type of change to production is much more common in macroeconomics, since it is how we typically represent technological improvements. If Crusoe's production function is $y = Al^{\alpha}$, then an increase in A will look exactly like this. Given a specific functional form for $u(\cdot)$ as well, it is straightforward to compute how Crusoe's choices of consumption c and labor l change for any given change in A.

For example, suppose $u(c, l) = \ln(c) + \ln(1 - l)$ as before. Above we showed that:

$$c^{\star} = A\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}.$$

Determining how $c^{\star}$ changes when A changes is called *comparative statics*. The typical exercise is to take the equation giving the optimal choice and to differentiate it with respect to the variable that is to change. In this case, we have an equation for Crusoe's optimal choice of $c^{\star}$, and we are interested in how that choice will change as A changes. That gives us:

$$\frac{\partial c^{\star}}{\partial A} = \left(\frac{\alpha}{1+\alpha}\right)^{\alpha}. \tag{2.7}$$

The derivative in equation (2.7) is positive, so Crusoe's optimal choice of consumption will increase when A increases.

The comparative statics exercise for Crusoe's optimal labor choice $l^\star$ is even easier. Above we derived:

$$l^\star = \frac{\alpha}{1+\alpha}.$$

There is no A on the right-hand side, so when we take the partial derivative with respect to A, the right-hand side is just a constant. Accordingly, $\partial l^\star / \partial A = 0$, i.e., Crusoe's choice of labor effort does not depend on his technology. This is precisely what Barro depicts in his Figure 2.10.

The intuition of this result is as follows. When A goes up, the marginal product of labor goes up, since the slope of the production function goes up. This encourages Crusoe to work harder. On the other hand, the increase in A means that for any l Crusoe has more output, so he is wealthier. As a result, Crusoe will try to consume more of any normal goods. To the extent that leisure $1 - l$ is a normal good, Crusoe will actually work less. Under these preferences and this production function, these two effects happen to cancel out precisely. In general, this will not be the case.

Variable	Definition
y	Income, in units of consumption
k	Capital
l	Labor, fraction of time spent on production
$f(l)$	Production function
α	A parameter of the production function
A	Technology of production
c	Consumption
$1-l$	Leisure, fraction of time spent recreating
$\mathcal{L}(\cdot)$	Lagrangean function
λ	Lagrange multiplier

Table 2.1: Notation for Chapter 2

Exercises

Exercise 2.1 (Easy)
An agent cares about consumption and leisure. Specifically, the agent's preferences are: $U = \ln(c) + \ln(l)$, where c is the agent's consumption, and l is the number of hours the agent

spends per day on leisure. When the agent isn't enjoying leisure time, the agent works, either for herself or for someone else. If she works n_s hours for herself, then she produces $y = 4n_s^{0.5}$ units of consumption. For each hour that she works for someone else, she gets paid a competitive wage w, in units of consumption.

Write out the agent's optimization problem.

Exercise 2.2 (Moderate)
Suppose Crusoe's preferences are given by: $u(c, l) = c^{\gamma}(1 - l)^{1-\gamma}$, for some γ between zero and one. His technology is: $y = f(l) = Al^{\alpha}$, just like before. Solve for Crusoe's optimal choices of consumption c and labor l. (You can use either substitution or a Lagrangean, but the former is easier in this sort of problem.)

Appendix: Calculus Notation

Suppose we have a function: $y = f(x)$. We can think of differentiation as an operator that acts on objects. Write $\frac{d}{dx}$ as the operator that differentiates with respect to x. We can apply the operator to both sides of any equation. Namely,

$$\frac{d}{dx}\big(y\big) = \frac{d}{dx}\big(f(x)\big).$$

We often write the left-hand side as $\frac{dy}{dx}$, and the right-hand side as $f'(x)$. These are just notational conventions.

When we have functions of more than one variable, we are in the realm of multivariate calculus and require more notation. Suppose we have $z = f(x, y)$. When we differentiate such a function, we will take *partial derivatives* that tell us the change in the function from changing only one of the arguments, while holding any other arguments fixed. Partial derivatives are denoted with curly dees (i.e., with ∂) to distinguish them from normal derivatives. We can think of partial differentiation as an operator as before:

$$\frac{\partial}{\partial x}\big(z\big) = \frac{\partial}{\partial x}\big(f(x, y)\big).$$

The left-hand side is often written as $\frac{\partial z}{\partial x}$, and the right-hand side as $f_1(x, y)$. The subscript 1 on f indicates a partial derivative with respect to the first argument of f. The derivative of f with respect to its second argument, y, can similarly be written: $f_2(x, y)$.

The things to remember about this are:

- Primes (f') and straight dees (df) are for functions of only one variable.
- Subscripts (f_1) and curly dees (∂f) are for functions of more than one variable.