Chapter 5

The Market-Clearing Model

Most of the models that we use in this book build on two common assumptions. First, we assume that there exist markets for all goods present in the economy, and that all markets clear. Second, we assume that all agents behave competitively, which means that they take prices as given. Models that satisfy these assumptions are called *general equilibrium* models. There are a number of important results that apply to all general equilibrium models, regardless of what kind of goods, agents, or technologies are used. In this chapter, we will demonstrate three of these results within a general setting. Many of the models that we use throughout the rest of the book will be special cases of the general model presented here. Since we omit most of the simplifying assumptions that we make in other chapters, the treatment is more formal and mathematical than usual.

Section 5.1 introduces our general equilibrium framework. In Section 5.2 we show that within this framework the general price level is undetermined. This implies that prices can be normalized without loss of generality. For example, in many models we set the price of the consumption good to be one. In Section 5.3 we show that in a general equilibrium model one market clearing constraint is redundant, a fact known as *Walras' Law*. Section 5.4 presents the *First Welfare Theorem*, which states that under certain conditions equilibria in a general equilibrium model are efficient.

5.1 A General Pure-Exchange Economy

We will consider an economy with many different goods and consumers. Instead of having a representative consumer, we allow for the possibility that each consumer has a different utility function. However, we make one simplification: there is no production in the economy. The consumers have endowments of goods and can trade their endowments in markets, but there is no possibility of producing any goods in excess of the endowments.[1]

There are N different goods in the economy, where N is any positive integer. For each good there is a market, and the price of good n is denoted p_n. There are I different consumers. Each consumer has a utility function over her consumption of the N goods in the economy. Consumption of good n by consumer i is denoted as c_n^i, and the utility function for consumer i is $u_i(c_1^i, c_2^i, \ldots, c_N^i)$. Notice that the utility function is indexed by i, so that it can be different for each consumer. The consumers also have endowments of the N goods, where e_t^i is the endowment of consumer i of good n.

All consumers meet at the beginning of time in a central marketplace. Here the consumers can sell their endowments and buy consumption goods. If consumer i sells all her endowments, her total income is $\sum_{n=1}^N p_n e_n^i$. Similarly, total expenditure on consumption goods is $\sum_{n=1}^N p_n c_n^i$. Consumer i maximizes utility subject to her budget constraint, which states that total expenditure on consumption has to equal total income from selling the endowment. Mathematically, the problem of consumer i is:

$$\max_{\{c_n^i\}_{n=1}^N} u_i(c_1^i, c_2^i, \ldots, c_N^i) \text{ subject to:} \qquad (5.1)$$

$$\sum_{n=1}^N p_n c_n^i = \sum_{n=1}^N p_n e_n^i.$$

We will also need a market-clearing constraint for each of the goods. The market-clearing condition for good n is:

$$\sum_{i=1}^I c_n^i = \sum_{i=1}^I e_n^i. \qquad (5.2)$$

Note that in the budget constraint we sum over all goods for one consumer, while in the market-clearing conditions we sum over all consumers for one good. The only additional assumptions that we will make throughout this chapter are: that I and N are positive integers, that all endowments e_n^i are positive and that all utility functions are strictly increasing in all arguments. The assumption of increasing utility functions is important because it implies that all prices are positive in equilibrium. We will use this fact below. Notice that we do not make any further assumptions like differentiability or concavity, and that we do not restrict attention to specific functional forms for utility. The results in this chapter rest solely on the general structure of the market-clearing model. We are now ready to define an equilibrium for this economy along the lines developed in Chapter 3.

An *allocation* is a set of values for consumption for each good and each consumer. A *competitive equilibrium* is an allocation $\{c_1^i, c_2^i, \ldots, c_N^i\}_{i=1}^I$ and a set of prices $\{p_1, p_2, \ldots, p_N\}$ such that:

[1] While this assumption may seem restrictive, in fact all results of this chapter can be shown for production economies as well. However, notation and algebra are more complicated with production, so we concentrate on the pure-exchange case.

- Taking prices as given, each consumer i chooses $\{c_1^i, c_2^i, \ldots, c_N^i\}$ as a solution to the maximization problem in equation (5.1); and
- Given the allocation, all market-clearing constraints in equation (5.2) are satisfied.

The model is far more general than it looks. For example, different goods could correspond to different points of time. In that case, the budget constraint would be interpreted as a present-value budget constraint, as introduced in Chapter 3. We can also incorporate uncertainty, in which case different goods would correspond to different states of the world. Good 1 could be consumption of sun-tan lotion in case it rains tomorrow, while good 2 could be sun-tan lotion in case it's sunny. Presumably, the consumer would want to consume different amounts of these goods, depending on the state of the world. By using such time- and state-contingent goods, we can adapt the model to almost any situation.

5.2 Normalization of Prices

In our model, the general level of prices is undetermined. For example, given any equilibrium, we can double all prices and get another equilibrium. We first ran into this phenomenon in the credit-market economy of Section 3.2, where it turned out that the price level P was arbitrary. An important application is the possibility of normalizing prices. Since it is possible to multiply prices by a positive constant and still have an equilibrium, the constant can be chosen such that one price is set to one. For example, if we want to normalize the price of the first good, we can choose the constant to be $1/p_1$. Then, when we multiply all prices by this constant, the normalized price of the first good becomes $(p_1)(1/p_1) = 1$. If for every equilibrium there is another one in which the price of the first good is one, there is no loss in generality in assuming that the price is one right away. Without always mentioning it explicitly, we make use of this fact in a number of places throughout this book. Normally the price of the consumption good is set to one, so that all prices can be interpreted in terms of the consumption good.[2] The good whose price is set to one is often called the *numéraire*.

In order to show that the price level is indeterminate, we are going to assume that we have already found an allocation $\{c_1^i, c_2^i, \ldots, c_N^i\}_{i=1}^I$ and a price system $\{p_1, p_2, \ldots, p_N\}$ that satisfy all the conditions for an equilibrium. We now want to show that if we multiply all prices by a constant $\gamma > 0$ we will still have an equilibrium. That is, the allocation $\{c_1^i, c_2^i, \ldots, c_N^i\}_{i=1}^I$ will still satisfy market-clearing, and the values for consumption will still be optimal choices for the consumers given the new price system $\{\gamma p_1, \gamma p_2, \ldots, \gamma p_N\}$.

It is obvious that the market-clearing constraints will continue to hold, since we have not changed the allocation and the prices do not enter in the market-clearing constraints. Therefore we only need to show that the allocation will still be optimal, given the new price

[2] Examples are the labor market model in Section 6.1 and the business-cycle model in Chapter 9. In both cases, the price of consumption is set to one.

system. We know already that the allocation is an optimal choice for the consumers given the old price system. If we can show that the new price system does not change the budget constraint of the consumer, then the consumer's problem with the new prices will be equivalent to the original problem, so it will have the same solution. The budget constraint with the new prices is:

$$\sum_{n=1}^{N}(\gamma p_n)c_n^i = \sum_{n=1}^{N}(\gamma p_n)e_n^i.$$

We can pull the common γ terms outside the summations, so we can divide each side by γ to yield:

$$\sum_{n=1}^{N} p_n c_n^i = \sum_{n=1}^{N} p_n e_n^i,$$

which is equal to the budget constraint under the original price system. The consumer's problem does not change, so it still has the same solution. This shows that the allocation $\{c_1^i, c_2^i, \ldots, c_N^i\}_{i=1}^I$ and prices $\{\gamma p_1, \gamma p_2, \ldots, \gamma p_N\}$ form an equilibrium as well.

The basic idea is that only relative prices matter, not absolute prices. If all prices are multiplied by a constant, income from selling the endowment will increase in the same proportion as the cost of consumption goods. As long as the relative prices are constant, such a change will not influence the decisions of consumers. Note that we did not need to look at any first-order conditions to prove our point. The possibility of normalizing prices derives from the basic structure of this market economy, not from specific assumptions about utility or technology.

5.3 Walras' Law

In a general equilibrium model, one market-clearing constraint is redundant. This means that if each consumer's budget constraint is satisfied and all but one market-clearing conditions hold, then the last market-clearing condition is satisfied automatically. This fact is of practical value, because it implies that we can omit one market-clearing constraint right away when computing an equilibrium. Without mentioning it, we made already use of this in Section 3.2. While the definition of equilibrium required the goods market to clear, the market-clearing constraints for goods were not actually used afterwards. This was possible because they were implied by the budget constraints and the fact that the bond market cleared. This feature of general equilibrium models is known as *Walras' Law*.

To see that Walras' law holds in our general pure-exchange economy, assume that the budget constraints for each of the I consumers and the market-clearing constraints for the first $N-1$ goods are satisfied. We want to show that the last market-clearing constraint for good N is also satisfied. Summing the budget constraints over all consumers yields:

$$\sum_{i=1}^{I}\sum_{n=1}^{N} p_n c_n^i = \sum_{i=1}^{I}\sum_{n=1}^{N} p_n e_n^i.$$

Rearranging gives:

$$\sum_{n=1}^{N}\sum_{i=1}^{I} p_n c_n^i = \sum_{n=1}^{N}\sum_{i=1}^{I} p_n e_n^i,$$

$$\sum_{n=1}^{N} p_n \sum_{i=1}^{I} c_n^i = \sum_{n=1}^{N} p_n \sum_{i=1}^{I} e_n^i, \text{ or:}$$

$$\sum_{n=1}^{N} p_n \left[\sum_{i=1}^{I} c_n^i - \sum_{i=1}^{I} e_n^i\right] = 0. \tag{5.3}$$

Inside the brackets we have the difference between the total consumption and the total endowment of good n. If the market for good n clears, this difference is zero. Since we assume that the first $N-1$ markets clear, equation (5.3) becomes:

$$p_N \left[\sum_{i=1}^{I} c_N^i - \sum_{i=1}^{I} e_N^i\right] = 0.$$

Since $p_N > 0$, this implies:

$$\sum_{i=1}^{I} c_N^i - \sum_{i=1}^{I} e_N^i = 0, \text{ or:}$$

$$\sum_{i=1}^{I} c_N^i = \sum_{i=1}^{I} e_N^i.$$

Thus the Nth market will clear as well.

The intuition behind this result is easiest to see when the number of markets is small. If there is only one good, say apples, the budget constraints of the consumers imply that each consumer eats as many apples as she is endowed with. Then the market-clearing constraint has to be satisfied as well, since it is already satisfied on the level of each consumer. Now assume there is one more good, say oranges, and the market-clearing constraint for apples is satisfied. That implies that total expenditures on apples equal total income from selling apples to other consumers. Since each consumer balances spending with income, expenditures have to equal income for oranges as well, so the market for oranges clears.

5.4 The First Welfare Theorem

The first two features of general equilibrium models that we presented in this chapter were technical. They are of some help in computing equilibria, but taken for themselves they do not provide any deep new insights that could be applied to the real world. The situation is different with the last feature that we are going to address, the efficiency of outcomes in general equilibrium economies. This result has important implications for the welfare properties of economic models, and it plays a key role in the theory of comparative economic systems.

Before we can show that equilibria in our model are efficient, we have to make precise what exactly is meant by efficiency. In economics, we usually use the concept of *Pareto efficiency*. Another term for Pareto efficiency is Pareto optimality, and we will use both versions interchangeably. An allocation is Pareto efficient if it satisfies the market-clearing conditions and if there is no other allocation that: (1) also satisfies the market-clearing conditions; and (2) makes everyone better off. In our model, an allocation $\{c_1^i, c_2^i, \dots, c_N^i\}_{i=1}^I$ is therefore Pareto efficient if the market-clearing constraint in equation (5.2) holds for each of the N goods and if there is no other allocation $\{\bar{c}_1^i, \bar{c}_2^i, \dots, \bar{c}_N^i\}_{i=1}^I$ that also satisfies market-clearing and such that:

$$u(\bar{c}_1^i, \bar{c}_2^i, \dots, \bar{c}_N^i) > u(c_1^i, c_2^i, \dots, c_N^i)$$

for every consumer i.[3] Notice that the concept of Pareto optimality does not require us to take any stand on the issue of distribution. For example, if utility functions are strictly increasing, one Pareto-optimal allocation is to have one consumer consume all the resources in the economy. Such an allocation is clearly feasible, and every alternative allocation makes this one consumer worse off. A Pareto-efficient allocation is therefore not necessarily one that many people would consider "fair" or even "optimal". On the other hand, many people would agree that it is better to make everyone better off as long as it is possible to do so. Therefore we can interpret Pareto efficiency as a minimal standard for a "good" allocation, rather than as a criterion for the "best" one.

We now want to show that any equilibrium allocation in our economy is necessarily Pareto optimal. The equilibrium consists of an allocation $\{c_1^i, c_2^i, \dots, c_N^i\}_{i=1}^I$ and a price system $\{p_1, p_2, \dots, p_N\}$. Since market-clearing conditions hold for any equilibrium allocation, the first requirement for Pareto optimality is automatically satisfied. The second part takes a little more work. We want to show that there is no other allocation that also satisfies market-clearing and that makes everyone better off. We are going to prove this by contradiction. That is, we will assume that such a better allocation actually exists, and then we will show that this leads us to a contradiction. Let us therefore assume that there is another allocation $\{\bar{c}_1^i, \bar{c}_2^i, \dots, \bar{c}_N^i\}_{i=1}^I$ that satisfies market-clearing and such that:

$$u(\bar{c}_1^i, \bar{c}_2^i, \dots, \bar{c}_N^i) > u(c_1^i, c_2^i, \dots, c_N^i)$$

[3] A weaker notion of Pareto efficiency replaces the strict inequality with weak inequalities plus the requirement that at least one person is strictly better off. The proof of the First Welfare Theorem still goes through with the weaker version, but for simplicity we use strict inequalities.

for every consumer i. We know that consumer i maximizes utility subject to the budget constraint. Since the consumer chooses $\{c_1^i, c_2^i, \ldots, c_N^i\}$ even though $\{\bar{c}_1^i, \bar{c}_2^i, \ldots, \bar{c}_N^i\}$ yields higher utility, it has to be the case that $\{\bar{c}_1^i, \bar{c}_2^i, \ldots, \bar{c}_N^i\}$ violates the consumer's budget constraint:

$$\sum_{n=1}^{N} p_n \bar{c}_n^i > \sum_{n=1}^{N} p_n e_n^i. \tag{5.4}$$

Otherwise, the optimizing consumers would not have chosen the consumptions in the allocation $\{c_1^i, c_2^i, \ldots, c_N^i\}_{i=1}^{I}$ in the first place. Summing equation (5.4) over all consumers and rearranging yields:

$$\begin{aligned}
\sum_{i=1}^{I}\sum_{n=1}^{N} p_n \bar{c}_n^i &> \sum_{i=1}^{I}\sum_{n=1}^{N} p_n e_n^i, \\
\sum_{n=1}^{N}\sum_{i=1}^{I} p_n \bar{c}_n^i &> \sum_{n=1}^{N}\sum_{i=1}^{I} p_n e_n^i, \\
\sum_{n=1}^{N} p_n \sum_{i=1}^{I} \bar{c}_n^i &> \sum_{n=1}^{N} p_n \sum_{i=1}^{I} e_n^i, \text{ so:} \\
\sum_{n=1}^{N} p_n \left[\sum_{i=1}^{I} \bar{c}_n^i - \sum_{i=1}^{I} e_n^i\right] &> 0.
\end{aligned}$$

We assumed that the allocation $\{\bar{c}_1^i, \bar{c}_2^i, \ldots, \bar{c}_N^i\}_{i=1}^{I}$ satisfied market-clearing. Therefore the terms inside the brackets are all zero. This implies $0 > 0$, which is a contradiction. Therefore, no such allocation $\{\bar{c}_1^i, \bar{c}_2^i, \ldots, \bar{c}_N^i\}_{i=1}^{I}$ can exist, and the original equilibrium allocation $\{c_1^i, c_2^i, \ldots, c_N^i\}_{i=1}^{I}$ is Pareto optimal.

Since any competitive equilibrium is Pareto optimal, there is no possibility of a redistribution of goods that makes everyone better off than before. Individual optimization together with the existence of markets imply that all gains from trade are exploited.

There is also a partial converse to the result that we just proved, the "Second Welfare Theorem". While the First Welfare Theorem says that every competitive equilibrium is Pareto efficient, the Second Welfare Theorem states that every Pareto optimum can be implemented as a competitive equilibrium, as long as wealth can be redistributed in advance. The Second Welfare Theorem rests on some extra assumptions and is harder to prove, so we omit it here. In economies with a single consumer there are no distributional issues, and the two theorems are equivalent.

Variable	Definition
N	Number of goods
p_n	Price of good n
I	Number of consumers
c_n^i	Consumption of good n by consumer i
$u_i(\cdot)$	Utility function of consumer i
e_n^i	Endowment with good n of consumer i
γ	Arbitrary proportionality factor

Table 5.1: Notation for Chapter 5

Exercises

Exercise 5.1 (Easy)
Show that Walras' law holds for the credit-market economy that we discussed in Chapter 3.2. That is, use the consumer's budget constraints and the market-clearing conditions for goods to derive the market-clearing condition for bonds in equation (3.9).

Exercise 5.2 (Hard)
Assume that the equilibrium price of one of the N goods is zero. What is the economic interpretation of this situation? Which of our assumptions ruled out that a price equals zero? Why? Does Walras' Law continue to hold? What about the First Welfare Theorem?