Chapter 6

The Labor Market

This chapter works out the details of two separate models. Section 6.1 contains a one-period model in which households are both demanders and suppliers of labor. Market clearing in the labor market determines the equilibrium wage rate. Section 6.2 further develops the two-period model from Chapter 3. In this case, the households are permitted to choose their labor supply in each period.

6.1 Equilibrium in the Labor Market

This economy consists of a large number of identical households. Each owns a farm on which it employs labor to make consumption goods, and each has labor that can be supplied to other farmers. For each unit of labor supplied to others, a household receives a wage w, which is paid in consumption goods. Households take this wage as given. In order to make the exposition clear, we prohibit a household from providing labor for its own farm. (This has no bearing on the results of the model.)

The first task of the representative household is to maximize the profit of its farm. The output of the farm is given by a production function $f(l_d)$, where l_d is the labor demanded (i.e., employed) by that farm. The only expense of the farm is its labor costs, so the profit of the farm is: $\pi = f(l_d) - wl_d$. The household that owns the farm chooses how much labor l_d to hire. The first-order condition with respect to l_d is:

$$\frac{\partial \pi}{\partial l_d} = f'(l_d^\star) - w = 0, \text{ so:}$$

$$w = f'(l_d^\star). \tag{6.1}$$

This implies that the household will continue to hire laborers until the marginal product of additional labor has fallen to the market wage. Equation (6.1) pins down the optimal labor input $l_d^\star$. Plugging this into the profit equation yields the maximized profit of the household: $\pi^\star = f(l_d^\star) - wl_d^\star$.

After the profit of the farm is maximized, the household must decide how much to work on the farms of others and how much to consume. Its preferences are given by $u(c, l_s)$, where c is the household's consumption, and l_s is the amount of labor that the household supplies to the farms of other households. The household gets income $\pi^\star$ from running its own farm and labor income from working on the farms of others. Accordingly, the household's budget is:

$$c = \pi^\star + wl_s,$$

so Lagrangean for the household's problem is:

$$\mathcal{L} = u(c, l_s) + \lambda[\pi^\star + wl_s - c].$$

The first-order condition with respect to c is:

$$u_1(c^\star, l_s^\star) + \lambda^\star[-1] = 0, \tag{FOC c}$$

and that with respect to l_s is:

$$u_2(c^\star, l_s^\star) + \lambda^\star[w] = 0. \tag{FOC l_s}$$

Solving each of these for λ and setting them equal yields:

$$-\frac{u_2(c^\star, l_s^\star)}{u_1(c^\star, l_s^\star)} = w, \tag{6.2}$$

so the household continues to supply labor until its marginal rate of substitution of labor for consumption falls to the wage the household receives.

Given particular functional forms for $u(\cdot)$ and $f(\cdot)$, we can solve for the optimal choices $l_d^\star$ and $l_s^\star$ and compute the equilibrium wage. For example, assume:

$$u(c, l) = \ln(c) + \ln(1 - l), \text{ and:}$$
$$f(l) = Al^\alpha.$$

Under these functional forms, equation (6.1) becomes:

$$w = A\alpha(l_d^\star)^{\alpha-1}, \text{ so:}$$
$$l_d^\star = \left(\frac{A\alpha}{w}\right)^{\frac{1}{1-\alpha}}. \tag{6.3}$$

This implies that the profit $\pi^\star$ of each household is:

$$\pi^\star = A\left(\frac{A\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{A\alpha}{w}\right)^{\frac{1}{1-\alpha}}.$$

After some factoring and algebraic manipulation, this becomes:

$$\pi^\star = A(1-\alpha)\left(\frac{A\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}. \tag{6.4}$$

Under the given preferences, we have $u_1(c,l) = 1/c$ and $u_2(c,l) = -1/(1-l)$. Recall, the budget equation implies $c = \pi + wl_s$. Plugging these into equation (6.2) gives us:

$$\frac{\pi^\star + wl_s^\star}{1-l_s^\star} = w,$$

which reduces to:

$$l_s^\star = \frac{1}{2} - \frac{\pi^\star}{2w}.$$

Plugging in $\pi^\star$ from equation (6.4) yields:

$$l_s^\star = \frac{1}{2} - \left(\frac{1}{2w}\right) A(1-\alpha)\left(\frac{A\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}},$$

which reduces to:

$$l_s^\star = \frac{1}{2} - \left(\frac{1-\alpha}{2\alpha}\right)\left(\frac{A\alpha}{w}\right)^{\frac{1}{1-\alpha}}.$$

Now we have determined the household's optimal supply of labor $l_s^\star$ as a function of the market wage w, and we have calculated the household's optimal choice of labor to hire $l_d^\star$ for a given wage. Since all household's are identical, equilibrium occurs where the household's supply equals the household's demand. Accordingly, we set $l_s^\star = l_d^\star$ and call the resulting wage $w^\star$:

$$\frac{1}{2} - \left(\frac{1-\alpha}{2\alpha}\right)\left(\frac{A\alpha}{w^\star}\right)^{\frac{1}{1-\alpha}} = \left(\frac{A\alpha}{w^\star}\right)^{\frac{1}{1-\alpha}}.$$

We gather like terms to get:

$$\frac{1}{2} = \left[1 + \left(\frac{1-\alpha}{2\alpha}\right)\right]\left(\frac{A\alpha}{w^\star}\right)^{\frac{1}{1-\alpha}}.$$

Further algebraic manipulation yields:

$$w^\star = A\alpha\left(\frac{1+\alpha}{\alpha}\right)^{1-\alpha}.$$

Finally, we plug this equilibrium wage back into our expressions for $l_s^\star$ and $l_d^\star$, which were in terms of w. For example, plugging the formula for $w^\star$ into equation (6.3) gives us:

$$l_d^\star = \left(\frac{A\alpha}{w^\star}\right)^{\frac{1}{1-\alpha}} = \left[\frac{A\alpha}{A\alpha\left(\frac{1+\alpha}{\alpha}\right)^{1-\alpha}}\right]^{\frac{1}{1-\alpha}} = \frac{\alpha}{1+\alpha}.$$

Of course, we get the same answer for $l_s^\star$, since supply must equal demand in equilibrium.

Given these answers for $l_s^\star$, $l_d^\star$, and $w^\star$, we can perform comparative statics to determine how the equilibrium values are influenced by changes in the underlying parameters. For example, suppose the economy experiences a positive shock to its productivity. This could be represented by an increase in the A parameter to the production function. We might be interested in how that affects the equilibrium wage:

$$\frac{\partial w^\star}{\partial A} = \alpha \left(\frac{\alpha + 1}{\alpha} \right)^{1-\alpha} > 0,$$

so the equilibrium wage will increase. Just by inspecting the formula for $l_d^\star$ and $l_s^\star$, we know that labor supply and labor demand will be unchanged, since A does not appear. The intuition of this result is straightforward. With the new, higher productivity, households will be more inclined to hire labor, but this is exactly offset by the fact that the new wage is higher. On the other side, households are enticed to work more because of the higher wage, but at the same time they are wealthier, so they want to enjoy more leisure, which is a normal good. Under these preferences, the two effects cancel.

Variable	Definition
w	Wage in consumption goods per unit of labor
l_d	Labor demanded by owner of farm
$f(l_d)$	Output of farm
π	Profit of farm
c	Consumption of household
l_s	Labor supplied by household
$u(c, l_s)$	Utility of household
λ	Lagrange multiplier
$\mathcal{L}$	Lagrangean
A	Parameter of the production function
α	Parameter of the production function

Table 6.1: Notation for Section 6.1

6.2 Intertemporal Labor Choice

The model in this section is a pure extension of that developed in Section 3.2. In that model the representative household lived for two periods. Each period, the household got an endowment, e_1 and e_2. The household chose each period's consumption, c_1 and c_2, and the number of dollars of bonds b_1 to carry from period 1 to period 2.

The model presented here is almost identical. The only difference is that the household exerts labor effort in order to acquire goods instead of having them endowed exogenously. In particular, the household has some production function: $y_t = f(l_t)$. The household chooses each period's labor, l_1 and l_2. The income y_t takes the place of the endowment e_t in the model from Chapter 3.

The household's maximization problem is:

$$\max_{c_1, c_2, l_1, l_2, b_1} \{u(c_1, l_1) + \beta u(c_2, l_2)\}, \text{ subject to:}$$
$$Pf(l_1) = Pc_1 + b_1, \text{ and:}$$
$$Pf(l_2) + b_1(1 + R) = Pc_2.$$

Refer to Chapter 3 for a discussion of: (i) the budget constraints, (ii) the meaning of the price level P and interest rate R, and (iii) how the bonds work. The Lagrangean is:

$$\mathcal{L} = u(c_1, l_1) + \beta u(c_2, l_2) + \lambda_1[Pf(l_1) - Pc_1 - b_1] + \lambda_2[Pf(l_2) + b_1(1 + R) - Pc_2].$$

There are seven first-order conditions:

$$\text{(FOC } c_1) \qquad u_1(c_1^\star, l_1^\star) + \lambda_1^\star[-P] = 0,$$
$$\text{(FOC } c_2) \qquad \beta u_1(c_2^\star, l_2^\star) + \lambda_2^\star[-P] = 0,$$
$$\text{(FOC } l_1) \qquad u_2(c_1^\star, l_1^\star) + \lambda_1^\star[Pf'(l_1^\star)] = 0,$$
$$\text{(FOC } l_2) \qquad \beta u_2(c_2^\star, l_2^\star) + \lambda_2^\star[Pf'(l_2^\star)] = 0, \text{ and:}$$
$$\text{(FOC } b_1) \qquad \lambda_1^\star[-1] + \lambda_2^\star[(1 + R)] = 0.$$

We leave off the FOCs with respect to λ_1 and λ_2 because we know that they reproduce the constraints. Solving equations (FOC c_1) and (FOC c_2) for the Lagrange multipliers and plugging into equation (FOC b_1) yields:

$$\frac{u_1(c_1^\star, l_1^\star)}{u_1(c_2^\star, l_2^\star)} = \beta(1 + R). \tag{6.5}$$

This is the same Euler equation we saw in Chapter 3. Solving equations (FOC l_1) and (FOC l_2) for the Lagrange multipliers and plugging into equation (FOC b_1) yields:

$$\frac{u_2(c_1^\star, l_1^\star)}{u_2(c_2^\star, l_2^\star)} = \frac{\beta(1 + R)f'(l_1^\star)}{f'(l_2^\star)}. \tag{6.6}$$

This is an Euler equation too, since it too relates marginal utilities in consecutive periods. This time, it relates the marginal utilities of labor.

We could analyze equations (6.5) and (6.6) in terms of the abstract functions, $u(\cdot)$ and $f(\cdot)$, but it is much simpler to assume particular functional forms and then carry out the analysis. Accordingly, assume:

$$u(c, l) = \ln(c) + \ln(1 - l), \text{ and:}$$
$$f(l) = Al^\alpha.$$

Plugging the utility function into equation (6.5) yields:

$$\frac{c_2^\star}{c_1^\star} = \beta(1+R),$$

just like in Chapter 3. All the analysis from that chapter carries forward. For example, this equation implies that a higher interest rate R implies that the household consumes more in period 2 relative to period 1. Equation (6.6) becomes:

$$\frac{(1-l_2^\star)(l_2^\star)^{1-\alpha}}{(1-l_1^\star)(l_1^\star)^{1-\alpha}} = \beta(1+R). \tag{6.7}$$

Analysis of this equation is somewhat tricky. As a first step, let $D(l) = (1-l)l^{1-\alpha}$ be a helper function. Then equation (6.7) can be written as:

$$\frac{D(l_1^\star)}{D(l_2^\star)} = \beta(1+R). \tag{6.8}$$

Now, let's consider how $D(l)$ changes when l changes:

$$\begin{aligned} D'(l) &= (1-l)(\alpha-1)l^{\alpha-2} + l^{\alpha-1}(-1) \\ &= l^{\alpha-2}(\alpha - \alpha l - 1 + l - l) \\ &= l^{\alpha-2}[\alpha(1-l)-1]. \end{aligned}$$

We know that $l^x > 0$ for all x, so $l^{\alpha-2} > 0$. Further, $\alpha(1-l) < 1$, since l and α are both between zero and one. Putting these together, we find that $D'(l) < 0$, so increasing l causes $D(l)$ to decrease.

Now, think about what must happen to $l_1^\star$ and $l_2^\star$ in equation (6.8) if the interest rate R increases. That means that the right-hand side increases, so the left-hand side must increase in order to maintain the equality. There are two ways that the left-hand side can increase: either (i) $D(l_2^\star)$ increases, or (ii) $D(l_1^\star)$ decreases (or some combination of both). We already determined that $D(l)$ and l move in opposite directions. Hence, either $l_2^\star$ decreases or $l_1^\star$ increases (or some combination of both). Either way, $l_2^\star/l_1^\star$ decreases. The intuition of this result is as follows. A higher interest rate means the household has better investment opportunities in period 1. In order to take advantage of those, the household works relatively harder in period 1, so it earns more money to invest.

Exercises

Exercise 6.1 (Hard)
This economy contains 1,100 households. Of these, 400 own type-a farms, and the other 700 own type-b farms. We use superscripts to denote which type of farm. A household of type $j \in \{a, b\}$ demands (i.e., it hires) l_d^j units of labor, measured in hours. (The "d" is for

Variable	Definition
$U(\cdot)$	Overall utility
t	Time
c_t	Consumption at period t
l_t	Labor at period t
$u(\cdot)$	Period utility
β	Household's discount factor
y_t	Household's income in period t, in units of consumption
$f(l_t)$	Production function
P	Cost of a unit of consumption
R	Nominal interest rate
b_t	Number of dollars of bonds bought at period t
λ_t	Lagrange multiplier in period t
$\mathcal{L}$	Lagrangean
N	Number of households
$D(l)$	Helper function, to simplify notation

Table 6.2: Notation for Section 6.2

demand.) The type-j household supplies l_s^j of labor. (The "s" is for supply.) The household need not use its own labor on its own farm. It can hire other laborers and can supply its own labor for work on other farms. The wage per hour of work in this economy is w. This is expressed in consumption units, i.e., households can eat w. Every household takes the wage w as given. Preferences are:

$$u(c^j, l_s^j) = \ln(c^j) + \ln(24 - l_s^j),$$

where c^j is the household's consumption. Production on type-a farms is given by:

$$y = (l_d^a)^{0.5},$$

and that on type-b farms is:

$$y = 2(l_d^b)^{0.5}.$$

We are going to solve for the wage that clears the market. In order to do that, we need to determine demand and supply of labor as a function of the wage.

If an owner of a type-a farm hires l_d^a hours of labor at wage w per hour, the farm owner will make profit:

$$\pi^a = (l_d^a)^{0.5} - wl_d^a.$$

1. Use calculus to solve for a type-a farmer's profit-maximizing choice of labor l_d^a to hire as a function of the wage w. Call this amount of labor $l_d^{a\star}$. It will be a function of w. Calculate the profit of a type-a farmer as a function of w. Call this profit $\pi^{a\star}$.
2. If an owner of a type-b farm hires l_d^b hours of labor at wage w per hour, the farm owner will make profit:

$$\pi^b = 2(l_d^b)^{0.5} - wl_d^b.$$

   Repeat part 1 but for type-b farmers. Call a type-b farmer's profit-maximizing choice of labor $l_d^{b\star}$. Calculate the profit of a type-b farmer as a function of w. Call this profit $\pi^{b\star}$.
3. If a type-a farmer works l_s^a, then that farmer's income will be: $\pi^{a\star} + wl_s^a$. Accordingly, the budget constraint for type-a farmers is:

$$c^a = \pi^{a\star} + wl_s^a.$$

   A type-a household chooses its labor supply by maximizing its utility subject to its budget. Determine a type-a household's optimal choice of labor to supply for any given wage w. Call this amount of labor $l_s^{a\star}$.
4. Repeat part 3 but for type-b households. Call this amount of labor $l_s^{b\star}$.
5. Aggregate labor demand is just the sum of the demands of all the farm owners. Calculate aggregate demand by adding up the labor demands of the 400 type-a farmers and the 700 type-b farmers. This will be an expression for hours of labor l in terms of the market wage w. Call the result $l_d^\star$.
6. Aggregate labor supply is just the sum of the supplies of all the households. Calculate aggregate supply, and call it $l_s^\star$.
7. Use your results from parts 5 and 6 to solve for the equilibrium wage $w^\star$. (Set the two expressions equal and solve for w.)

Exercise 6.2 (Hard)
Consider an economy with many identical households. Each household owns a business that employs both capital (machinery) k and labor l_d to produce output y. (The "d" is for demand.) Production possibilities are represented by $y = Ak^{\frac{3}{10}}(l_d)^{\frac{7}{10}}$. The stock of capital that each household owns is fixed. It may employ labor at the prevailing wage w per unit of labor l_d. Each household takes the wage as given. The profit of each household from running its business is:

$$\pi = y - wl_d = Ak^{\frac{3}{10}}(l_d)^{\frac{7}{10}} - wl_d. \tag{6.9}$$

1. Determine the optimal amount of labor for each household to hire as a function of its capital endowment k and the prevailing wage w. Call this amount of labor $l_d^\star$.

2. Plug $l_d^\star$ back into equation (6.9) to get the maximized profit of the household. Call this profit $\pi^\star$.

3. Each household has preferences over its consumption c and labor supply l_s. These preferences are represented by the utility function: $u(c, l_s) = c^{\frac{1}{2}}(1 - l_s)^{\frac{1}{2}}$. Each household has an endowment of labor can be used in the household's own business or rented to others at the wage w. If the household supplies labor l_s, then it will earn labor income wl_s. Output, wages, and profit are all quoted in terms of real goods, so they can be consumed directly. Set up the household's problem for choosing its labor supply l_s. Write it in the following form:

$$\max_{\text{choices}} \{\text{objective}\} \quad \text{subject to: constraints}$$

4. Carry out the maximization from part 3 to derive the optimal labor supply $l_s^\star$.

5. Determine the equilibrium wage $w^\star$ in this economy.

6. How does the equilibrium wage $w^\star$ change with the amount of capital k owned by each household?

7. What does this model imply about the wage differences between the U.S. and Mexico? What about immigration between the two countries?