Epistemic Game Theory:
Online Appendix

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Preliminaries

Fix a finite type structure $\mathcal{T} = \left(I, (S_{-i}, T_i, \beta_i)_{i\in I}\right)$ and a probability $\mu \in \Delta(S \times T)$. Let $\mathcal{T}^{\mu} = \left(I, (S_{-i}, T_i^{\mu}, \beta_i^{\mu})_{i\in I}\right)$ be a type structure that admits μ as a common prior and such that $T_i^{\mu} \subseteq T_i$ for every i.

Fix a player i and a type profile $t^* \in T^{\mu}$. Define

$$E^1(t_i^*) = \{(s, t_{-i}, t_i^*) : \mu(s, t_{-i}, t_i^* | t_i^*) > 0\}$$

Suppose $E^k(t_i^*)$ has been defined for every $1 < k \leq n$ and let

$$E^{n+1}(t_i^*) = \left\{(s,t) : \exists \left(s', t'\right) \in E^n, j \in I \text{ s.t. } t_j = t'_j \text{ and } \mu\left(s, t_{-j}, t'_j | t'_j\right) > 0\right\}$$

Let $E(t_i^*) = \cup_{n=1}^{\infty} E^n(t_i^*)$ and $E(t^*) = \cup_{i\in I} E(t_i^*)$.

Proposition. *Let t_i^* be in $CP_i(\mu)$ and $\nu = \mu(\cdot|E(t_i^*))$. Define $T_j^{\nu} = proj_{T_j} E(t_i^*)$ for every j and let $\mathcal{T}^{\nu} = \left(I, \left(S_{-j}, T_j^{\nu}, \beta_j^{\nu}\right)_{j\in I}\right)$ be the type structure generated by the common prior ν. Then t_i^* is in $CP_i(\nu)$. In particular, if μ is minimal for t_i^* then $\nu = \mu$.*

Proof. We first prove that for all j and all $t_j \in T_j^{\mu}$,

$$\nu\left(S \times T_{-j}^{\mu} \times \{t_j\}\right) > 0 \implies \text{marg}_{S_{-j}\times T_{-j}} \nu(\cdot|t_j) = \text{marg}_{S_{-j}\times T_{-j}} \mu(\cdot|t_j).$$

By definition, if $\mu\left(S \times T_{-j}^{\mu} \times \{t_j\}\right) > 0$ then

$$\mu(s_{-j}, t_{-j}|t_j) = \frac{\mu((s_{-j}, t_{-j}) \times S_j \times \{t_j\})}{\sum_{\left(s'_{-j}, t'_{-j}\right)\in S_{-j}\times T_{-j}} \mu\left(\left(s'_{-j}, t'_{-j}\right) \times S_j \times \{t_j\}\right)}$$

For every $(s_{-j},t_{-j})\in S_{-j}\times T_{-j}$. If $\nu\left(S\times T^{\mu}_{-j}\times\{t_j\}\right)>0$ then $t_j\in \text{proj}_{T_j}E^k(t_i^*)$ for some k. For every $\left(s'_{-j},t'_{-j}\right)\in S_{-j}\times T_{-j}$, if

$$\mu\left(\left(s'_{-j},t'_{-j}\right)\times S_j\times\{t_j\}\right)>0$$

then $\left(s'_{-j},t'_{-j}\right)\times S_j\times\{t_j\}\subseteq E^{k+1}(t_i^*)\subseteq E(t_i^*)$, thus

$$\nu\left(\left(s'_{-j},t'_{-j}\right)\times S_j\times\{t_j\}\right)=\frac{\mu\left(\left(s'_{-j},t'_{-j}\right)\times S_j\times\{t_j\}\right)}{\mu\left(E\left(t_i^*\right)\right)}$$

Therefore

$$\begin{aligned}\mu\left(s_{-j},t_{-j}|t_j\right) &= \frac{\frac{1}{\mu(E(t_i^*))}}{\frac{1}{\mu(E(t_i^*))}}\frac{\mu\left(\left(s_{-j},t_{-j}\right)\times S_j\times\{t_j\}\right)}{\sum_{\left(s'_{-j},t'_{-j}\right)\in S_{-j}\times T_{-j}}\mu\left(\left(s'_{-j},t'_{-j}\right)\times S_j\times\{t_j\}\right)}\\ &= \frac{\nu\left(\left(s_{-j},t_{-j}\right)\times S_j\times\{t_j\}\right)}{\sum_{\left(s'_{-j},t'_{-j}\right)\in S_{-j}\times T_{-j}}\nu\left(\left(s'_{-j},t'_{-j}\right)\times S_j\times\{t_j\}\right)}\\ &= \nu\left(s_{-j},t_{-j}|t_j\right)\end{aligned}$$

We can conclude that for every $j\in I$, $\beta^{\nu}_j(t_j)=\beta^{\mu}_j(t_j)$ for each $t_j\in T^{\nu}_j$.

It remains to prove that $\varphi_i(\mathcal{T}^{\nu})(t_i^*)=\varphi_i(\mathcal{T}^{\mu})(t_i^*)$. For every j, $t_j\in T^{\nu}_j$ and $k\geq 0$, let $\varphi^k_j(\mathcal{T}^{\nu})(t_j)$ be the k-th order belief of type t_j in the type structure $\mathcal{T}^{\nu}$. Define $\varphi^k_j(\mathcal{T}^{\mu})$ analogously. For every $j\in I$ and $t_j\in T^{\nu}_j$, we have $\beta^{v}(t_j)=\beta^{\mu}(t_j)$, hence $\varphi^1_j(\mathcal{T}^{\nu})(t_j)=\varphi^1_j(\mathcal{T}^{\mu})(t_j)$. Suppose $\varphi^k_j(\mathcal{T}^{\nu})(t_j)=\varphi^k_j(\mathcal{T}^{\mu})(t_j)$ for all j, $k\leq K$ and $t_j\in T^{\nu}_j$. Then

$$\beta^{\nu}(t_j)\left(\left\{(s_{-j},t_{-j}):\varphi^K_{-j}(\mathcal{T}^{\nu})(t_{-j})=h^K_{-j}\right\}\right)=\beta^{\mu}(t_j)\left(\left\{(s_{-j},t_{-j}):\varphi^K_{-j}(\mathcal{T}^{\mu})(t_{-j})=h^K_{-j}\right\}\right)$$

for every $h^K_{-j}\in\Delta\left(X^{K-1}_{-j}\right)$. Therefore $\varphi^{K+1}_j(\mathcal{T}^{\mu})(t_j)=\varphi^{K+1}_j(\mathcal{T}^{\nu})(t_j)$. Since this is true for every K, we have $\varphi_j(\mathcal{T}^{\mu})(t_j)=\varphi_j(\mathcal{T}^{\nu})(t_j)$ for every $t_j\in T^{\nu}_j$, in particular, for t_i^*. This concludes the proof that t_i^* is in $CP_i(\nu)$. □

An analogous result holds for type profiles. We omit the proof, which is an almost exact replica of the proof of Proposition 1.

Proposition. *Let t^* be in $CP(\mu)$ and define $\nu=\mu(\cdot|E(t^*))$. Define $T^{\nu}_i=proj_{T_i}E(t^*)$ for every $i\in I$ and let $\mathcal{T}^{\nu}=\left(I,(S_{-i},T^{\nu}_i,\beta^{\nu}_i)_{i\in I}\right)$ be the type space generated by the common prior ν. Then t^* is in $CP(\nu)$. In particular, if μ is minimal for t^* then $\nu=\mu$.*

Events across type structures

Let R^{μ}, $B^{k}R^{\mu}$ and CBR^{μ} be the events corresponding to, respectively, "rationality", "k-th order belief in rationality" and "common belief in rationality" in the type structure $\mathcal{T}^{\mu}$. In the proofs we will not formally distinguish between CBR and CBR^{μ}. This is justified by the next result.

Proposition. *If* $(s_i, t_i) \in CP_i(\mu) \cap CBR_i$*, then* $(s_i, t_i) \in CBR_i^{\mu}$.

Proof. Let R^{*}, $B^{k}R^{*}$ and CBR^{*} be the events corresponding to, respectively, *rationality*, *k-th order belief in rationality* and *common belief in rationality* in the universal type structure $\mathcal{H} = \left(I, (S_{-i}, H_i, f_i)_{i \in I}\right)$. For every i, let $\psi_i(\mathcal{T}) : S_i \times T_i \to S_i \times H_i$ be the map defined as

$$\psi_i(\mathcal{T})(s_i, t_i) = (s_i, \varphi(\mathcal{T})(t_i))$$

for every (s_i, t_i). As is well known, $B^{k}R^{*}$ and R^{*} are measurable events, and $\psi_i(\mathcal{T}^{\mu})$ and $\psi_i(\mathcal{T})$ are measurable maps. Furthermore, for every i, every event $E_{-i} \subseteq S_{-i} \times H_{-i}$ and every type $t_i \in T_i$,

$$f_i(\varphi_i(\mathcal{T})(t_i))(E_{-i}) = \beta_i(t_i)\left(\psi_{-i}(\mathcal{T})^{-1}(E_{-i})\right)$$

where $\psi_{-i}(\mathcal{T}) = \prod_{j \neq i} \psi_j(\mathcal{T})$. Define analogously the functions $(\psi_i(\mathcal{T}^{\mu}))_{i \in I}$.

Let $\psi = \prod_{i \in I} \psi_i$. It can be easily checked that $R = \psi(\mathcal{T})^{-1}(R^{*})$ and $R^{\mu} = \psi(\mathcal{T}^{\mu})^{-1}(R^{*})$. Suppose for every $k \leq K$ we have $B^{k}R = \psi(\mathcal{T})^{-1}\left(B^{k}R^{*}\right)$ and $B^{k}R^{\mu} = \psi(\mathcal{T}^{\mu})^{-1}\left(B^{k}R^{*}\right)$. It follows from

$$\begin{aligned} \beta_i(t_i)\left(B^{K}R\right) &= \beta_i(t_i)\left(\psi_{-i}(\mathcal{T})^{-1}\left(B^{K}R^{*}\right)\right) \\ &= f_i(\varphi_i(\mathcal{T})(t_i))\left(B^{K}R^{*}\right) \end{aligned}$$

that $(s_i, t_i) \in B_i\left(B^{K}R\right)$ if and only if $(s_i, \varphi(\mathcal{T})(t_i)) \in B_i\left(B^{K}R^{*}\right)$. Equivalently, $B_i\left(B^{K}R\right) = \psi_i(\mathcal{T})^{-1}\left(B_i\left(B^{K}R^{*}\right)\right)$ for every i. Therefore $B\left(B^{K}R\right) = \psi(\mathcal{T})^{-1}\left(B\left(B^{K}R^{*}\right)\right)$. Hence

$$B^{K+1}R = B^{K}R \cap B\left(B^{K}R\right) = \psi(\mathcal{T})^{-1}\left(B^{K}R^{*}\right) \cap \psi(\mathcal{T})^{-1}\left(B\left(B^{K}R^{*}\right)\right) = \psi(\mathcal{T})^{-1}\left(B^{K+1}R^{*}\right)$$

By induction, we can conclude that $B^{k}R^{\mu} = \psi(\mathcal{T}^{\mu})^{-1}\left(B^{k}R^{*}\right)$ for every k. Moreover,

$$CBR = \cap_k B^{k}R = \cap_k \psi(\mathcal{T})^{-1}\left(B^{k}R^{*}\right) = \psi(\mathcal{T})^{-1}\left(\cap_k B^{k}R^{*}\right) = \psi(\mathcal{T})^{-1}(CBR^{*})$$

The exact same arguments apply to the type structure $\mathcal{T}^{\mu}$, therefore we have $CBR^{\mu} =$

$\psi(\mathcal{T}^{\mu})^{-1}(CBR^{*})$.

Let $(s_i, t_i) \in CP_i(\mu) \cap CBR_i$. By applying the results above and the assumption $\varphi(\mathcal{T})(t_i) = \varphi(\mathcal{T}^{\mu})(t_i)$, we can conclude

$$\begin{aligned}
1 &= \beta_i(t_i)(CBR) \\
&= \beta_i(t_i)\left(\psi_{-i}(\mathcal{T})^{-1}(CBR^{*})\right) \\
&= f_i(\varphi_i(\mathcal{T})(t_i))(CBR^{*}) \\
&= f_i(\varphi_i(\mathcal{T}^{\mu})(t_i))(CBR^{*}) \\
&= \beta_i^{\mu}(t_i)\left(\psi_{-i}(\mathcal{T}^{\mu})^{-1}(CBR^{*})\right) \\
&= \beta_i^{\mu}(t_i)(CBR^{\mu})
\end{aligned}$$

therefore $(s_i, t_i) \in CBR_i^{\mu}$. □

Other events of interest which appear in the next proofs are $CB([\phi])$ and $CB([n])$. The argument behind the previous proposition can be easily adapted to show that we do not need to distinguish between these events and their counterparts in the type structure $\mathcal{T}^{\mu}$.

Proof of Theorem 4

(1)

Claim. For every k, $E^k(t_i^*) \subseteq CBR$.

Proof. For every profile (s_{-i}, t_{-i}), if $\mu(s_{-i}, t_{-i}|t_i^*) > 0$ then $\beta^{\mu}(t_i^*)(s_{-i}, t_{-i}) > 0$ and since t_i^* is in $CBR_i \subseteq B_i(CBR)$ then $(s_{-i}, t_{-i}) \in CBR_{-i}$. Therefore $E^1(t_i^*) \subseteq CBR$.

Suppose the claim is proved for every $k \leq K$. If $(s, t) \in E^{K+1}(t_i^*)$ there exist $(s', t') \in E^K(t_i^*)$ and a player j such that $t_j = t_j'$ and $\beta^{\mu}\left(t_j'\right)(s_{-j}, t_{-j}) > 0$. Since t_j' is in $B_j(CBR)$ then $(s_{-j}, t_{-j}) \in CBR_{-j}$. Therefore $(s, t) \in CBR$. Therefore, by induction, we conclude that for every k, $E^k(t_i^*) \subseteq CBR$. □

We now show that $\mu \in \Delta(S \times T)$ defines a correlated equilibrium. Let $\mu(s_j, t_j) > 0$ for some player j and pair (s_j, t_j). Then $(s_{-j}, t_{-j}, s_j, t_j) \in E^k(t_i^*)$ for some k and some $(s_{-j}, t_{-j}) \in S_{-j} \times T_{-j}$. Pick $l \neq j$. Then

$$\mu(s_j, t_j|t_l) = \beta^{\mu}(t_l)(s_j, t_j) > 0$$

Since t_l is in $CBR_l \subseteq B_l R$ then $(s_j, t_j) \in R_j$. Therefore s_j is a best response to

$$\text{marg}_{S_{-j}} \beta_j^{\mu}(t_j) = \text{marg}_{S_{-j}} \mu(\cdot|t_j) = \text{marg}_{S_{-j}} \mu(\cdot|s_j, t_j)$$

where the last equality follows from AI independence. Therefore $\mu \in \Delta(S \times T)$ is a correlated equilibrium.

(2)

Let $\nu \in \Delta(S)$ be a correlated equilibrium distribution. Then

$$\sum_{s_{-i} \in S_{-i}} u(s_i, s_{-i}) \nu(s_{-i}|s_i) \geq \sum_{s_{-i} \in S_{-i}} u(s_i', s_{-i}) \nu(s_{-i}|s_i)$$

for every $s_i' \in S_i$. Let $T_i^{\mu} = \{s_i : \nu(s_i) > 0\}$ and $T^{\mu} = \prod_{i \in I} T_i^{\mu}$. Define the prior $\mu \in \Delta(S \times T)$ as

$$\mu(s, t) = \nu(s)$$

if $s = t$ and

$$\mu(s, t) = 0$$

otherwise. Define β^{μ} to be generated by μ, that is

$$\beta_i^{\mu}(t_i)(s_{-i}, t_{-i}) = \mu(s_{-i}, t_{-i}|t_i)$$

for every i and every (s, t). We have a well defined type structure $\mathcal{T}^{\mu} = \left(I, (S_{-i}, T_i^{\mu}, \beta_i^{\mu})_{i \in I}\right)$ admitting μ as a common prior. The prior satisfies Condition AI trivially, since for every s_i and t_i if $\mu(s_i, t_i) > 0$ then $s_i = t_i$.

If $\mu(s_i, t_i) > 0$ then $s_i = t_i$ and s_i is a best response to $\nu(\cdot|s_i)$, hence $(s_i, t_i) \in R_i$. Moreover, if $\beta_i(t_i)(s_{-i}, t_{-i}) > 0$ then $\mu(s_{-i}, t_{-i}) > 0$ hence $(s_{-i}, t_{-i}) \in R_{-i}$. Therefore, if $\mu(s, t) > 0$ then $(s, t) \in RCBR$.

Proof of Theorem 8

It is enough to prove that if $(s_i, t_i^*) \in CP_i(\mu) \cap CB([n])$ and μ is minimal for t_i^* then μ satisfies AI. As before, it is immediate to check that for every k, $E^k(t_i^*) \subseteq CB([n])$.

Let $\mu(s_j, t_j) > 0$. There exist (s_{-j}, t_{-j}) such that $(s, t) \in E^k(t_i^*)$ for some k and

$$\mu(s_{-j}, t_{-j}|s_j, t_j) > 0.$$

Let $l \neq j$. Then $\mu(s_j, t_j|t_l) > 0$ and since $(s_j, t_j) \in CB([n])$, then t_l is in $B([n])$, hence $s_j = n_j(t_j)$. To conclude, if $\mu(s_j, t_j) > 0$ then $s_j = n_j(t_j)$. Therefore μ satisfies AI.

Proof of Theorem 7

It is convenient to prove here a slightly stronger result.

Theorem. *(7b) If there is a probability $\mu \in \Delta(S \times T)$, a tuple $t^* \in CP(\mu) \cap [\phi] \cap CB([\phi]) \cap B(R)$ and $\nu = \mu(\cdot|E(t^*))$ satisfies AI, then there exist $\sigma_i \in \Delta(S_i)$ for all i such that $\sigma = (\sigma_i)_{i \in I}$ is a Nash Equilibrium and $\phi_i = \prod_{k \neq i} \sigma_k$.*

As in the proof of Theorem 2, if $t^* \in CB([\phi])$ then for every i and every k, $E^k(t_i^*) \subseteq CB([\phi])$. The rest of the proof is based on Aumann and Brandenburger (1995).

Claim. For every (s_i, t_i) if $\nu(s_i, t_i) > 0$ then $\nu(s_{-i}) = \nu(s_{-i}|s_i, t_i) = \phi_i(s_{-i})$.

Proof. For every (s_i, t_i), if $\nu(s_i, t_i) > 0$ then $(s_{-i}, t_{-i}, s_i, t_i) \in E^k(t^*)$ for some k. Since $E^k(t^*) \subseteq CB([\phi])$ and $E^k(t^*) \subseteq E^{k+1}(t^*)$ then $(s_i, t_i) \in [\phi]_i$. Hence

$$\nu(s_{-i}|s_i, t_i) = \nu(s_{-i}|t_i) = \beta^\nu(t_i)(s_{-i}) = \phi_i(s_{-i})$$

where the first equality follows from AI. Therefore

$$\nu(s_{-i}) = \sum_{(s_i, t_i)} \nu(s_{-i}|s_i, t_i)\,\nu(s_i, t_i) = \sum_{(s_i, t_i)} \phi_i(s_{-i})\,\nu(s_i, t_i) = \phi_i(s_{-i}).$$

□

Claim. For every s, $\nu(s) = \prod_{i=1}^{I} \nu(s_i)$.

Proof. Suppose for $K < |I|$ and every $s \in S$ and $i \in I$,

$$\nu(s_1, s_2, .., s_K, ..., s_I) = \prod_{i=1}^{K} \nu(s_i)\,\nu(s_{K+1}, ..., s_I)$$

We know from the previous claim that this is true for $K = 1$. Suppose it is true for some $K > 1$. Then

$$\begin{aligned}
\nu(s_1, ..., s_I) &= \text{marg}_{S_{-(K+1)}} \nu(s_1, ..., s_K, s_{K+2}, ..., s_I | s_{K+1}) \nu(s_{K+1}) \\
&= \text{marg}_{S_{-(K+1)}} \nu(s_1, ..., s_K, s_{K+2}, ..., s_I) \nu(s_{K+1}) \\
&= \sum_{s'_{K+1} \in S_{K+1}} \nu\left(s_1, ..., s_K, s'_{K+1}, s_{K+2}, ..., s_I\right) \nu(s_{K+1}) \\
&= \sum_{s'_{K+1} \in S_{K+1}} \nu(s_1) \cdots \nu(s_K) \, \text{marg}_{S_{K+1} \times ... \times S_I} \nu\left(s'_{K+1}, s_{K+2}, ..., s_I\right) \nu(s_{K+1}) \\
&= \nu(s_1) \cdots \nu(s_K) \nu(s_{K+1}) \sum_{s'_{K+1} \in S_{K+1}} \text{marg}_{S_{K+1} \times ... \times S_I} \nu\left(s'_{K+1}, s_{K+2}, ..., s_I\right) \\
&= \nu(s_1) \cdots \nu(s_{K+1}) \nu(s_{K+2}, ..., s_I)
\end{aligned}$$

Therefore the claim holds for every $K \leq I$. □

Claim. If $\nu(s_{-i}) > 0$ then $\phi_i(s_{-i}) = \prod_{k \neq i} \nu(s_k)$

Proof. By combining the previous two claims, if $\nu(s_i, t_i) > 0$ then

$$\nu(s_{-i} | s_i, t_i) = \phi_i(s_{-i}) = \nu(s_{-i}) = \prod_{k \neq i} \nu(s_k).$$

□

Define $\sigma_i = \text{marg}_{S_i} \nu$. Let $\sigma_i(s_i) > 0$. Fix a player $j \neq i$ and the type t_j^* in the tuple t^*. By assumption $t_j^* \in [\phi]_j$. By the claims above and AI independence,

$$\nu\left(s_i | t_j^*\right) = \nu\left(s_i | s_j, t_j^*\right) = \phi_j(s_i) = \sigma(s_i)$$

hence $\nu\left(s_i | t_j^*\right) > 0$. Let t_i be a type such that $\nu\left(s_i, t_i | t_j^*\right) > 0$. Since $t_j^* \in B(R)_j$, then s_i is a best response to the first order belief of type t_i. Because $t_j^* \in CB([\phi])_j$, then $t_i \in [\phi]_i$, i.e. the first order belief of t_i is given by the conjecture $\phi_i = \prod_{k \neq i} \sigma_k$. To conclude, for every player i, every strategy in the support of σ_i is a best response to the conjecture $\prod_{k \neq i} \sigma_i$. Therefore, σ is a Nash Equilibrium.

Proof of Theorem 9

Let t^* belong to

$$CP(\mu) \cap [\phi] \cap CB([\phi]) \cap B(R) \cap CB([n])$$

Notice that μ is not assumed to be minimal. Let $\nu = \mu\left(\cdot|E\left(t^*\right)\right)$. From Proposition 2, we have that if $t^* \in CP\left(\mu\right)$ then $t^* \in CP\left(\nu\right)$. Therefore, t^* is in

$$CP\left(\nu\right) \cap [\phi] \cap CB\left([\phi]\right) \cap B\left(R\right) \cap CB\left([n]\right)$$

As before, it is immediate to check that for every i and every k, $E^k\left(t_i^*\right) \subseteq CB\left([n]\right)$. Let $\nu\left(s_j, t_j\right) > 0$. There exist $\left(s_{-j}, t_{-j}\right)$ such that $\left(s_j, t_j, s_{-j}, t_{-j}\right) \in E^k\left(t_i^*\right)$ for some k and i, and

$$\nu\left(s_{-j}, t_{-j}|s_j, t_j\right) > 0.$$

Let $l \neq j$. Then $\nu\left(s_j, t_j|t_l\right) > 0$ and since $\left(s_j, t_j\right) \in CB\left([n]\right)$, then t_l is in $B\left([n]\right)$, hence $s_j = n_j\left(t_j\right)$. To conclude, if $\nu\left(s_j, t_j\right) > 0$ then $s_j = n_j\left(t_j\right)$. Therefore ν satisfies AI. We can now apply Theorem 7b.

Proof of Theorem 14

By standard arguments, we can find two types $\left(\bar{t}_1, \bar{t}_2\right) \in \left[\mathcal{T}^\Theta\right] \cap CB\left(\left[\mathcal{T}^\Theta\right]\right) \cap CB\left([\psi]\right) \cap CB\left(\Theta \times R\right)$. Let $\bar{t}_i^\Theta = \varphi_{i,\Theta}\left(\bar{t}_i\right)$ for every i.

Definition. A type t_i^Θ of player i is *reachable in N steps* if there exists a sequence $t_{i(1)}^{\Theta,1}, ..., t_{i(N)}^{\Theta,N}$ such that:

- $t_{i(1)}^{\Theta,1} = \bar{t}_{i(1)}^\Theta$
- $i\left(N\right) = i$ and $t_{i(N)}^\Theta = t_i^\Theta$
- For all $n \leq N$, $\beta_{i(n-1)}^\Theta\left(t_{i(n-1)}^\Theta\right)\left(\left[t_{i(n)}^\Theta\right]\right) > 0$

Let RE^N be the set of types reachable in N steps. Since the type structure $\mathcal{T}^\Theta$ is minimal, every type is reachable in a finite number of steps.

We need to show that for every N every player i and type t_i^Θ in RE^N, if $\psi_i\left(t_i^\Theta\right)\left(s_i\right) > 0$ then s_i is optimal to the conjecture $\phi\left(t_i^\Theta\right)$ defined as

$$\phi\left(t_i^\Theta\right)\left(s_{-i}\right) = \sum_{t_{-i}^\Theta \in T_{-i}^\Theta} \beta_i^\Theta\left(t_i^\Theta\right)\left(t_{-i}^\Theta\right)\psi\left(t_{-i}^\Theta\right)\left(s_{-i}\right)$$

for every $s_{-i} \in S_{-i}$.

Let t_i^Θ be in RE^N. Since $\mathcal{T}^\Theta$ is minimal, it is without loss of generality to assume $N > 2$. Let $\bar{t}_{i(1)}^\Theta, ..., t_{i(N)}^{\Theta,N}$ be a sequence reaching t_i^Θ in N-steps.

Claim. There exist a sequence $\bar{t}_{i(1)}, t_{i(2)}, ..., t_{i(N)}$ in T such that $i\left(N\right) = i$, $\varphi_{i(n),\Theta}\left(t_{i(n)}\right) = \varphi_{i(n)}^\Theta\left(t_{i(n)}^\Theta\right)$ for all $n \leq N$ and $\beta\left(t_{i(n)}\right)\left(\left[t_{i(n+1)}\right]\right) > 0$ for every $n \leq N - 1$.

Proof. Since $\bar{t}^{\Theta}_{i(1)} = \varphi_{i,\Theta}\left(\bar{t}_{i(1)}\right)$ and $\beta^{\Theta}_{i(1)}\left(\bar{t}^{\Theta}_{i(1)}\right)\left(\left[t^{\Theta}_{i(2)}\right]\right) > 0$ then there must exist a type $t_{i(2)}$ such that $\varphi_{i(2),\Theta}\left(t_{i(2)}\right) = \varphi^{\Theta}_{i(2)}\left(t^{\Theta}_{i(2)}\right)$ and $\beta_{i(1)}\left(t_{i(1)}\right)\left(\left[t_{i(2)}\right]\right) > 0$. A simple argument by induction concludes the proof. □

Claim. For every $2 < n \leq N$, $t_{i(n)}$ is in $[\psi] \cap R \cap CB\left(\left[\mathcal{T}^{\Theta}\right]\right) \cap CB\left([\psi]\right) \cap CB\left(\Theta \times R\right)$.

Proof. It can be easily proved by induction. □

Suppose $\psi_i\left(t^{\Theta}_i\right)\left(s_i\right) > 0$. Since $t_{i(N-1)}$ is in $B\left([\psi]\right)$, then $\beta\left(t_{i(N-1)}\right)\left(t_i, s_i\right) > 0$. Since $t_{i(N-1)}$ is in $B\left(R\right)$ then s_i is a best response to the first order belief over strategies of type t_i, defined as the conjecture

$$\phi\left(t_i\right)\left(s_{-i}\right) = \sum_{t_{-i} \in T_{-i}} \beta\left(t_i\right)\left(t_{-i}, s_{-i}\right)$$

For every $s_{-i} \in S_{-i}$.

Since t_i is in $B\left(\left[\mathcal{T}^{\Theta}\right]\right) \cap B\left([\psi]\right)$, for every $\left(t_{-i}, s_{-i}\right)$ such that $\beta\left(t_i\right)\left(t_{-i}, s_{-i}\right) > 0$ there is a type $t^{\Theta}_{-i} \in T^{\Theta}_{-i}$ such that $\varphi_{-i,\Theta}\left(t_{-i}\right) = \varphi^{\Theta}_{-i}\left(t^{\Theta}_{-i}\right)$ and $s_{-i} = \psi_{-i}\left(t^{\Theta}_{-i}\right)$. Therefore

$$\begin{aligned}
\phi\left(t_i\right)\left(s_{-i}\right) &= \sum_{t_{-i} \in T_{-i}} \beta\left(t_{i(N)}\right)\left(t_{-i}, s_{-i}\right) \\
&= \sum_{t^{\Theta}_{-i} \in T^{\Theta}_{-i}} \sum_{t_{-i}:\varphi_{-i,\Theta}\left(t_{-i}\right)=\varphi^{\Theta}_{-i}\left(t^{\Theta}_{-i}\right)} \beta\left(t_i\right)\left(t_{-i}, \psi\left(t^{\Theta}_{-i}\right)\right) \\
&= \sum_{t^{\Theta}_{-i} \in T^{\Theta}_{-i}} \beta^{\Theta}_i\left(t^{\Theta}_i\right)\left(t^{\Theta}_{-i}\right)\psi\left(t^{\Theta}_{-i}\right)\left(s_{-i}\right)
\end{aligned}$$