Foundations for Structural Preferences

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June 18, 2026

Abstract

This paper provides a behavioral foundation for structural rationality (Siniscalchi, 2022), a notion of optimality that models agents with well-defined beliefs at all decision points in dynamic choice problems, even following unexpected events. The concept of *implausible events* is introduced to characterize circumstances that the agent considers unlikely, but still possible. Implausible events have zero prior probability, but can nevertheless influence ex-ante choices in certain circumstances. Theorem 1 establishes that structural rationality is the minimal departure from expected utility maximization that still allows the identification of conditional beliefs following implausible events. Theorem 2 offers a tight characterization of structural rationality via an additional continuity axiom.

Keywords: conditional probability systems, structural rationality, implausible events.

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1 Introduction

Consider an expected-utility (EU) maximizing agent facing a dynamic decision problem under uncertainty. Ex-ante, the agent may assign zero probability to reaching a particular interim decision point. In this case, the EU-maximizing agent is indifferent ex-ante among all courses of action that only differ starting from that decision point. Furthermore, her beliefs conditional upon unexpectedly reaching such a decision point are undefined.

Structural rationality (Siniscalchi, 2022, SRDG henceforth) instead models agents who hold well-defined beliefs at all decision points, and have strict *ex-ante* preferences for plans that specify conditionally optimal continuations even after unexpected realizations of the uncertainty. SRDG shows that this feature is essential for constructing an incentive-compatible procedure that enables an experimenter to elicit players' beliefs at all information sets in a dynamic game.

The present paper provides a behavioral characterization of structural rationality in general dynamic choice problems. This yields an axiomatic foundation for the analysis in SRDG. By highlighting its behavioral underpinnings, it indicates how structural rationality can provide a principled model of choice for situations where decision-makers may value contingent planning in the face of events that are both a priori unlikely and hard to quantify.

To illustrate, consider research and development in novel technologies. Figure 1 depicts the problem faced by an automobile manufacturer who, perhaps in the early 2000s, must decide whether to invest in battery (b) or fuel cell-powered (c) electric vehicles, or neither (n). A priori, the firm views battery and fuel-cell EVs as "moonshots," which are unlikely to be viable but may yield a high payoff if, unexpectedly, they are. The firm also receives a subsidy that partially or fully covers research costs for the first technology it explores; taking this subsidy into account, the firm's *net* initial research costs are $\gamma_1 \geq 0$. By researching a technology $t \in \{b, c\}$, the firm learns whether or not it t is viable. If it is, the firm can deploy it (d) and obtain a profit of $\pi_t > 0$, or explore the other technology, bearing the incremental research cost $\gamma_2 - \gamma_1 > 0$ but potentially reaping higher profits $\pi_{b,c} > \max(\pi_b, \pi_c)$ if both technologies are viable. If instead t is not viable, the firm can explore the other technology, again at an incremental cost $\gamma_2 - \gamma_1$, or stop experimenting (s).

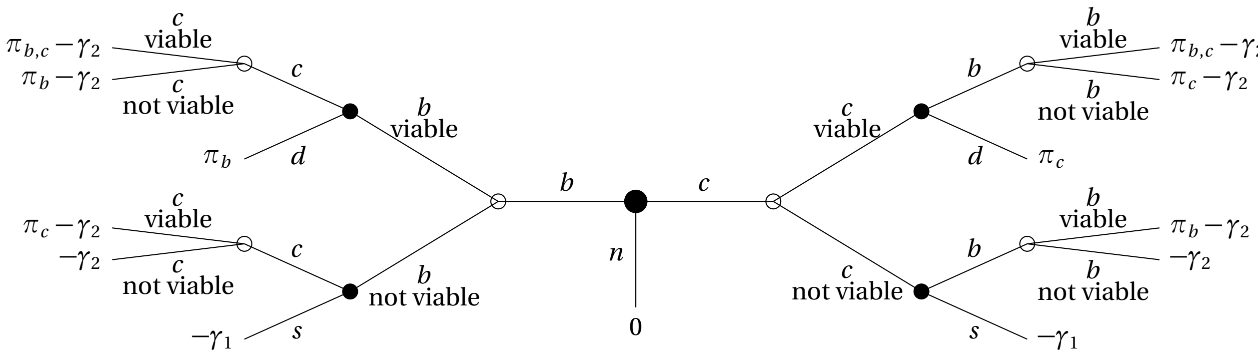

Figure 1: A decision tree in the EV engine example.

In this scenario, structural rationality allows one to model the following assumptions:

- First, initially, the firm is "virtually certain" that neither technology is viable. If the research subsidy is partial, so $\gamma_1 > 0$, the firm prefers n to b or c at the initial node. However, if the subsidy is full, so $\gamma_1 = 0$, the firm prefers both b and c to n, because it anticipates the possibility of a positive payoff if the chosen technology is unexpectedly viable.

- Second, despite being virtually certain that either technology is infeasible, the firm holds well-defined beliefs conditional on the event that, say, battery EVs are viable. For instance, the firm may assign high or low conditional probability to the event that fuel-cell EVs are also viable. Depending on payoffs, this has implications for the firm's optimal continuation strategy—e.g., whether to choose c or d in case battery EVs are viable.

- Third, in some cases the firm may have too little information about either technology to confidently assess which one is the more promising "moonshot." In other instances, the firm may instead view one of them as a strictly more promising (though still unlikely) moonshot. Thus, the firm may or may not be able to rank b and c at the initial node.

The main result of this paper shows that, in a precise axiomatic sense, structural rationality is the minimal weakening of expected-utility (EU) maximization (Savage, 1972; Anscombe and Aumann, 1963; Fishburn, 1970) required to model decision-makers with the characteristics described above and identify probabilistic beliefs given all conditioning events in the dynamic decision problem at hand.

A key ingredient in the axiomatization is the formal definition of an *implausible* event—one that, in the eyes of the individual, is virtually certain not to happen. In the EV example, this is captured by the assumption that, with only a partial subsidy, the firm is unwilling to invest in either technology, no matter how profitable it might be if viable. More generally, an event N is implausible if the individual is unwilling to take a bet that entails a (possibly small) loss if the event N does not occurs, even if the bet yields a gain if N does occur (see Definition 4). For both EU and structurally rational individuals, this behavior implies that event N, and hence all its sub-events, have zero prior probability. For an EU maximizer, a zero-probability event is "Savage-null" (see §4), i.e., irrelevant. On the other hand, as noted above, a structurally rational agent may, for instance, deem it implausible that battery EVs are viable, and yet have a strict preference for betting on both technologies, rather than only battery EVs, being viable. In other words, implausible events need not be Savage-null for structurally rational agents.

This paper is organized as follows. Section 2 introduces the setup. Section 3 formalizes structural preferences. Section 4 presents the axiomatic characterization. Finally, Section 5 discusses the related literature and extensions. All proofs are in the Appendix.

2 Setup

The axiomatic analysis in this paper concerns preferences over acts, i.e., maps from uncertain states to outcomes, as in Savage (1972) and Anscombe and Aumann (1963). It is implicitly assumed that the individual compares strategies by "reducing" them to acts: see Example 1 for an illustration. Relaxing the reduction assumption is left to future work.

Fix a *state space* Ω, endowed with a sigma-algebra Σ. The class of *conditioning events* for the agent is a *finite* collection $\mathscr{F} \subset \Sigma$ such that $\Omega \in \mathscr{F}$. This is the set of events that the individual may observe in the course of the dynamic decision problem of interest.

In many applications, $\mathscr{F}$ is a *laminar family* (cf. Schrijver, 2003, §13.4): for every $E, F \in \mathscr{F}$, either $E \cap F = \emptyset$, or $E \subseteq F$, or $F \subseteq E$. In particular, $\mathscr{F}$ is laminar if it consists of all subsets in an event tree (a sequence of progressively finer measurable partitions of Ω). However, when the individual's actions can influence the information she receives, $\mathscr{F}$ may fail to be laminar. This is the case for the EV engine scenario (see Example 1 below for details), and is often the case in game-theoretic applications. This paper does not require $\mathscr{F}$ to be laminar; however, if it is, one axiom (Axiom 9a) admits a simpler formulation.

Denote by $B_0(\Sigma)$ the set of Σ-measurable real functions with finite range, and by $B(\Sigma)$ its sup-norm closure. For $\phi, \psi \in B(\Sigma)$, $\phi \leq \psi$ means $\phi(\omega) \leq \psi(\omega)$ for all ω. Denote by $\Delta(\Sigma)$ the set of *finitely* additive probability measures on Σ. For $p \in \Delta(\Sigma)$ and $\phi \in B_0(\Sigma)$, the integral of ϕ with respect to p is $\int \phi \, dp = \sum_{x \in \phi(\Omega)} x \, p(\phi^{-1}(x))$, which is a finite sum by the definition of $B_0(\Sigma)$. For $\phi \in B(\Sigma)$, the integral $\int \phi \, dp$ is defined as e.g. in Rao and Rao (1983), §4.5; in particular, it satisfies $\int \phi \, dp = \sup\{\int \phi_0 \, dp \,:\, \phi_0 \in B_0(\Sigma), \phi_0 \leq \phi\}$. When convenient, I write $\mathrm{E}_p \phi$ instead of $\int \phi \, dp$.

Endow $\Delta(\Sigma)$ with the topology induced by $B(\Sigma)$ (i.e., the weak* topology): in particular, for a sequence $(p^k)_{k \geq 1}$ in $\Delta(\Sigma)$, $p^k \to p \in \Delta(\Sigma)$ means that $\int \phi \, dp^k \to \int \phi \, dp$ for all $\phi \in B(\Sigma)$; equivalently, $p^k \to p$ if and only if $\lim_k p^k(E)$ exists (and equals $p(E)$) for every $E \in \Sigma$ (cf. Dunford and Schwartz, 1957, Ex. IV.13.18).

Let X be a convex set of outcomes; for instance, the set of simple lotteries on some prize space Y, as in Anscombe and Aumann (1963). The axiomatic analysis focuses primarily on the set $\mathscr{A}_0$ of **simple acts**, i.e., functions $f : \Omega \to X$ such that $f(\Omega)$ is finite and $f^{-1}(x) \in \Sigma$ for all $x \in X$. As usual, for every $x \in X$, denote the simple act that delivers x at every $\omega \in \Omega$ simply by x. The main object of interest is a binary relation $\succ$ on $\mathscr{A}_0$, interpreted as the individual's ex-ante preference. Section 5.D describes the extension to a larger class of acts.

3 Conditional Beliefs and Structural Preferences

As in SRDG, the agent's beliefs are modeled as a collection of conditional probabilities, one for each conditioning event in $\mathscr{F}$ (cf. Myerson, 1986; Ben-Porath, 1997; Battigalli and Siniscalchi, 1999 and Rényi, 1955). These beliefs are required to be consistent across different conditioning events, in the sense that they can be approximated by a sequence of finitely additive probability measures that converge in the weak* topology.

Definition 1 *$\mu = \left(\mu(\cdot|F)\right)_{F\in\mathscr{F}} \in \Delta(\Sigma)^{\mathscr{F}}$ is a* ***consistent conditional probability system (CCPS)*** *if there is a sequence $(p^k)_{k\geq 1}$ in $\Delta(\Sigma)$ such that, for every $F \in \mathscr{F}$, (i) $p^k(F) > 0$ for all $k \geq 1$, and (ii) $p^k(\cdot|F) \to \mu(\cdot|F)$ in the weak* topology. In this case, say that $(p^k)_{k\geq 1}$ is a* ***perturbation*** *of μ.*

Denote by $\Delta(\Sigma, \mathscr{F})$ the set of all CCPSs. Section 5.E analyzes the extension to countably additive CCPSs.

In a dynamic game, SRDG calls a strategy s_i of player i "structurally rational" given μ if it is a best reply to every element of some perturbation of μ. Proposition 1 in SRDG shows that s_i is structurally rational given μ if and only if there is no (mixed) strategy σ_i that eventually yields a strictly higher payoff than s_i under all perturbations of μ. This paper takes the decision-theoretic analog of the latter condition as the definition of structural preference.

Definition 2 *Consider a binary relation $\succ$ on $\mathscr{A} \subseteq X^{\Omega}$, and assume that $\mathscr{A}_0 \subseteq \mathscr{A}$. Fix a non-constant, affine $u : X \to \mathbb{R}$ and a CCPS $\mu \in \Delta(\Sigma, \mathscr{F})$. Then $\succ$ is the* ***structural preference represented by*** *(u, μ)* ***on*** *$\mathscr{A}$ if $\{u \circ h : h \in \mathscr{A}\} \subseteq B(\Sigma)$ and, for every pair $f, g \in \mathscr{A}$, $f \succ g$ if and only if, for every perturbation $(p^k)_{k\geq 1}$ of μ, eventually $\int u \circ f \, dp^k > \int u \circ g \, dp^k$.*

The above formulation accommodates both preferences over simple acts, as well as preferences over the more general class introduced in Section 5.D.

Example 1 To formally illustrate the preference statements in the EV example described in the Introduction, let $\Omega = \{nn, nv, vn, vv\}$, where nn means that neither battery nor fuel-cell

EVs are viable, nv means that only fuel-cell EVs are viable, and so on. Let $\Sigma = 2^{\Omega}$. The set of conditioning events is $\mathscr{F} = \{\Omega, B, \Omega \setminus B, C, \Omega \setminus C\}$, where $B = \{vn, vv\}$ is the event that battery EVs are viable, and $C = \{nv, vv\}$ is the event that fuel-cell EVs are viable. (The firm observes either B or $\Omega \setminus B$ if it chooses to research battery EVs, and C or $\Omega \setminus C$ if it chooses to research fuel-cell EVs.) The set $\mathscr{F}$ is not laminar, because $B \cap C = \{vv\} \notin \mathscr{F}$. Let $X = \mathbb{R}$ and $u(x) = x$.

Consider first the belief array $\mu = \big(\mu(\cdot|F)\big)_{F \in \mathscr{F}}$ such that

$$\mu(\{nn\}|\Omega) = \mu(\{nn\}|\Omega \setminus B) = \mu(\{nn\}|\Omega \setminus C) = 1, \quad \mu(\{vn\}|B) = \mu(\{nv\}|C) = 1. \tag{1}$$

This array μ is a CCPS: two perturbations (referred to in forthcoming examples) are given by

$$p^k(\{nn\}) = 1 - \frac{1}{k+1} - \frac{1}{(k+1)^2}, p^k(\{vn\}) = \frac{1}{k+1}, \quad p^k(\{nv\}) = \frac{1}{(k+1)^2}, \tag{2}$$

$$q^k(\{nn\}) = 1 - \frac{1}{k+1} - \frac{1}{(k+1)^2}, q^k(\{vn\}) = \frac{1}{(k+1)^2}, \quad q^k(\{nv\}) = \frac{1}{k+1}. \tag{3}$$

Consider the structural preference $\succ^{\mu}$ represented by the identity utility and the CCPS μ. With reference to Figure 1, denote by $f_{b,a',a''}$ the act induced by the strategy "choose b at the initial node, then $a' \in \{c, d\}$ if b is viable and $a'' \in \{c, s\}$ if not," and by $f_{c,a',a''}$ the act induced by the strategy "choose c at the initial node, then $a' \in \{b, d\}$ if c is viable and $a'' \in \{b, s\}$ if not." Naturally, the payoffs delivered by these acts depend upon the parameters $\pi_b, \pi_c, \pi_{b,c}, \gamma_2$ and, crucially, γ_1. Finally, let $f_n \equiv 0$ be the (constant) act induced by the choice of n.

Since events B and C have zero prior probability, if $\gamma_1 > 0$ in Figure 1, then $f_n \succ^{\mu} f_{b,a',a''}$ and $f_n \succ^{\mu} f_{c,a',a''}$ for all a', a''. This is because, for any perturbation $(\tilde{p}^k)_{k \geq 1}$ of μ, $\mathrm{E}_{\tilde{p}^k} f_{b,a',a''} \leq \pi_{b,c} \tilde{p}^k(B \cup C) - \gamma_1 \tilde{p}^k(\{nn\}) \to -\gamma_1 < 0$, and similarly for $f_{c,a',a''}$. Thus, for $\gamma_1 > 0$, n is the unique optimal ex-ante choice. However, if $\gamma_1 = 0$ (the firm receives a full subsidy), then for instance $\mathrm{E}_{\tilde{p}^k} f_{b,d,s} = \pi_b \tilde{p}^k(B) > 0$ for every k, and so $f_{b,d,s} \succ^{\mu} f_n$, i.e., the firm strictly prefers to experiment with battery EVs. Furthermore, again assuming $\gamma_1 = 0$, $f_{b,d,s} \succ^{\mu} f_{b,c,s}$: that is, *ex-ante*, if battery EVs are viable, the firm strictly prefers to develop them rather than to also explore fuel cells, even though B has zero prior probability. This is because, for any perturbation $(\tilde{p}^k)_{k \geq 1}$ of μ, for k large enough, $\mathrm{E}_{\tilde{p}^k} f_{b,d,s} = \pi_b \tilde{p}^k(B) > (\pi_{b,c} - \gamma_2)\tilde{p}^k(\{vv\}) + (\pi_b - \gamma_2)\tilde{p}^k(\{vn\}) = \mathrm{E}_{\tilde{p}^k} f_{b,c,s}$, because $\gamma_2 > 0$ and $\frac{\tilde{p}^k(\{vv\})}{\tilde{p}^k(B)} \to \mu(\{vv\}|B) = 0$. Finally, if $\pi_b = \pi_c$, then for the perturbation in Eq. (2), $\mathrm{E}_{p^k} f_{b,d,s} = \pi_b p^k(B) > \pi_c p^k(C) = \mathrm{E}_{p^k} f_{c,d,s}$ for all k, whereas for the perturbation in Eq. (3), $\mathrm{E}_{q^k} f_{b,d,s} = \pi_b q^k(B) < \pi_c q^k(C) = \mathrm{E}_{q^k} f_{c,d,s}$ for all k: that is, exploring only battery EVs or only fuel cell EVs are not comparable under $\succ^\mu$.

Next, consider the belief array $\nu = (\nu(\cdot|F))_{F\in\mathscr{F}}$ such that

$$\nu(\{nn\}|\Omega) = \nu(\{nn\}|\Omega \setminus B) = \nu(\{nn\}|\Omega \setminus C) = 1, \quad \nu(\{vv\}|B) = \frac{1}{2}, \quad \nu(\{vv\}|C) = 1. \tag{4}$$

This array ν is also a CCPS; one perturbation is given by $r^k(\{nn\}) = 1 - \frac{1}{k+1}$ and $r^k(\{vn\}) = r^k(\{vv\}) = \frac{1}{k+1} \cdot \frac{1}{2}$. Under this CCPS, again assuming that $\pi_b = \pi_c$, the firm now strictly prefers exploring only battery EVs rather than only fuel cells: $f_{b,d,s} \succ^\nu f_{c,d,s}$. This is because, for any perturbation $(\tilde{p}^k)_{k\geq 1}$ of ν,

$$\frac{\tilde{p}^k(B)}{\tilde{p}^k(C)} = \frac{\tilde{p}^k(B)}{\tilde{p}^k(\{vv\})} \cdot \frac{\tilde{p}^k(\{vv\})}{\tilde{p}^k(C)} \to \frac{1}{\nu(\{vv\}|B)} \cdot \nu(\{vv\}|C) = 2;$$

thus, $\mathrm{E}_{\tilde{p}^k} f_{b,d,s} = \pi_b \tilde{p}^k(B) > \pi_c \tilde{p}^k(C) = \mathrm{E}_{\tilde{p}^k} f_{c,d,s}$ for k sufficiently large. □

4 Behavioral analysis

Throughout this section, as in Definition 2, $\mathscr{A}$ denotes a subset of X^Ω that contains $\mathscr{A}_0$.

The starting point for the axiomatization of structural preferences is Theorem 13.3 in Fishburn (1970) (see also Kreps, 1988, Chapter 7). I reproduce this result for ease of reference. Recall that $\succ$ is **irreflexive** if not $f \succ f$; it is **asymmetric** if $f \succ g$ implies not $g \succ f$; and **negatively transitive** if, for all $f, g, h \in \mathscr{A}$, not $f \succ g$ and not $g \succ h$ imply not $f \succ h$. An asymmetric, negatively transitive preference is irreflexive and transitive, and an irreflexive, transitive relation is asymmetric (cf. Fishburn, 1970, §2). Also, event $E \in \Sigma$ is **null (for $\mathscr{A}$)** (Savage, 1972) if, for all $f, g \in \mathscr{A}$ such that $f(\omega) = g(\omega)$ for all $\omega \notin E$, neither $f \succ g$ nor $g \succ f$ hold.

Theorem 0 (Fishburn, 1970) *Suppose that the following axioms hold for all $f, g, h \in \mathscr{A}_0$:*[1]

Axiom B1 (Weak Order). *$\succ$ is asymmetric and negatively transitive;*

Axiom B2 (Independence). *$f \succ g$ and $\alpha \in (0,1)$ imply $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$;*

Axiom B3 (Archimedean Continuity). *$f \succ g$ and $g \succ h$ imply that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$ for some $\alpha, \beta \in (0,1)$;*

Axiom B4 (Non-Degeneracy). *$x \succ y$ for some $x, y \in X$;*

Axiom B5 (Monotonicity). *If $E \in \Sigma$ is not null for $\mathscr{A}_0$, then for all $x, y \in X$, $xEf \succ yEf$ iff $x \succ y$.*

Then there are a non-constant $u : X \to \mathbb{R}$ and $p \in \Delta(\Sigma)$ such that $f \succ g$ if and only if $\int u \circ f \, dp > \int u \circ g \, dp$. Furthermore, u is unique up to positive linear transformations, and p is unique.

Structural preferences share many of the key properties of EU preferences. First, preferences over X are consistent with EU. Second, Independence and Monotonicity (with a suitable formulation) are maintained. Third, preferences are irreflexive and transitive. However, to allow for implausible, but non-null conditioning events, with associated well-defined conditional beliefs, Archimedean continuity and negative transitivity must be restricted, and non-degeneracy must be strengthened.

Example 2 Consider the CCPS μ in Eq. (1) of Example 1, and its perturbations $(p^k)_{k\geq 1}$ and $(q^k)_{k\geq 1}$ in Eqs. (2) and (3). Denot by 1_E the indicator function of event $E \in \Sigma$. For any $x > 0$, $\mathrm{E}_{p^k} x 1_B = \frac{x}{k+1} > \frac{1}{(k+1)^2} = \mathrm{E}_{p^k} 1_C$ for large k, and $\mathrm{E}_{q^k} 1_C = \frac{1}{k+1} > \frac{x}{(k+1)^2} = \mathrm{E}_{q^k} x 1_B$ for large k. Thus, $x 1_B$ and 1_C are not strictly ranked under $\succ^\mu$ for any $x > 0$. In particular, $2 \cdot 1_B \not\succ^\mu 1_C$ and $1_C \not\succ^\mu 1_B$. However, $2 \cdot 1_B \succ^\mu 1_B$, because for any perturbation $(\tilde{p}^k)_{k\geq 1}$ of μ, $\mathrm{E}_{\tilde{p}^k} 2 \cdot 1_B = 2\tilde{p}^k(B) > \tilde{p}^k(B) = \mathrm{E}_{\tilde{p}^k} 1_B > 0$ for every k. Thus, negative transitivity fails. Furthermore, $1 \succ^\mu 2 \cdot 1_B$, because for any perturbation $(\tilde{p}^k)_{k\geq 1}$ of μ, $\mathrm{E}_{\tilde{p}^k} 1_B = \tilde{p}^k(B) \to 0$. As was just shown, $2 \cdot 1_B \succ^\mu 1_B$. However, there is no $\beta \in (0,1)$ for which $2 \cdot 1_B \succ^\mu \beta \cdot 1 + (1-\beta) 1_B$, as for any perturbation $(\tilde{p}^k)_{k\geq 1}$ of μ,

[1] I restate Fishburn's result using the present notation. Also, Fishburn considers preferences over non-simple acts, which requires an additional axiom.

$\mathrm{E}_{\tilde{p}^k}\big(\beta\cdot 1+(1-\beta)1_B\big)\to\beta>0$. Thus, Archimedean continuity fails as well. $\square$

This motivates the following axioms.

Axiom 1 (Strict Partial Order) *$\succ$ is irreflexive and transitive on $\mathscr{A}$.*

Axiom 2 (Negative Transitivity for Prizes) *For all $x,y,z\in X$, not $y\succ x$ and not $z\succ y$ imply not $z\succ x$.*

Axiom 3 (Independence) *For all $f,g,k\in\mathscr{A}$ and $\alpha\in(0,1)$, $f\succ g$ if and only if $\alpha f+(1-\alpha)k\succ\alpha g+(1-\alpha)k$*

Axiom 4 (Monotonicity) *For all $f,g,h\in\mathscr{A}$: if, for all $\omega\in\Omega$, not $g(\omega)\succ f(\omega)$ [resp. not $f(\omega)\succ g(\omega)$], then $g\succ h$ implies $f\succ h$ [resp. $h\succ g$ implies $h\succ f$]*

Observation 5 in the Appendix shows that Axiom 4 implies Fishburn's B5. Furthermore, recall that Axiom 3 implies Savage's Sure-Thing Principle (Postulate P2):

Remark 1 *Assume Axiom 3. For all $f,g,k,k'\in\mathscr{A}$, and all $E\in\Sigma$: $fEk\succ gEk$ if and only if $fEk'\succ gEk'$.*

Hence, one can define *conditional preferences* following Savage (1972):

Definition 3 *For all event $E\in\Sigma$, the individual's **conditional preference** $\succ_E$ **given** E is defined as follows: for every $f,g\in\mathscr{A}$, $f\succ_E g$ if, for some $k\in\mathscr{A}$, $fEk\succ gEk$.*

Remark 1 ensures that any $k\in\mathscr{A}$ will yield the same conditional ranking of the strategies f,g; $\succ_\Omega$ is the prior preference $\succ$. The remaining axioms focus on conditional preferences.

Axiom 5 (Nondegeneracy) *For all $F\in\mathscr{F}$, there exist $x,y\in X$ such that $x\succ_F y$.*

Axiom 6 (Prize Continuity) *For all $F\in\mathscr{F}$ and $x,y,z\in X$: if $x\succ_F y$ and $y\succ_F z$, then there exist $\alpha,\beta\in(0,1)$ such that $\alpha x+(1-\alpha)z\succ_F y$ and $y\succ_F\beta x+(1-\beta)z$.*

Prize continuity is in the spirit of Blume, Brandenburger, and Dekel (1991a), except that conditioning events in that paper are singleton sets. Non-degeneracy is a substantive requirement: it implies that all conditioning events "matter" for preferences (even though some may be assigned zero ex-ante probability in the structural-preference representation).

Next, I formalize the key notion of implausible event.

Definition 4 *Fix an event $E \in \Sigma$. An event $N \in \Sigma$ is **implausible** given E if, for all $x, y, z \in X$, $y \succ_E z$ implies $y \succ_E xNz$; otherwise, it is **plausible** given E. Let* $\mathrm{imp}(E) = \cup\{F \in \mathscr{F} :$ *F is implausible given E*$\}$.

In words, when comparing the certain outcome y with the binary act xNz conditional on E, the assumption that $y \succ_E z$ is enough to induce the individual to strictly prefer y. Intuitively, the individual is so confident that, if E were to occur, N would not, that she simply disregards the corresponding prize x, no matter how attractive it may be. Instead, she focuses on the fact that, by choosing the bet on N, she would forego the prize y and instead receive the strictly worse prize z in case N does not occur.

Yet, this condition does not imply that N is null for $\succ_E$. In particular, if $N \in \mathscr{F}$, then Axiom 5 implies that there are $x, y \in X$ such that $x \succ_N y$, i.e., by Definition 3, $xNy \succ y$. Thus, N is not null for $\succ$, even though it may be implausible given Ω.

Under Axioms 1–6, the union of two implausible events is implausible: thus, $\mathrm{imp}(E)$ is implausible given E. Furthermore, if N is implausible given E and $M \subset N$, then M is also implausible given E: thus, $F \in \mathscr{F}$ is implausible given E if and only if $F \subseteq \mathrm{imp}(E)$.[2]

Example 3 (continued from Ex. 1) In the EV engine example, events B and C are implausible given Ω for $\succ^\mu$ and $\succ^\nu$ (cf. Eqs. 1 and 4). Consider $\succ^\mu$ and B for definiteness. For every perturbation $(\tilde{p}^k)_{k\geq 1}$ of μ, $\tilde{p}^k(\Omega \setminus B) \geq \tilde{p}^k(\{nn\}) \to 1$ as $k \to \infty$. Hence, for any $x, y, z \in \mathbb{R}$ with

[2] See Lemma 12 in the Appendix.

$y > z$, $y > z\tilde{p}^k(\Omega \setminus B) + x\tilde{p}^k(B) = \mathrm{E}_{\tilde{p}^k} xBz$ for all sufficiently large k, so $y \succ^\mu xBz$. However, B is not null for $\succ^\mu$: for instance, $1_B \succ^\mu 0$, because $\mathrm{E}_{\tilde{p}^k} 1_B = \tilde{p}^k(B) > 0$ for all k. $\square$

Together with the preceding axioms, the next two axioms ensure that, if we "carve out" the implausible set $\mathrm{imp}(F)$ from a conditioning event F, preferences conditional on the residual set $F \setminus \mathrm{imp}(F)$ satisfy the axioms for EU in Theorem 0. This formalizes the intuition that structural preferences only differ from EU preferences in that they take implausible conditioning events into account: if one carves those out, there is no difference.

Axiom 7 (Plausible Negative Transitivity) *For all $F \in \mathscr{F}$, $\succ_{F\setminus \mathrm{imp}(F)}$ is negatively transitive on $\mathscr{A}$.*

Axiom 8 (Plausible Continuity) *For all $F \in \mathscr{F}$ and $e, f, g \in \mathscr{A}$, if $e \succ_{F\setminus \mathrm{imp}(F)} f$ and $f \succ_{F\setminus \mathrm{imp}(F)} g$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha e + (1-\alpha)g \succ_{F\setminus \mathrm{imp}(F)} f$ and $f \succ_{F\setminus \mathrm{imp}(F)} \beta e + (1-\beta)g$.*

The first main result of this paper is that Axioms 1–8 uniquely identify the individual's utility u and conditional beliefs μ. Furthermore, the structural preference $\succ^{u,\mu}$ represented by (u,μ) is the *minimal*, or coarsest, relation that satisfies these axioms, and thus shares key properties of EU preferences (strict preorder, independence, monotonicity, and EU risk preferences) yet allows for implausible, but non-null conditioning events, with associated well-defined conditional beliefs.

Theorem 1 *Let $\succ$ be a binary relation on $\mathscr{A}_0$.*

1. *If $\succ$ is a structural preference, then it satisfies Axioms 1–8 for $\mathscr{A} = \mathscr{A}_0$.*
2. *If $\succ$ satisfies Axioms 1–8 for $\mathscr{A} = \mathscr{A}_0$, then there exists a non-constant, affine, cardinally unique[3] function $u : X \to \mathbb{R}$ and a unique CCPS μ such that the structural preference $\succ^{u,\mu}$ represented by (u,μ) on $\mathscr{A}_0$ satisfies $f \succ^{u,\mu} g \Longrightarrow f \succ g$ for every $f, g \in \mathscr{A}_0$.*

In both 1 and 2, $N \in \Sigma$ is implausible given $F \in \mathscr{F}$ if and only if $\mu(N|F) = 0$.

[3]That is, unique up to a positive affine transformation.

Section 5.A sketches the proof of sufficiency.

Consistently with Theorem 1, there exist preferences that are finer (i.e., rank more acts than) structural preferences, yet satisfy Axioms 1–8:

Example 4 (continued from Ex. 1) Consider the preference $\succ^*$ defined by $f \succ^* g$ if and only if $\mathrm{E}_{p^k} f > \mathrm{E}_{p^k} g$ for all k, where (p^k) is as in Eq. (2); recall that (p^k) is a perturbation of the CCPS μ in Eq. (1). Then $\succ^*$ satisfies Axioms 1–8: in particular, to verify Axioms 7 and 8, note that $F \setminus \mathrm{imp}(F) = \{nn\}$ for $F \in \{\Omega, \Omega \setminus B, \Omega \setminus C\}$, $B \setminus \mathrm{imp}(B) = \{vn\}$, and $C \setminus \mathrm{imp}(C) = \{nv\}$, so preferences conditional upon such events coincide with preferences over prizes at a single state. However, it is not the structural preference represented by μ: in particular, $1_B \succ^* 1_C$, whereas these acts are not ranked by $\succ^\mu$. Finally, if $\succ^*$ is a structural preference represented by a different CCPS μ', there exist a conditioning event $F \in \mathscr{F}$ and events $A, A' \subseteq F$ with $\mu(A|F) > \mu(A'|F)$ and $\mu'(A|F) < \mu'(A'|F)$; thus, $p^k(A|F) > p^k(A'|F)$ eventually, whereas for all perturbations $(\tilde{p}^k)_{k\ge1}$ of μ', $\tilde{p}^k(A|F) < \tilde{p}^k(A'|F)$ eventually, which imply $1_A \succ^*_F 1_{A'}$ and, respectively, $1_A \prec^{\mu'}_F 1_{A'}$. Thus, $\succ^*$ is not a structural preference. □

One final axiom is required to pin down structural preferences. To build intuition, together with Independence, the Archimedean Continuity axiom implies that strict EU preferences are robust to mixing with strictly ranked, but "nearly indifferent" prizes. In particular, if $f \succ g$, there are prizes x, y such that y is "only slightly" preferred to x, such that $\frac{1}{2}f + \frac{1}{2}x \succ \frac{1}{2}g + \frac{1}{2}y$.

Structural preferences satisfy a conditional version of this property. If $\mathscr{F}$ is a laminar family, this takes a particularly simple form:

Axiom 9a (Conditional Continuity – laminar $\mathscr{F}$) *For all $f, g \in \mathscr{A}$, if $f \succ g$, then there exists at least one $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ x$ such that $\frac{1}{2}f + \frac{1}{2}x \succ_F \frac{1}{2}g + \frac{1}{2}y$. Furthermore, for every $\omega \in \Omega$ such that $f(\omega) \prec g(\omega)$, there is one such F with $\omega \in F$.*

EU preferences satisfy Axiom 9a with $F = \Omega$, in which case the requirement that $\omega \in F$ for states where g delivers better prizes than f holds trivially. For structural preferences, a suitable conditioning event exists, but may be a non-trivial subset of Ω.

If $\mathscr{F}$ is not a laminar family, Axiom 9a is no longer sufficent. The reason is that, when conditioning events can overlap without being nested, the following preference configuration can arise: $f \succ g$, $\frac{1}{2}f + \frac{1}{2}x \succ_F \frac{1}{2}g + \frac{1}{2}y$ and $\frac{1}{2}f + \frac{1}{2}x \succ_G \frac{1}{2}g + \frac{1}{2}y$ for $F, G \in \mathscr{F}$ that intersect but are not nested; $F \cup G$ is plausible given both F and G, and yet $\frac{1}{2}f + \frac{1}{2}x \prec_{F\cup G} \frac{1}{2}g + \frac{1}{2}y$: Section 5.F provides an example. To rule out this possibility, I require that $\frac{1}{2}f + \frac{1}{2}x$ be preferred to $\frac{1}{2}g + \frac{1}{2}y$ conditional upon a (minimal) event that, by construction, contains as subsets all the conditioning events $G \in \mathscr{F}$ given which it is plausible (in the sense of Definition 4):

Definition 5 *For $F \in \mathscr{F}$, let $\mathscr{F}(F)$ be the smallest subset of $\mathscr{F}$ that contains F and such that, if $G \in \mathscr{F}$, and $\cup\mathscr{F}(F)$ is plausible given G, then $G \in \mathscr{F}(F)$.*[4]

Axiom 9b (Conditional Continuity – general $\mathscr{F}$) *For all $f, g \in \mathscr{A}$: if $f \succ g$, then there is at least one $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ x$ such that $\frac{1}{2}f + \frac{1}{2}x \succ_{\cup\mathscr{F}(F)} \frac{1}{2}g + \frac{1}{2}y$. Furthermore, for every $\omega \in \Omega$ such that $f(\omega) \prec g(\omega)$, there is one such F such that $\omega \in \cup\mathscr{F}(F)$.*

In Example 4, preference $\succ^*$ fails Axiom 9b. Indeed, consider acts $1_B, 1_C$ and state $\omega = nv$. This state belongs to $\cup\mathscr{F}(\Omega) = \Omega$ and $\cup\mathscr{F}(C) = C$, but not to $\cup\mathscr{F}(B) = B$. As noted in Ex. 4, $1_B \succ^* 1_C$, but $1_B(\omega) \prec^* 1_C(\omega)$. For any $x, y \in \mathbb{R}$, $\mathrm{E}_{p^k}[\frac{1}{2}1_B + \frac{1}{2}x] \to \frac{1}{2}x$ and $\mathrm{E}_{p^k}[\frac{1}{2}1_C + \frac{1}{2}y] \to \frac{1}{2}y$, so for $y > x$, not $\frac{1}{2}1_B + \frac{1}{2}x \succ^* \frac{1}{2}1_C + \frac{1}{2}y$. Moreover, $\mathrm{E}_{p^k}[\frac{1}{2}1_B + \frac{1}{2}x]C0 = \frac{1}{2}p^k(B \cap C) + \frac{1}{2}p^k(C)x = \frac{1}{2}p^k(C)x$ because $p^k(B \cap C) = p^k(\{vv\}) = 0$, whereas $\mathrm{E}_{p^k}[\frac{1}{2}1_C + \frac{1}{2}y]C0 = \frac{1}{2}p^k(C) + \frac{1}{2}p^k(C)y = \frac{1}{2}p^k(C)(1+y) > \frac{1}{2}p^k(C)x$ for all k, so again, not $\frac{1}{2}1_B + \frac{1}{2}x \succ^*_C \frac{1}{2}1_C + \frac{1}{2}y$ if $y > x$.

The characterization of structural preferences can now be stated.

Theorem 2 *Let $\succ$ be a binary relation on $\mathscr{A}_0$. The following statements are equivalent:*

1. *$\succ$ satisfies Axioms 1–8 and 9b (or 9a if $\mathscr{F}$ is laminar);*

[4] Remarks 2 and 3 in the Appendix imply that, if $\mathscr{F}$ is laminar, then $\cup\mathscr{F}(F) \in \mathscr{F}$, so Axiom 9b implies Axiom 9a. Lemma 18 also establishes the converse by showing that, for laminar $\mathscr{F}$, $\frac{1}{2}f + \frac{1}{2}x \succ_F \frac{1}{2}g + \frac{1}{2}y$ for some $F \in \mathscr{F}$ implies that $\frac{1}{2}f + \frac{1}{2}x \succ_{\cup\mathscr{F}(FG)} \frac{1}{2}g + \frac{1}{2}y$ for some (possibly different) $G \in \mathscr{F}$ with $\cup\mathscr{F}(G) \supseteq F$.

2. $\succ$ admits a structural-preference representation (u,μ) on $\mathscr{A}_0$, where $u: X \to \mathbb{R}$ is non-constant affine and $\mu \in \Delta(\Sigma, \mathscr{F})$.

Furthermore, in (2), u is cardinally unique and μ is unique.

5 Discussion and Extensions

5.A Proof Structure (sufficiency). There are two main steps in the construction of the representation of structural preferences in Definition 2. The first adapts standard arguments to show that preferences over the (convex) set X of outcomes are represented by a Bernoulli utility function u (Lemma 10) and that preferences conditional on the plausible part $F \setminus \mathrm{imp}(F)$ of each conditioning event $F \in \mathscr{F}$ admit an EU representation (Lemma 16), with utility u and a probability $p_F \in \Delta(\Sigma)$ satisfying $p_F(F \setminus \mathrm{imp}(F)) = 1$. Furthermore, an event N is implausible given a conditioning event F if and only if $p_F(N) = 0$.

The second step is to show that the probability array μ defined by setting $\mu(\cdot|F) = p_F$ for every $F \in \mathscr{F}$ is indeed a CCPS, and that preferences admit the representation in Definition 2. The argument consists of two sub-steps. The first is to show that both CCPSs (Proposition 2) and structural preferences (Proposition 3) admit alternative representations that do not involve perturbations, but are stated directly in terms of the probability array μ. This sub-step does not involve preferences directly.

The second sub-step then connects the above alternative representations with properties of the preference relation $\succ$. The argument "builds up" a representation of the unconditional preference $\succ$ from the conditional preferences $\succ_F$, for $F \in \mathscr{F}$. The key argument shows that preferences conditional upon overlapping events $F, G \in \mathscr{F}$, of which at least one is plausible given the other, uniquely identify preferences conditional upon $F \cup G$ (Lemmata 13–15). This step leverages the fact that, since Savage's Postulate P2 holds (Remark 1), the conditional preferences $\succ_F$ and $\succ_G$ must be the updates of conditional preference $\succ_{F\cup G}$. Iterating this argument uniquely pins down preferences conditional upon any event of the form $\cup\mathscr{F}(F)$ for some $F \in \mathscr{F}$ (cf. Definition 5). Finally, the proof shows that such preferences uniquely pin down the unconditional preference $\succ$, and that the latter has the representation provided in Proposition 3, which is equivalent to Definition 2.

The argument just given does *not* invoke Axiom 9a or 9b. As discussed in Section 4, the latter are only needed to rule out richer preference relations $\succ'$ that do not admit the representation in Definition 2, but that nonetheless satisfy Axioms 1–8.

5.B Elicitation In SRDG, each player is required to play the dynamic game of interest, and their conditional beliefs are elicited via side bets. Similarly, in the setting of this paper, preferences over acts can be elicited as the player actually faces the dynamic decision problem of interest. Since only ex-ante preferences need be elicited, this can be done via a simple randomized incentive scheme (cf. Luce and Raiffa, 1957, §13.6). To illustrate, consider again the example of Figure 1 and fix two arbitrary acts f, g. At the initial node, the firm (or the subject) is asked to choose between actions b and c, and also between acts f and g. Then, at the end of the experiment, the subject receives an equal-probability lottery between the outcome resulting from its choice of actions in the decision tree and from its choice of act at the initial node. If the subject's ex-ante preferences satisfy the Independence axiom (and she reduces plans to acts), the subject will choose f (resp. g) if $f \succ g$ (resp. $g \succ f$). This scheme can be extended to comparisons of multiple pairs of acts.

When alternatives are not strictly ranked, forced-choice mechanisms such as the above can generate spurious observations. This is the case for any preference model that allows for indifference or incomparability[5]. Recent contributions (Nielsen and Rigotti, 2025; Halevy, Walker-Jones, and Zrill, 2025) discuss elicitation schemes that allow subjects *not* to make a (unique) choice, and may be adapted to the present setting; this is left to future work.

That said, Axioms 9a / 9b identify further restrictions that must be satisfied if two acts are

[5]The belief elicitation mechanism in SRDG is incentive-compatible, under the assumption that players are structurally rational, because it relies on observing specific patterns of strict preferences.

in fact strictly ranked. Testing such restrictions provides one way to weed out at least some spurious forced choices.

5.C Related Literature Myerson (1986) (see also Myerson, 1991, Chapter 1) introduces and axiomatizes conditional expected-utility maximization with respect to a conditional probability system à la Rényi (1955). In Myerson (1986), a conditioning event F may have zero prior probability; however, in this case event F is Savage-null for the ex-ante preference. This implies that the individual's conditional preference $\succ_F$ cannot be derived from their prior preference. For this reason, Myerson (1986) takes as input the collection of all conditional preferences. However, one concern is that it is not possible to observe choices conditional upon disjoint events. This paper instead takes only the individual's ex-ante preferences as input.

Lexicographic expected-utility preferences (Blume et al., 1991a) can also model events with "infinitely small" probability that are relevant for decisions. Lexicographic (weak) preferences are complete, whereas for structural preferences, $f \not\succ g$ and $g \not\succ f$ can mean that f and g are not comparable, rather than indifferent. In addition, Blume et al. (1991a) is motivated by the analysis of solution concepts for games with simultaneous moves that rule out weakly dominated strategies (Blume, Brandenburger, and Dekel, 1991b). Correspondingly, their paper devotes special attention to preferences represented by "full-support" lexicographic probability systems, for which no state is Savage-null, and the relative likelihood of any two events is pinned down. By way of contrast, structural preferences, are designed to analyze dynamic decision problems and games, and are entirely pinned down by the individual's conditional beliefs; consequently, as noted in the Introduction, the relative likelihood of certain conditioning events may be undetermined.

In addition, a structurally rational individual is not just concerned about the possibility that zero-probability events may in fact occur: she also seeks choices that are robust to slight misspecifications of positive prior or conditional probabilities. As a result, structural preferences are not lexicographic, even if all non-empty mesurable subsets of the state space are conditioning events:

Example 5 Let $\Omega = \{a, b, c\}$ and $\mathscr{F} = 2^{\Omega} \setminus \{\emptyset\}$; consider the CCPS $\mu \in \Delta(\Sigma, \mathscr{F})$ uniquely defined by $\mu(\{a\}|\Omega) = \mu(\{b\}|\Omega) = \frac{1}{2}$ and $\mu(\{a\}|\{a,c\}) = \mu(\{b\}|\{b,c\}) = 1$, and let $\succ$ be the structural preference represented by μ and the identity utility function on $X = \mathbb{R}$. Fix a lexicographic probability system (LPS) $\lambda = (p_1, \ldots, p_N) \in \Delta(\Sigma)^N$ and denote by $\succ^{\lambda}$ the (strict) lexicographic preference induced by λ and by the identity utility function.[6] I claim that $\succ^{\lambda} \neq \succ$. Since λ is arbitrary, this implies that $\succ$ does not admit any lexicographic expected utility representation.

By contradiction, suppose $\succ^{\lambda} = \succ$. Since $(q^k)_{k\geq 1} = \left((\frac{1}{2} - \frac{1}{k+2}, \frac{1}{2} - \frac{1}{k+2}, \frac{2}{k+2})\right)_{k\geq 1}$ is a perturbation of μ for which $(1,-1,0) \cdot q^k = (-1,1,0) \cdot q^k$ for all $k \geq 1$, neither $(1,-1,0) \succ (-1,1,0)$ nor $(-1,1,0) \succ (1,-1,0)$. Let $n \in \{1, \ldots, N+1\}$ be such that $p_m(\{a\}) = p_m(\{b\})$ for all $m = 1, \ldots, n$: if $n \leq N$, then either $(1,-1,0) \succ^{\lambda} (-1,1,0)$ or $(-1,1,0) \succ^{\lambda} (1,-1,0)$; thus, it must be that $p_n(\{a\}) = p_n(\{b\})$ for all $n = 1, \ldots, N$. Furthermore, $(1,1,0) \succ (0,0,x)$ for all x, because all perturbations $(q^k)_{k\geq 1}$ of μ satisfy $q^k(\{a,b\}) \to 1$. If $p_1(\{c\}) > 0$, then for $x > \frac{1-p_1(\{c\})}{p_1(\{c\})}$, $(0,0,x) \succ^{\lambda} (1,1,0)$: thus, $p_1(\{c\}) = 0$.

To sum up, $\succ = \succ^{\lambda}$ requires $p_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $p_n = (\pi_n, \pi_n, 1 - 2\pi_n)$, with $\pi_n \in [0, \frac{1}{2}]$, for $n = 2, \ldots, N$. In addition, since $q^k(\{c\}) > 0$ for all perturbations $(q^k)_{k\geq 1}$ of μ, $(0,0,1) \succ (0,0,0)$: to ensure that also $(0,0,1) \succ^{\lambda} (0,0,0)$, it must be that $\pi_n < \frac{1}{2}$ for some n. It is without loss to assume that $p_2 = (\pi_2, \pi_2, 1 - 2\pi_2)$ with $0 < \pi_2 < \frac{1}{2}$.

Finally, consider the perturbation $(q^k)_{k\geq 1} = \left((\frac{1}{2}, \frac{1}{2} - \frac{1}{k+2}, \frac{1}{k+2})\right)_{k\geq 1}$: then $(1,-1,0) \cdot q^k = (0,0,1) \cdot q^k$ for all k, so neither $(1,-1,0) \succ (0,0,1)$ nor $(0,0,1) \succ (1,-1,0)$. However, $(1,-1,0) \cdot p_1 = (0,0,1) \cdot p_1$ and, since $\pi_2 < \frac{1}{2}$, $(1,-1,0) \cdot p_2 < (0,0,1) \cdot p_2$, so $(0,0,1) \succ^{\lambda} (1,-1,0)$: contradiction.

It is currently an open question whether strengthening the axioms in Section 4 in addition to imposing a specific structure on the set $\mathscr{F}$ of conditioning events can ensure that structural preferences also admit a lexicographic expected utility representation (outside of the trivial case of standard EU preferences).

[6] Recall that, for $f, g \in \mathbb{R}^{\Omega}$, $f \succ^{\lambda} g$ if and only if there is $n \in \{1, \ldots, N\}$ such that $\int f \, dp_n > \int g \, dp_n$ and $\int f \, dp_m = \int g \, dp_m$ for $m = 1, \ldots, n$.

On a technical note, for disjoint events F and G, Blume et al. (1991a, Definition 5.1) formalizes the meaning of "event F is infinitely more likely than event G" for lexicographic preferences. This is conceptually related to implausibility: if G is implausible given $F \cup G$, then intuitively F is infinitely more likely than G. However, when applied to structural preferences, the definition in Blume et al. (1991a) is more demanding:

Example 6 Let $\Omega = \{a, b, c\}$, with $\mathscr{F} = \{\Omega, \{b\}\}$, and $\mu(\{a\}|\Omega) = 1$; assume that $X = \mathbb{R}$ and u is the identity, and let $\succ$ be the structural preference represented by (u, μ). Then $\{c\}$ is implausible given Ω. However, $\{a, b\} = \Omega \setminus \{c\}$ is not infinitely more likely than $\{c\}$ in the sense of Blume et al. (1991a) for $\succ$. For instance, let $f = (0, 1, 0)$ and $g = (0, 0, 0)$: then $f \succ_{\{a,b\}} g$ (because one can take $h = g$ in Definition 3, and every perturbation of μ must assign positive probability to b). However, for example, $f = f\{a, b\}0 \not\succ (0, 0, 1) = g\{a, b\}1$, because the relative likelihood of b vs. c is not pinned down by μ. □

Both Myerson (1986) and Blume et al. (1991a) assume finite state spaces; the present paper handles state spaces of arbitrary cardinality, provided the set $\mathscr{F}$ of conditioning events is finite.

5.D Extension to bounded acts As anticipated above, the structural-preference representation on the set $\mathscr{A}_0$ of simple acts extends to a larger set of acts. Following e.g. Schmeidler (1989), given a binary relation $\succ$ on $\mathscr{A}_0$ that satisfies Axiom 2 (in particular, a structural preference on $\mathscr{A}_0$), call a function $f : \Omega \to X$ *$\succ$-measurable* if, for every $x \in X$, the sets $\{\omega : f(\omega) \not\succ x\}$ and $\{\omega : x \not\succ f(\omega)\}$ belong to Σ. A simple act is $\succ$-measurable, because each of these sets is necessarily a finite union of elements of Σ. Then, let $\mathscr{A}_b$ be the set of all $\succ$-measurable functions $f : \Omega \to X$ for which, in addition, there exist $x, y \in X$ such that $x \not\succ f(\omega) \not\succ y$ for all $\omega \in \Omega$. A simple act is $\succ$-bounded because, by Axiom 2, the relation $\not\succ$, i.e., "not $\succ$," is transitive on X. Thus, $\mathscr{A}_0 \subseteq \mathscr{A}_b$. Call the elements of $\mathscr{A}_b$ **$\succ$-bounded acts**.

Proposition 1 *Fix a non-constant $u : X \to \mathbb{R}$ and $\mu \in \Delta(\Sigma, \mathscr{F})$. If (u, μ) represent $\succ$ on $\mathscr{A}_b$, then $\succ$ satisfies Axioms 1–8 and 9b for $\mathscr{A} = \mathscr{A}_b$. If $\succ'$ also satisfies Axioms 1–8 and 9b for $\mathscr{A} = \mathscr{A}_b$*

and coincides with $\succ$ *on* $\mathscr{A}_0$*, then* $\succ = \succ'$*.*

In other words, the extension of a structural preference from $\mathscr{A}_0$ to $\mathscr{A}_b$ is unique.

5.E Extension to Countably additive CCPSs. Denote by $\Delta_c(\Sigma)$ the set of countably additive probabilities on (Ω,Σ). Say that a CCPS $\mu \in \Delta(\Sigma,\mathscr{F})$ is *countably additive* if $\mu(\cdot|F) \in \Delta_c(\Sigma)$ for all $F \in \mathscr{F}$. Structural preferences represented by a countably additive CCPS are characterized by the following additional axiom, adapted from Arrow (1974):

Axiom 10 (Plausible Monotone Continuity) *For all* $F \in \mathscr{F}$*,* $f,g \in \mathscr{A}_0$ *with* $f \succ_{F\setminus \mathrm{imp}(F)} g$*,* $x \in X$*, and* $E_1,E_2,\ldots \in \Sigma$ *with* $E_1 \supseteq E_2 \supseteq \ldots$ *and* $\cap_{\ell\geq 1} E_\ell = \emptyset$*, there is* $L \geq 1$ *such that* $x E_L f \succ_{F\setminus \mathrm{imp}(F)} g$ *and* $f \succ_{F\setminus \mathrm{imp}(F)} x E_L g$*.*

Lemma 16 in Appendix D shows that, for every $F \in \mathscr{F}$, preferences conditional on $F \setminus \mathrm{imp}(F)$ are consistent with EU and represented by a probability $p_F \in \Delta(\Sigma)$. By Arrow (1974), Axiom 10 holds if and only if each p_F is countably additive. Since the representing CCPS in Theorems 1 and 2 is defined by setting $\mu(\cdot|F) = p_F$ for every $F \in \mathscr{F}$, the result follows.

If a CCPS μ is countably additive, then it admits a perturbation consisting of countably additive probabilities; furthermore, in Definition 2 it is without loss to restrict attention to such perturbations.[7] However, it is crucial that the (countably additive) perturbations converge in the topology induced by $B(\Sigma)$, and not in the usual weak convergence topology—the topology induced by bounded, continuous functions:

Example 7 Let $\Omega = [0,1]$, endowed with the usual topology; let Σ be the Borel sigma-algebra, and $\mathscr{F} = \{\Omega\}$. In this case, by design, structural preferences coincide with EU preferences: see SRDG, §4. For instance, Finally, take $X = \mathbb{R}$, let u be the identity, and consider the (trivial) CCPS $\mu = \big(\mu(\cdot|\Omega)\big)$ defined by $\mu(\{0\}|\Omega) = 1$. Then, for example $1_{\{0\}} \succ 0$, consistently with the preferences of an EU individual with beliefs $\mu(\cdot|\Omega)$.

[7] These facts follow from Corollary 1 and Propositions 2 and 3 in the Appendix.

Now consider the sequence $(p^k)_{k\geq 1}$ in $\Delta_c(\Sigma)$ such that $p^k(\{1/k\}) = 1$. Then $(p^k)_{k\geq 1}$ is not a perturbation of μ, because it does not converge pointwise to $\mu(\cdot|\Omega)$. However, for every (bounded) continuous function $\phi : \Omega \to \mathbb{R}$, $\int \phi \, dp^k = \phi(1/k) \to \phi(0) = \int \phi \, d\mu(\cdot|\Omega)$. If one deemed $(p^k)_{k\geq 1}$ a perturbation of μ, then it would not be the case that $1_{\{0\}} \succ 0$, because for every k, $\int 1_{\{0\}} \, dp^k = 0 = \int 0 \, dp^k$. Thus, relaxing the notion of perturbation would result in preferences that are weaker (less restrictive) than EU in this example.

5.F Conditional continuity for non-laminar families As noted in Section 4, if $\mathscr{F}$ is a laminar family, the conditional continuity property takes the simpler form of Axiom 9a. However, if $\mathscr{F}$ is not laminar, the formulation in Axiom 9b is necessary.

Example 8 Let $\Omega = \{a, b, c, d\}$, $F = \{b, c\}$, $G = \{c, d\}$ and $\mathscr{F} = \{\Omega, F, G\}$. Define μ by $\mu(\{a\}|\Omega) = 1$ and $\mu(\{b\}|F) = \mu(\{c\}|G) = \frac{1}{2}$; then μ is a CCPS. Let $X = \mathbb{R}$, let $\succ$ be the structural preference represented by the identity u on X and μ, and consider acts $f = (0, 4, 4, 4)$ and $g = (0, 0, 10, 0)$. For every perturbation (p^k) of μ, $p^k(\{b\})/p^k(\{c\}) \to 1$ and similarly $p^k(\{c\})/p^k(\{d\}) \to 1$, which implies that $p^k(\cdot|F \cup G) \to P$, where $P(\{b\}) = P(\{c\}) = P(\{d\}) = \frac{1}{3}$; since $\mathrm{E}_P \, u \circ f = 4 > \frac{10}{3} = \mathrm{E}_P \, u \circ g$, even though $f(c) < g(c)$, there exist $x, y \in X$ with $y \succ x$ such that $\mathrm{E}_P \, u \circ [\frac{1}{2} f + \frac{1}{2} x] > \mathrm{E}_P \, u \circ [\frac{1}{2} g + \frac{1}{2} y]$, and thus eventually also $\mathrm{E}_{p^k(\cdot|F\cup G)} u \circ [\frac{1}{2} f + \frac{1}{2} x] > \mathrm{E}_{p^k(\cdot|F\cup G)} u \circ [\frac{1}{2} g + \frac{1}{2} y]$. Hence, eventually $\mathrm{E}_{p^k} u \circ \{[\frac{1}{2} f + \frac{1}{2} x](F \cup G) f\} > \mathrm{E}_{p^k} u \circ \{[\frac{1}{2} g + \frac{1}{2} y](F \cup G) f\}$ for all perturbations of μ, and so $[\frac{1}{2} f + \frac{1}{2} x](F \cup G) f \succ [\frac{1}{2} g + \frac{1}{2} y](F \cup G) f$, i.e., $\frac{1}{2} f + \frac{1}{2} x \succ_{F\cup G} \frac{1}{2} g + \frac{1}{2} y$. Recall from §5.A that E is plausible given $H \in \mathscr{F}$ iff $\mu(E|H) > 0$: thus, $\mathscr{F}(F) = \{F, G\}$ according to Definition 5, and so $\succ$ satisfies Axiom 9b.

On the other hand, $\mathrm{E}_{\mu(\cdot|H)} u \circ f \leq \mathrm{E}_{\mu(\cdot|H)} u \circ g$ for all $H \in \mathscr{F}$, so for the same $x, y \in X$ with $y \succ x$, $\mathrm{E}_{\mu(\cdot|H)} u \circ [\frac{1}{2} f + \frac{1}{2} x] < \mathrm{E}_{\mu(\cdot|H)} u \circ [\frac{1}{2} g + \frac{1}{2} y]$ for all $H \in \mathscr{F}$, and therefore $\mathrm{E}_{p^k(\cdot|H)} u \circ [\frac{1}{2} f + \frac{1}{2} x] < \mathrm{E}_{p^k(\cdot|H)} u \circ [\frac{1}{2} g + \frac{1}{2} y]$ eventually for all perturbations (p^k) of μ. This implies that $\frac{1}{2} f + \frac{1}{2} x \prec_H \frac{1}{2} g + \frac{1}{2} y$ for all $H \in \mathscr{F}$. Thus, the structural preference $\succ$ does not satisfy Axiom 9a. □

A Main result: Preliminaries

For the following definitions and results, up to and including Lemma 2, fix an array $\mu \in \Delta(\Sigma)^{\mathscr{F}}$ such that $\mu(F|F) = 1$ for $F \in \mathscr{F}$ (not necessarily a CCPS).

For every $F \in \mathscr{F}$, define the *covered set* $C_\mu(F)$ as follows:

$$C_\mu^0(F) = F, \quad C_\mu^m(F) = \cup\{G \in \mathscr{F} : \mu(C_\mu^{m-1}(F)|G) > 0\}, \quad C_\mu(F) = \cup_{m\geq 0} C_\mu^m(F). \tag{5}$$

Also let[8]

$$\mathscr{C}_\mu^0(F) = \{F\}, \qquad \mathscr{C}_\mu^m(F) = \{G \in \mathscr{F} : \mu(C_\mu^{m-1}(F)|G) > 0\}, \qquad \mathscr{C}_\mu(F) = \cup_{m\geq 0} \mathscr{C}_\mu^m(F). \tag{6}$$

Thus, by definition, $C_\mu^m(F) = \cup \mathscr{C}_\mu^m(F)$ for all $m \geq 0$, and $C_\mu(F) = \cup \mathscr{C}_\mu(F)$.

Observation 1 For every $F \in \mathscr{F}$, $(\mathscr{C}_\mu^m(F))_{m\geq 0}$ and $(C_\mu^m(F))_{m\geq 0}$ are setwise increasing. Thus, there is $M \geq 1$ such that $\mathscr{C}_\mu^m(F) = \mathscr{C}_\mu^M(F)$, hence $C_\mu^m(F) = C_\mu^M(F) = C_\mu(F)$, for all $m \geq M$. Consequently, for all $G \in \mathscr{F}$, $G \subseteq C_\mu(F)$ implies $G \in \mathscr{C}_\mu(F)$.

Proof: If $G \in \mathscr{C}_\mu^m(F)$ for some $m \geq 0$, then $\mu(C_\mu^m(F)|G) = \mu(\cup \mathscr{C}_\mu^m(F)|G) \geq \mu(G|G) = 1$, so also $G \in \mathscr{C}_\mu^{m+1}(F)$. The second claim follows from the fact that $\mathscr{F}$ is finite. Finally, if $G \subseteq C_\mu(F) = C_\mu^M(F)$, then $\mu(C_\mu^M(F)|G) \geq \mu(G|G) = 1$, so $G \in \mathscr{C}_\mu^{M+1}(F) = \mathscr{C}_\mu^M(F) = \mathscr{C}_\mu(F)$. ■

Remark 2 *For every $F \in \mathscr{F}$, there is an ordered list $F_1, \ldots, F_L \in \mathscr{F}$ such that $F_1 = F$, $\mathscr{C}_\mu(F) = \{F_1, \ldots, F_L\}$, and for every $\ell = 2, \ldots, L$, $\mu(\cup_{m=1}^{\ell-1} F_m | F_\ell) > 0$. Hence, if $\mathscr{F}$ is laminar, then $C_\mu(F) \in \mathscr{F}$.*

Proof: Fix $F \in \mathscr{F}$ and let M be as in Observation 1. For every $m = 0, \ldots, M$, let $L(m) = |\mathscr{C}_\mu^m(F)|$. Enumerate the elements of $\mathscr{C}_\mu^M(F)$ as follows: let $F_1 = F$, so $\mathscr{C}_\mu^0(F) = \{F_1\}$; inductively, if $\mathscr{C}_\mu^m(F) = \{F_1, \ldots, F_{L(m)}\}$ and $m < M$, enumerate the elements of $\mathscr{C}_\mu^{m+1}(F) \setminus \mathscr{C}_\mu^m(F)$ as

[8] To clarify the notation: if $\mathscr{S}$ is a collection of sets, then $\cup\mathscr{S}$ denotes the union of all $S \in \mathscr{S}$; and if $\mathscr{S}_0, \mathscr{S}_1, \ldots$ are collections of sets, then $\cup_{m\geq 0}\mathscr{S}_m = \{S : \exists m \geq 0, S \in \mathscr{S}_m\}$, which is also a collection of sets.

$F_{L(m)+1,\dots,F_{L(m+1)}}$, so that $\mathscr{C}_\mu^{m+1}(F)=\{F_1,\dots,F_{L(m+1)}\}$: this is possible because, by Observation 1, the sets $\mathscr{C}_\mu^m(F)$ are monotonically increasing. Finally, for $\ell=2,\dots,L(M)$, let m^* be such that $F_\ell\in\mathscr{C}_\mu^{m^*+1}(F)\setminus\mathscr{C}_\mu^{m^*}(F)$: then $\mu(\cup_{n=1}^{\ell-1}F_n|F_\ell)\geq\mu(\cup_{n=1}^{L(m^*)}F_n|F_\ell)=\mu(C_\mu^{m^*}(F)|F_\ell)>0$.

To prove the second claim, I show inductively that $\cup_{m=1}^{\ell}F_m\in\mathscr{F}$ for $\ell=1,\dots,L(M)$. For $\ell=1$, $F_1=F\in\mathscr{F}$. By induction, assume the result is true for some $\ell\in\{1,\dots,L(M)-1\}$ and consider $\ell+1$. Since $\mu(\cup_{m=1}^{\ell}F_m|F_{\ell+1})>0$, $F_{\ell+1}\cap\left(\cup_{m=1}^{\ell}F_m\right)\neq\emptyset$, so by laminarity the sets $F_{\ell+1}$ and $\cup_{m=1}^{\ell}F_m$ are nested. By the inductive hypothesis, $\cup_{m=1}^{\ell}F_m\in\mathscr{F}$, so $\cup_{m=1}^{\ell+1}F_m=F_{\ell+1}\cup\left(\cup_{m=1}^{\ell}F_m\right)\in\left\{F_{\ell+1},\cup_{m=1}^{\ell}F_m\right\}\subseteq\mathscr{F}$. ∎

The next result is related to Definition 5.

Remark 3 *For every $F\in\mathscr{F}$, $\mathscr{C}_\mu(F)$ is the smallest subset $\mathscr{C}$ of $\mathscr{F}$ such that (i) $F\in\mathscr{C}$, and (ii) for every $G\in\mathscr{F}$, $\mu(\cup\mathscr{C}|G)>0$ implies $G\in\mathscr{C}$.*

Proof: Let $\mathscr{C}$ satisfy (i) and (ii) for $F\in\mathscr{F}$, and let M be such that $\mathscr{C}_\mu^M(F)=\mathscr{C}_\mu^{M+1}(F)=\mathscr{C}_\mu(F)$, per Observation 1.

I claim that $\mathscr{C}_\mu(F)\subseteq\mathscr{C}$. For $m=0$, $\mathscr{C}_\mu^0(F)=\{F\}$ and by (i) $F\in\mathscr{C}$, so $\mathscr{C}_\mu^0(F)\subseteq\mathscr{C}$. Inductively, suppose $\mathscr{C}_\mu^m(F)\subseteq\mathscr{C}$ for some $m\geq 0$. By definition, $G\in\mathscr{C}_\mu^{m+1}(F)$ iff $\mu(\cup\mathscr{C}_\mu^m(F)|G)>0$: but by the inductive hypothesis, this implies $\mu(\cup\mathscr{C}|G)>0$ as well, so $G\in\mathscr{C}$ by (ii). Thus, $\mathscr{C}_\mu^{m+1}(F)\subseteq\mathscr{C}$. Since this holds for $m=M$ as well, the claim is proved.

Next, I claim that $\mathscr{C}_\mu(F)=\mathscr{C}_\mu^M(F)=\mathscr{C}_\mu^{M+1}(F)$ satisfies (i) and (ii), so that $\mathscr{C}\subseteq\mathscr{C}_\mu(F)$, which proves the remark. By Observation 1, $\{F\}=\mathscr{C}_\mu^0(F)\subseteq\mathscr{C}_\mu^M(F)$, so $F\in\mathscr{C}_\mu(F)$, i.e., (i) holds. Next, suppose $G\in\mathscr{F}$ satisfies $\mu(\cup\mathscr{C}_\mu(F)|G)>0$, i.e., $\mu(\cup\mathscr{C}_\mu^M(F)|G)>0$: then $G\in\mathscr{C}_\mu^{M+1}(F)$ by definition; since $\mathscr{C}_\mu(F)=\mathscr{C}_\mu^{M+1}(F)$ as well, (ii) holds. ∎

Lemma 1 *For every $F\in\mathscr{F}$, there is at most one $P\in\Delta(\Sigma)$ such that*

$$P(F)>0,\quad P(C_\mu(F))=1,\quad \forall G\in\mathscr{F},\ E\in\Sigma,\ E\subseteq G\subseteq C_\mu(F)\Rightarrow P(E)=\mu(E|G)P(G). \tag{7}$$

Proof: I show inductively that, for every $m \geq 0$, if $P \in \Delta(\Sigma)$ satisfies

$$P(F) > 0, \quad P(C_\mu^m(F)) = 1, \quad \forall G \in \mathscr{F},\ E \in \Sigma,\ E \subseteq G \subseteq C_\mu^m(F) \Rightarrow P(E) = \mu(E|G)P(G). \tag{8}$$

then it is unique. Since $C_\mu(F) = C_\mu^M(F)$ for some $M \geq 0$, this implies the claim.

For $m = 0$, suppose $P \in \Delta(\Sigma)$ satisfies Eq. (8). Then $P(F) = P(C_\mu^0(F)) = 1$ and furthermore, taking $G = F$, for every measurable $E \subseteq F$, $P(E) = \mu(E|F)P(F) = \mu(E|F)$. Thus, if such a measure P exists, it must be $\mu(\cdot|F)$.

Inductively, suppose the claim is true for some $m \geq 0$. Suppose $P \in \Delta(\Sigma)$ satisfies Eq. (8) for $m+1$. Then $P(C_\mu^m(F)) \geq P(F) > 0$, so $P(\cdot|C_\mu^m(F))$ is well-defined. Furthermore, $P(F|C_\mu^m(F)) > 0$, $P(C_\mu^m(F)|C_\mu^m(F)) = 1$ by definition, and if $E, G \in \Sigma$ satisfy $G \in \mathscr{F}$ and $E \subseteq G \subseteq C_\mu^m(F)$, then by Observation 1 also $E \subseteq G \subseteq C_\mu^{m+1}(F)$, so $P(E) = \mu(E|G)P(G)$ and so, dividing both sides by $P(C_\mu^m(F)) > 0$, $P(E|C_\mu^m(F)) = \mu(E|G)P(G|C_\mu^m(F))$. Thus, $P^m \equiv P(\cdot|C_\mu^m(F))$ satisfies Eq. (8) for m, and so by the induction hypothesis it is the only measure that does.

Now consider $G \in \mathscr{C}_\mu^{m+1}(F)$, so $\mu(G \cap C_\mu^m(F)|G) = \mu(C_\mu^m(F)|G) > 0$. If $P^m(G) = P(G|C_\mu^m(F)) > 0$, then $P(G) \geq P(G \cap C_\mu^m(F)) > 0$, and so by Eq. (8), $0 < P(G \cap C_\mu^m(F)|G) = \mu(G \cap C_\mu^m(F)|G)$ and $P(G \setminus C_\mu^m(F)|G) = \mu(G \setminus C_\mu^m(F)|G)$. Therefore,

$$P(G \setminus C_\mu^m(F)) = \frac{\mu(G \setminus C_\mu^m(F)|G)}{\mu(G \cap C_\mu^m(F)|G)} P(G \cap C_\mu^m(F)) = \frac{\mu(G \setminus C_\mu^m(F)|G)}{\mu(G \cap C_\mu^m(F)|G)} P^m(G) P(C_\mu^m(F)). \tag{9}$$

If instead $P^m(G) = 0$, suppose by contradiction that $P(G \setminus C_\mu^m(F)) > 0$. Since $G \in \mathscr{C}_\mu^{m+1}(F)$, $G \subseteq C_\mu^{m+1}(F)$, and by definition $G \setminus C_\mu^m(F) \subseteq G$: thus, applying Eq. (8) to $E = G \setminus C_\mu^m(F)$ yields $P(G \setminus C_\mu^m(F)) = \mu(G \setminus C_\mu^m(F)|G)P(G)$, i.e., $P(G \setminus C_\mu^m(F)|G) = \mu(G \setminus C_\mu^m(F)|G)$. Since by assumption $\mu(C_\mu^m(F)|G) > 0$, $P(G \setminus C_\mu^m(F)|G) = \mu(G \setminus C_\mu^m(F)|G) < 1$, and so $P(C_\mu^m(F)|G) > 0$. But then $P(G \cap C_\mu^m(F)) > 0$ and hence $P(G|C_\mu^m(F)) = P^m(G) > 0$, contradiction. Therefore, if $P^m(G) = 0$, then $P(G \setminus C_\mu^m(F)) = 0$, so Eq. (9) still holds.

It follows that, for every $G \in \mathscr{C}_\mu^{m+1}(F)$ and $E \in \Sigma$ with $E \subseteq G$,

$$P(E) = \mu(E|G)\left(1 + \frac{\mu(G \setminus C_\mu^m(F)|G)}{\mu(G \cap C_\mu^m(F)|G)}\right) P^m(G) P(C_\mu^m(F)). \tag{10}$$

To complete the proof, consider an arbitrary $E \in \Sigma$ with $E \subseteq C_\mu^{m+1}(F)$. Since $\mathscr{F}$ is finite, so is $\mathscr{C}_\mu^{m+1}(F)$, so there are $G_1, \dots, G_N \in \mathscr{C}_\mu^{m+1}(F)$ such that $E \subseteq \cup_n G_n$. Let $E_1 = E \cap G_1$ and then, inductively, $E_n = (E \cap G_n) \setminus \cup_{\ell=1}^{n-1} E_\ell$. The sets $E_1, \dots, E_N$ are disjoint, $E_n \subseteq G_n$ for each n, and $\cup_n E_n = \cup_n (E \cap G_n) = E \cap (\cup_n G_n) = E$. Then, by Eq. (10),

$$P(E) = \sum_n P(E_n) = P(C_\mu^m(F)) \sum_n \mu(E_n|G_n)\left(1 + \frac{\mu(G_n \setminus C_\mu^m(F)|G_n)}{\mu(G_n \cap C_\mu^m(F)|G_n)}\right) P^m(G_n) \tag{11}$$

In particular, this applies to $E = C_\mu^{m+1}(F)$, so, by Eq. (8)

$$1 = P(C_\mu^{m+1}(F)) = P(C_\mu^m(F)) \sum_n \mu(E_n|G_n)\left(1 + \frac{\mu(G_n \setminus C_\mu^m(F)|G_n)}{\mu(G_n \cap C_\mu^m(F)|G_n)}\right) P^m(G_n)$$

which implies that $P(C_\mu^m(F))$ is uniquely determined by P^m and μ. Substituting for $P(C_\mu^m(F))$ in Eq. (11), it follows that $P(E)$ is also uniquely determined by P^m and μ for every $E \in \Sigma$ with $E \subseteq C_\mu^{m+1}(F)$. Since $P(C_\mu^{m+1}(F)) = 1$, this proves the claim. ■

The following result is used to extend structural preferences from $\mathscr{A}_0$ to $\mathscr{A}_b$.

Corollary 1 *If $\mu(\cdot|G)$ is countably additive for every $G \in \cup\mathscr{C}_\mu(F)$, then P in Eq. (7) is countably additive.*

Proof: Consider a sequence $(E^n)_{n\geq 1}$ in Σ such that $\cap_n E_n = \emptyset$. It must be shown that $P(E_n) \to 0$.

Suppose first that $E_1 \subseteq G$ for some $G \in \mathscr{F}$ with $G \subseteq C_\mu(F)$; then $G \in \cup\mathscr{C}_\mu(F)$ by Observation 1. Then, from Eq. (7), $P(E^n) = \mu(E^n|G)P(G) \to 0$ because $\mu(\cdot|G)$ is countably additive.

Now suppose E_1 is arbitrary. Since $P(C_\mu(F)) = 1$, it is without loss to assume that $E_1 \subseteq C_\mu(F) = \cup\mathscr{C}^M(F)$, where M is as in Observation 1. But then

$$P(E^n) \leq \sum_{G \in \mathscr{C}^M(F)} P(E^n \cap G) \to 0$$

because $\mathscr{C}^M(F)$ is finite and, for every $G \in \mathscr{C}^M(F)$, $E^1 \cap G \subseteq G$, so $P(E^n \cap G) \to 0$. ■

Observation 2 For every $F, G \in \mathscr{F}$ and $m \geq 0$, if $G \in \mathscr{C}^m(F)$, then

1. $C_\mu(G) \subseteq C_\mu(F)$;
2. if $P \in \Delta(\Sigma)$ satisfies Eq. (7) for F, and $P(G) > 0$, then $C_\mu(G) = C_\mu(F)$.

Proof: For part 1, for $G' \in \mathscr{C}_\mu^0(G) = \{G\}$, by assumption $G' = G \in \mathscr{C}_\mu^m(F) = \mathscr{C}_\mu^{m+0}(F)$. Inductively, assume that, for $n \geq 0$, $\mathscr{C}_\mu^n(G) \subseteq \mathscr{C}_\mu^{m+n}(F)$ and consider $G' \in \mathscr{C}_\mu^{n+1}(G)$. By definition, $\mu(C_\mu^n(G)|G') > 0$, so a fortiori $\mu(C_\mu^{m+n}(F)|G') > 0$, and so $G' \in \mathscr{C}_\mu^{m+n+1}(F)$. Taking $M(G)$ as in Observation 1 for G, in particular $G' \in \mathscr{C}_\mu(G) = \mathscr{C}_\mu^{M(G)}(G)$ implies $G' \in \mathscr{C}_\mu^{M(G)+m}(F)$, and therefore, a fortiori, $G' \in \mathscr{C}_\mu(F)$. Thus, $C_\mu(G) = \cup\mathscr{C}_\mu(G) \subseteq \cup\mathscr{C}_\mu(F) = C_\mu(F)$.

For part 2, if $m = 0$ then the claim is trivially true because $\mathscr{C}^0(F) = \{F\}$. Assume it is true for some $m \geq 0$ and consider $G \in \mathscr{C}_\mu^{m+1}(F)$, so $\mu(C_\mu^m(F)|G) > 0$. Since $C_\mu^m(F) = \cup\mathscr{C}_\mu^m(F)$ and $\mathscr{C}_\mu^m(F)$ is finite, there is $F' \in \mathscr{C}_\mu^m(F)$ with $\mu(F'|G) > 0$. By Eq. (7) and the assumption that $P(G) > 0$, $P(F') > P(F' \cap G) = \mu(F'|G)P(G) > 0$. By the induction hypothesis, $P(F') > 0$ implies $C_\mu(F') = C_\mu(F)$. Furthermore, by Eq. (7), $\mu(G|F') = P(G \cap F'|F') > 0$, so $F' \in \mathscr{C}^1(G)$, and by part 1 $C_\mu(F') \subseteq C_\mu(G)$. Hence, $C_\mu(F) \subseteq C_\mu(G)$; since $C_\mu(G) \subseteq C_\mu(F)$ by part 1, $C_\mu(F) = C_\mu(G)$. ■

Lemma 2 *Fix $F, G \in \mathscr{F}$ and suppose that there exist $P, Q \in \Delta(\Sigma)$ that satisfy Eq. (7) for F and G respectively. Then*

1. *$Q(C_\mu(F)) > 0$ implies $C_\mu(F) \supseteq C_\mu(G)$.*
2. *$C_\mu(F) \supsetneq C_\mu(G)$ implies $P(C_\mu(G)) = 0$.*
3. *$C_\mu(F) = C_\mu(G)$ implies $P(G) > 0$, and so $P = Q$.*
4. *$P\left(\cup\{G \in \mathscr{F} : C_\mu(G) = C_\mu(F)\}\right) = 1$.*

Proof: 1: Let $M(F)$ and $M(G)$ be as in Observation 1 for F and G respectively. Since $\mathscr{C}_\mu^{M(F)}(F)$ is finite, $Q(C_\mu(F)) = Q(\cup\mathscr{C}_\mu^{M(F)}(F)) > 0$ implies that there is $F' \in \mathscr{C}_\mu^{M(F)}(F)$ such that $Q(F') > 0$. Similarly, since $\mathscr{C}_\mu^{M(G)}(G)$ is finite and by assumption $Q(\cup\mathscr{C}_\mu^{M(G)}(G)) = 1$, $Q(F') = Q(F' \cap \cup\mathscr{C}_\mu^{M(G)}(G)) > 0$ implies that there is $G' \in \mathscr{C}_\mu^{M(G)}(G)$ such that $Q(F' \cap G') > 0$. A fortiori, $Q(G') >$

0, so by Observation 2 part 2 $C_\mu(G) = C_\mu(G')$. Moreover, by Eq. (7), $\mu(F'|G') = Q(F'|G') > 0$, so a fortiori $\mu(C_\mu^{M(F)}(F)|G') > 0$, and so $G' \in \mathscr{C}_\mu^{M(F)+1}(F) = \mathscr{C}_\mu^{M(F)}(F)$, and Observation 2 part 1 then implies that $C_\mu(G') \subseteq C_\mu(F)$. Thus, $C_\mu(G) \subseteq C_\mu(F)$.

For part 2, if $P(C_\mu(G)) > 0$, then by part 1 $C_\mu(G) \supseteq C_\mu(F)$. This contradicts $C_\mu(F) \supsetneq C_\mu(G)$, so $P(C_\mu(G)) = 0$.

For part 3, since $C_\mu(F) = C_\mu(G)$, necessarily $P(C_\mu(G)) = 1$, and for any $G' \in \mathscr{F}$ and $E \in \Sigma$, $E \subseteq G' \subseteq C_\mu(G) = C_\mu(F)$ implies that $P(E) = \mu(E|G')P(G')$ because Eq. (7) holds for P and F. Since Q is the unique measure that satisfies Eq. (7) for G, it suffices to show that $P(G) > 0$.

Since $P(C_\mu(G)) = P(\cup \mathscr{C}^{M(G)}(G)) > 0$, where $M(G)$ is as in Observation 1, there is $G' \in \mathscr{C}_\mu^{M(G)}(G)$ with $P(G') > 0$. Let $G_1 = G'$, so $G_1 \in \mathscr{C}_\mu^{M(G)}(G) = \mathscr{C}_\mu^{M(G)+1-1}(G)$ and $P(G_1) > 0$. Inductively, suppose $G_m \in \mathscr{C}_\mu^{M(G)+1-m}(G)$ and $P(G_m) > 0$ for some $m \leq M(G)$. Then, by definition, $\mu(C_\mu^{M(G)-m}(G)|G_m) > 0$, and since $C_\mu^{M(G)-m}(G) = \cup \mathscr{C}_\mu^{M(G)-m}(G)$, there is $G_{m+1} \in \mathscr{C}_\mu^{M(G)-m}(G) = \mathscr{C}_\mu^{M(G)+1-(m+1)}(G)$ such that $\mu(G_{m+1}|G_m) > 0$. Since $G_m \in \mathscr{C}_\mu^{M(G)+1-m}(G)$, $G_m \subseteq C_\mu(G) = C_\mu(F)$, so $\mu(G_{m+1}|G_m) = P(G_{m+1}|G_m)$. Hence, $P(G_{m+1}) \geq P(G_{m+1} \cap G_m) > 0$, which completes the inductive step. In particular, conclude that there is $G_{M(G)+1} \in \mathscr{C}_\mu^{M(G)+1-(M(G)+1)}(G) = \mathscr{C}_\mu^0(G) = \{G\}$ with $P(G_{M(G)+1}) > 0$: that is, $P(G) > 0$, as claimed.

For part 4, let $\mathscr{C} = \{G \in \mathscr{F} : C_\mu(G) = C_\mu(F)\}$. By contradiction, suppose that $P(\cup \mathscr{C}) < 1$. Since $P(\cup \mathscr{C}_\mu(F)) = 1$, there is $G \in \mathscr{C}_\mu(F) \setminus \mathscr{C}$ such that $P(G) > 0$. By part 2, $C_\mu(F) \subseteq C_\mu(G)$. But by observation 2 part 1, $G \in \mathscr{C}_\mu(F)$ implies $C_\mu(G) \subseteq C_\mu(F)$, so $C_\mu(G) = C_\mu(F)$ and so $G \in \mathscr{C}$, contradiction. Thus, $P(\cup \mathscr{C}) = 1$. ■

The remaining results lead to alternative characterizations of CCPSs and structural preferences that do not rely on perturbations.

Lemma 3 *Fix a CCPS $\mu \in \Delta(\Sigma, \mathscr{F})$ and a perturbation $(p^k)_{k\geq 1}$ of μ. Consider an ordered list $F_1, \ldots, F_L \in \mathscr{F}$ such that, for every $\ell = 2, \ldots, L$, $\mu(\cup_{m=1}^{\ell-1} F_m | F_\ell) > 0$. Then $p \equiv \lim_k p^k(\cdot | \cup_\ell F_\ell)$ is well-defined, $p(F_1) > 0$, and for all $\ell = 1, \ldots, L$ and $E \subseteq F_\ell$, $p(E) = \mu(E|F_\ell)p(F_\ell)$.*

Proof: Argue by induction on L. For $L=1$, the claim follows from the definition of perturbation by setting $p=\mu(\cdot|F_1)$. Thus, assume the claim holds for some $L\geq 1$, let $p_L=\lim_k p^k(\cdot|\cup_{\ell=1}^L F_\ell)$, and assume that $F_{L+1}\in\mathscr{F}$ is such that $\mu(\cup_{\ell=1}^L F_\ell|F_{L+1})>0$.

Let $E=\cup_{\ell=1}^L F_\ell$ and $F=F_{L+1}$, so $\mu(E|F)>0$. This immediately implies that $E\cap F\neq\emptyset$; furthermore, from the definition of perturbation, eventually $p^k(E)>0$, $p^k(F)=p^k(F_{L+1})>0$, and $p^k(\cdot|F)\to\mu(\cdot|F)$. Hence, eventually $p^k(E\cap F)>0$.

Assume first that also $p_L(F)=p_L(F_{L+1})>0$, so $p_L(E\cap F)>0$ as well because $p_L(E)=1$ by definition. For every k,

$$p^k(E\cap F|E\cup F)=p^k(E\cap F|E)\cdot p^k(E|E\cup F)\quad\text{and}\quad p^k(E\cap F|E\cup F)=p^k(E\cap F|F)\cdot p^k(F|E\cup F).$$

Consider a subsequence $(p^{k(\ell)})_{\ell\geq1}$ of $(p^k)_{k\geq1}$ along which the numerical sequence $(p^{k(\ell)}(E\cap F|E\cup F))_{\ell\geq1}$ converges, and let its limit be $\pi\in[0,1]$. Since by the inductive hypothesis $p^k(E\cap F|E)\to p_L(E\cap F)>0$, and by definition $p^k(E\cap F|F)\to\mu(E\cap F|F)>0$, it follows that

$$p^{k(\ell)}(E|E\cup F)=\frac{p^{k(\ell)}(E\cap F|E\cup F)}{p^{k(\ell)}(E\cap F|E)}\to\frac{\pi}{p_L(E\cap F)}\text{ and }p^{k(\ell)}(F|E\cup F)=\frac{p^{k(\ell)}(E\cap F|E\cup F)}{p^{k(\ell)}(E\cap F|F)}\to\frac{\pi}{\mu(E\cap F|F)}.$$

Now $p^{k(\ell)}(E|E\cup F)+p^{k(\ell)}(F|E\cup F)-p^{k(\ell)}(E\cap F|E\cup F)=p^{k(\ell)}(E\cup F|E\cup F)=1$, so taking limits as $\ell\to\infty$ yields

$$\frac{\pi}{p_L(E\cap F)}+\frac{\pi}{\mu(E\cap F|F)}-\pi=1\quad\Leftrightarrow\quad\pi=\frac{1}{\frac{1}{p_L(E\cap F)}+\frac{1}{\mu(E\cap F|F)}-1}\in(0,1].$$

To sum up, for any subsequence $(p^{k(\ell)})_{\ell\geq1}$ of $(p^k)_{k\geq1}$ along which $(p^{k(\ell)}(E\cap F|E\cup F))_{\ell\geq1}$ converges, its limit is the same, as it is determined by $p_L(E\cap F)$ and $\mu(E\cap F|F)$. Therefore, also $p^k(E\cap F|E\cup F)\to\pi$. But then, considering the trivial subsequence defined by $k(\ell)=\ell$, also $p^k(E|E\cup F)\to\frac{\pi}{p_L(E\cap F)}>0$ and $p^k(F|E\cup F)\to\frac{\pi}{\mu(E\cap F|F)}>0$. Finally, for any measurable $G\subseteq E\cup F$,

$$p^k(G|E\cup F)=p^k(G\cap E|E\cup F)+p^k(G\cap[F\setminus E]|E\cup F)=p^k(G\cap E|E)\cdot p^k(E|E\cup F)+p^k(G\cap[F\setminus E]|F)\cdot p^k(F|E\cup F)$$

and all the terms in the r.h.s. have well-defined limits.

Hence, $\lim_k p^k(G|E\cup F)$ exists for all measurable $G \subseteq E\cup F$; let $p_{L+1} \equiv \lim_k p^k(\cdot|E\cup F)$. Finally, $p^k(F_1|E\cup F) = p^k(F_1|E)\cdot p^k(E|E\cup F) \to p_L(F_1)\cdot p(E|E\cup F) > 0$, and if $G \subseteq F_\ell$ for some $\ell = 1,\dots,L+1$, then $p^k(G|E\cup F) = p^k(G|F_\ell)\cdot p^k(F_\ell|E\cup F)$ for every k, so $p(G|E\cup F) = \mu(G|F_\ell)\cdot p(F_\ell|E\cup F)$. Thus, p has the requisite properties for step $L+1$.

To complete the proof, suppose instead that $p_L(F) = p_L(F_{L+1}) = 0$, so a fortiori $p_L(E\cap F) = 0$. Then

$$p^k(E\cap F|E\cup F) \le p^k(E\cap F|E) \to p_L(E\cap F) = 0.$$

Suppose that, for some subsequence $(p^{k(\ell)})_{\ell\ge1}$, and $\pi > 0$, $p^{k(\ell)}(F|E\cup F) \ge \pi$ for all ℓ. Then $\liminf_\ell p^{k(\ell)}(E\cap F|E\cup F) = \liminf_\ell p^{k(\ell)}(E\cap F|F)\cdot p^{k(\ell)}(F|E\cup F) \ge \lim_\ell p^{k(\ell)}(E\cap F|F)\cdot\pi = \mu(E\cap F|F)\cdot\pi > 0$, contradiction: thus, $p^k(F|E\cup F) \to 0$. But then $p^k(E|E\cup F) \ge 1 - p^k(F|E\cup F) \to 1$ and, for every measurable $G \subseteq E\cup F$, $p^k(G\cap F|E\cup F) \to 0$ and therefore

$$\begin{aligned} p^k(G|E\cup F) =& p^k(G\cap E|E\cup F) + p^k(G\cap[F\setminus E]|E\cup F) \\ &= p^k(G\cap E|E)\cdot p^k(E|E\cup F) + p^k(G\cap[F\setminus E]|E\cup F) \to p_L(G\cap E) = p_L(G). \end{aligned}$$

In other words, $p_L = \lim_k p^k(\cdot|E\cup F)$. Finally, for $\ell = 1,\dots,L$ and measurable $G \subseteq F_\ell$, by the induction hypothesis $p_L(G) = \mu(G|F_\ell)p_L(F_\ell)$; and for measurable $G \subseteq F_{L+1}$, since $p_L(F_{L+1}) = 0$ by assumption, a fortiori $p_L(G) = 0$ and $\mu(G|F_{L+1})p_{L+1}(F_{L+1}) = \mu(G|F_{L+1})p_L(F_{L+1}) = 0$. Thus, again, the claim holds for $L+1$ by taking $p_{L+1} = p_L$. ■

Corollary 2 *Fix a CCPS $\mu \in \Delta(\Sigma,\mathscr{F})$, a perturbation $(p^k)_{k\ge1}$ of μ, and $F, G \in \mathscr{F}$:*

1. *$p^k(C_\mu(F)) > 0$ for all $k \ge 1$, and $\lim_k p^k(\cdot|C_\mu(F))$ exists and satisfies Eq. (7) for F.*
2. *for all $G \in \cup_m \mathscr{C}^m_\mu(F)$, $\limsup_k \frac{p^k(G)}{p^k(F)} < \infty$; in particular, this holds if $C_\mu(F) \supseteq C_\mu(G)$.*
3. *if $C_\mu(F) \supsetneq C_\mu(G)$, then $\lim_k p^k(C_\mu(G)|C_\mu(F)) = 0$ and $\lim_k \frac{p^k(G)}{p^k(F)} = 0$.*

Proof: 1: By Remark 2, there are $F_1,\dots,F_L \in \mathscr{F}$ such that $\cup_\ell F_\ell = C_\mu(F)$ and, for all $\ell = 2,\dots,L$, $\mu(\cup_{m=1}^{\ell-1} F_m|F_\ell) > 0$. Then, by Lemma 3, $p \equiv \lim_k p^k(\cdot|C_\mu(F))$ is well-defined, $p(F_1) > 0$, $p(C_\mu(F)) = 1$, and for all $E \in \Sigma$ and $\ell = 1,\dots,L$, $E \subseteq F_\ell$ implies $p(E) = \mu(E|F_\ell)p(F_\ell)$. Suppose $G \in \mathscr{F}$ satisfies

$G \subseteq C_\mu(F)$: then $\mu(C_\mu(F)|G) > 0$, so by definition $G \in \mathscr{C}_\mu(F)$ and therefore $G = F_\ell$ for some $\ell = 1, \ldots, L$. Hence, p satisfies Eq. (7).

2: from part 1,

$$\limsup_k \frac{p^k(G)}{p^k(F)} = \limsup_k \frac{p^k(G|C_\mu(F))}{p^k(F|C_\mu(F))} \le \limsup_k \frac{1}{p^k(F|C_\mu(F))} = \frac{1}{\lim_k p^k(F|C_\mu(F))} < \infty$$

because $\lim_k p^k(F|C_\mu(F))$ exists and is strictly positive.

3: from part 1, $p \equiv \lim_k p^k(\cdot|C_\mu(F))$ satisfies Eq. (7) for F, and $C_\mu(F) \supsetneq C_\mu(G)$. By Lemma 2 $p(C_\mu(G)) = 0$, so $\lim_k p^k(C_\mu(G)|C_\mu(F)) = 0$; furthermore, since $p(G) \le p(C_\mu(G)) = 0$ as well,

$$\frac{p^k(G)}{p^k(F)} = \frac{p^k(G|C_\mu(F))}{p^k(F|C_\mu(F))} \to \frac{p(G)}{p(F)} = 0.$$

■

Lemma 4 *Fix an array $\mu \in \Delta(\Sigma)^{\mathscr{F}}$ such that $\mu(F|F) = 1$ for all $F \in \mathscr{F}$. For arbitrary $L \ge 1$ and $F_1, \ldots, F_L \in \mathscr{F}$, define $\pi : \{1, \ldots, L\} \to \mathbb{R}_{++}$ by letting*

$$\pi(\ell) = \Big| m \in \{1, \ldots, L\} : C_\mu(F_m) \supset C_\mu(F_\ell) \Big| + \frac{\ell}{L+1}.$$

Then π is one-to-one and, for all ℓ, m, $C_\mu(F_\ell) \supset C_\mu(F_m)$ implies $\pi(\ell) < \pi(m)$.

Thus, π measures how "surprising" $F_1, \ldots, F_L$ are: the fewer the sets $C_\mu(F_m)$ that strictly contain $C_\mu(F_\ell)$, the less surprising F_ℓ is. Ties are broken as a function of the index ℓ.

Proof: Let $\hat\pi(\ell) = \Big| m \in \{1, \ldots, L\} : C_\mu(F_m) \supset C_\mu(F_\ell) \Big|$, so $\pi(\ell) = \hat\pi(\ell) + \frac{\ell}{L+1}$. Consider $\ell, m \in \{1, \ldots, L\}$ with $\ell \ne m$. If $\hat\pi(\ell) = \hat\pi(m)$, then $\pi(\ell) - \pi(m) = \frac{\ell - m}{L+1} \ne 0$. Otherwise, $|\hat\pi(\ell) - \hat\pi(m)| \ge 1$, and $\frac{\ell-m}{L+1} \in \left(\frac{1-L}{L+1}, \frac{L-1}{L+1}\right) \subset (-1, 1)$. Thus, $\pi(\ell) \ne \pi(m)$, i.e., π is one-to-one.

Now suppose $C_\mu(F_\ell) \supset C_\mu(F_m)$, so $\ell \ne m$. For all $n = 1, \ldots, L$, $C_\mu(F_n) \supset C_\mu(F_\ell)$ implies $C_\mu(F_n) \supset C_\mu(F_m)$, so $\hat\pi(\ell) \le \hat\pi(m)$. In addition, $C_\mu(F_\ell) \supset C_\mu(F_m)$ but $C_\mu(F_\ell) \not\supset C_\mu(F_\ell)$, so $\hat\pi(\ell) < \hat\pi(m)$. As just shown above, $\hat\pi(m) - \hat\pi(\ell) \ge 1$ and $-1 < \frac{\ell - m}{L+1} < 1$, so also $\pi(\ell) < \pi(m)$. ■

Lemma 5 *Fix an array $\mu \in \Delta(\Sigma)^{\mathscr{F}}$ such that $\mu(F|F) = 1$ for all $F \in \mathscr{F}$, events $F_1, \dots, F_L \in \mathscr{F}$, probabilities $P_1, \dots, P_L \in \Delta(\Sigma)$, and a function $\pi : \{1, \dots, L\} \to \mathbb{R}_{++}$. Suppose that*

1. *$C_\mu(F_\ell) \neq C_\mu(F_m)$ for all $\ell \neq m$, and for every $F \in \mathscr{F}$, there is $\ell(F)$ with $C_\mu(F) = C_\mu(F_{\ell(F)})$;*
2. *for every $\ell = 1, \dots, L$, P_ℓ satisfies Eq. (7) for F_ℓ; and*
3. *for all ℓ, m, $C_\mu(F_\ell) \supset C_\mu(F_m)$ implies $\pi(\ell) < \pi(m)$.*

Then the sequence $(p^k)_{k \geq 1}$ defined by

$$p^k(E) = \sum_{\ell=1}^{L} \frac{k^{-\pi(\ell)}}{K_\pi^k} P_\ell(E), \qquad K_\pi^k = \sum_{\ell=1}^{L} k^{-\pi(\ell)}$$

is a perturbation of μ, so μ is a CCPS.

Proof: Since each p^k is a weighted average of $P_1, \dots, P_L$, it belongs to $\Delta(\Sigma)$. Now fix $F \in \mathscr{F}$. By assumption, there is ℓ such that $C_\mu(F) = C_\mu(F_\ell)$. By Lemma 2 part 3, P_ℓ satisfies Eq. (7) for F as well, and in particular $P_\ell(F) > 0$: hence, $p^k(F) > 0$ for all k. It remains to be shown that $\lim_k p^k(\cdot|F)$ exists and equals $\mu(\cdot|F)$.

By Eq. (7), since $F \subseteq C_\mu(F) = C_\mu(F_\ell)$, $\mu(\cdot|F) = P_\ell(\cdot|F)$. Furthermore, consider m such that $P_m(F) > 0$. By Lemma 2 part 1, $C_\mu(F_\ell) \supseteq C_\mu(F_m)$. If $m \neq \ell$, then $C_\mu(F_\ell) \neq C_\mu(F_m)$ by construction, so in fact $C_\mu(F_\ell) \supset C_\mu(F_m)$. Then, by the choice of π, $\pi(\ell) < \pi(m)$. Thus, for all measurable $E \subseteq F$,

$$\begin{aligned} p^k(E|F) &= \frac{\sum_{m:P_m(F)>0} \frac{k^{-\pi(m)}}{K_\pi^k} P_m(E)}{\sum_{m:P_m(F)>0} \frac{k^{-\pi(m)}}{K_\pi^k} P_m(F)} = \frac{\frac{k^{-\pi(\ell)}}{K_\pi^k} P_\ell(E) + \sum_{m\neq\ell:P_m(F)>0} \frac{k^{-\pi(m)}}{K_\pi^k} P_m(E)}{\frac{k^{-\pi(\ell)}}{K_\pi^k} P_\ell(F) + \sum_{m\neq\ell:P_m(F)>0} \frac{k^{-\pi(m)}}{K_\pi^k} P_m(F)} \\ &= \frac{P_\ell(E) + \sum_{m\neq\ell:P_m(F)>0} \frac{k^{-\pi(m)}}{k^{-\pi(\ell)}} P_m(E)}{P_\ell(F) + \sum_{m\neq\ell:P_m(F)>0} \frac{k^{-\pi(m)}}{k^{-\pi(\ell)}} P_m(F)} \to \frac{P_\ell(E)}{P_\ell(F)} = \mu(E|F). \end{aligned}$$

Thus, as claimed, $p^k(\cdot|F) \to \mu(\cdot|F)$. ■

Combining the preceding results:

Proposition 2 *An array $\mu \in \Delta(\Sigma)^{\mathscr{F}}$ such that $\mu(F|F) = 1$ for all $F \in \mathscr{F}$ is a CCPS if and only if, for every $F \in \mathscr{F}$, there is $P(F) \in \Delta(\Sigma)$ that satisfies Eq. (7) for F.*

Proof: If μ is a CCPS and $(p^k)_{k\geq 1}$ is a perturbation of μ, Corollary 2 part 1 implies that, for every $F \in \mathscr{F}$, $\lim_k p^k(\cdot|C_\mu(F))$ exists and satisfies Eq. (7) for F. Conversely, if the condition in the Proposition holds, fix a collection $F_1,\ldots,F_L \in \mathscr{F}$ that satisfies condition 1 of Lemma 5, and define π as in Lemma 4. By construction, π satisfies condition 3 of Lemma 5, and by assumption the probabilities $P(F_1),\ldots,P(F_L)$ satisfy condition 2 of that Lemma. Therefore, Lemma 5 implies that μ is a CCPS. ■

Henceforth, if μ is a CCPS, for every $F \in \mathscr{F}$, denote by $P_\mu(F)$ the (unique, by Lemma 1) measure that satisfies Eq. (7) for F.

The following result covers the representation of $\succ$ on both $\mathscr{A}_0$ and $\mathscr{A}_b$ (cf. 5).

Proposition 3 *Fix a CCPS $\mu \in \Delta(\Sigma,\mathscr{F})$ and $u : X \to \mathbb{R}$. For $f,g \in \mathscr{A}_b$, $\mathrm{E}_{p^k}[u\circ f - u\circ g] > 0$ eventually for all perturbations $(p^k)_{k\geq 1}$ of μ iff there exist $L \geq 1$ and $F_1,\ldots,F_L \in \mathscr{F}$ such that $\mathrm{E}_{P_\mu(F_\ell)} u\circ f > \mathrm{E}_{P_\mu(F_\ell)} u\circ g$ for all $\ell = 1,\ldots,L$, and $u(f(\omega)) \geq u(g(\omega)$ for all $\omega \notin \cup_{\ell=1}^L C_\mu(F_\ell)$.*

Proof: (If:) Suppose that there exist $L \geq 1$ and $F_1,\ldots,F_L \in \mathscr{F}$ as in the statement, and fix a perturbation $(p^k)_{k\geq 1}$ of μ. It is wlog to assume that that $C_\mu(F_m) \supseteq C_\mu(F_\ell)$ for no $m,\ell \in \{1,\ldots,L\}$ with $m \neq \ell$. Let $D_1 = C_\mu(F_1)$ and, inductively, $D_\ell = C_\mu(F_\ell) \setminus \cup_{m=1}^{\ell-1} D_m$ for $\ell = 2,\ldots,L$. Then $D_1,\ldots,D_L$ are pairwise disjoint, and $\cup_\ell D_\ell = \cup_\ell C_\mu(F_\ell)$.

By Corollary 2 part 1, for every ℓ, $p^k(\cdot|C_\mu(F_\ell)) \to P_\mu(F_\ell)$. By definition, $P_\mu(F_1)(D_1) = P_\mu(F_1)(C_\mu(F_1)) = 1$. Moreover, for $\ell = 2,\ldots,L$ and $m = 1,\ldots,\ell-1$, if $P_\mu(F_\ell)(C_\mu(F_m)) > 0$, then $C_\mu(F_m) \supseteq C_\mu(F_\ell)$ by Lemma 2 part 1, which contradicts the choice of $F_1,\ldots,F_L$: thus, $P_\mu(F_\ell)(\cup_{m=1}^{\ell-1} C_\mu(F_m)) = 0$, which implies that $P_\mu(F_\ell)(D_\ell) = 1$.

It follows that, for all $\ell = 1,\ldots,L$, eventually $p^k(D_\ell|C_\mu(F_\ell)) > 0$, and a fortiori $p^k(D_\ell) > 0$. Then, eventually, for all measurable $E \subseteq D_\ell$,

$$p^k(E|D_\ell) = \frac{p^k(E)}{p^k(D_\ell)} = \frac{p^k(E|C_\mu(F_\ell))}{p^k(D_\ell|C_\mu(F_\ell))} \to \frac{P_\mu(F_\ell)(E)}{P_\mu(F_\ell)(D_\ell)} = P_\mu(F_\ell)(E).$$

Thus, $\lim_k p^k(\cdot|D_\ell) = P_\mu(F_\ell)$. Hence, $\int \phi\, d p^k(\cdot|D_\ell) \to \int \phi\, d P_\mu(F_\ell)$ for all $\phi \in B(\Sigma)$. In particular, this holds for $\phi = u \circ f$ and $\phi = u \circ g$, because $f, g \in \mathscr{A}_b(\succ)$. Since $\mathrm{E}_{P_\mu(F_\ell)} u \circ f > \mathrm{E}_{P_\mu(F_\ell)} u \circ g$ for all $\ell = 1, \ldots, L$, eventually also $\mathrm{E}_{p^k(\cdot|D_\ell)} u \circ f > \mathrm{E}_{p^k(\cdot|D_\ell)} u \circ g$ for all $\ell = 1, \ldots, L$. Since in addition $u(f(\omega)) \geq u(g(\omega))$ for all $\omega \notin \cup_\ell C_\mu(F_\ell)$, eventually

$$\begin{aligned}\mathrm{E}_{p^k} u \circ f &= \sum_{\ell=1}^{L} p^k(D_\ell) \mathrm{E}_{p^k(\cdot|D_\ell)} u \circ f + \int_{\Omega \setminus \cup_\ell C_\mu(F_\ell)} u \circ f \, dp^k \\ &> \sum_{\ell=1}^{L} p^k(D_\ell) \mathrm{E}_{p^k(\cdot|D_\ell)} u \circ g + \int_{\Omega \setminus \cup_\ell C_\mu(F_\ell)} u \circ g \, dp^k = \mathrm{E}_{p^k} u \circ g.\end{aligned}$$

(Only If): Suppose that $\mathrm{E}_{p^k} u \circ f > \mathrm{E}_{p^k} u \circ g$ eventually for all perturbations $(p^k)_{k \geq 1}$ of μ. Let π be as in Lemma 4, and let $F_1, \ldots, F_L$ satisfy condition 1 in Lemma 5. Recall that $P_\mu(F_1), \ldots, P_\mu(F_L)$ satisfy condition 2 in that Lemma, and $F_1, \ldots, F_L$ and π jointly satisfy condition 3 therein.

I prove that (i) there is at least one $F \in \mathscr{F}$ such that $\mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g] > 0$, and (ii) for all $\omega \in \Omega$, if $u(f(\omega)) < u(g(\omega))$, then there is $F \in \mathscr{F}$ such that $\mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g] > 0$ and $\omega \in C_\mu(F)$. If so, the subcollection $\{F_\ell : \mathrm{E}_{P_\mu(F_\ell)}[u \circ f - u \circ g] > 0\}$ satisfies the conditions of statement.

If (i) does not hold, then in particular $\mathrm{E}_{P_\mu(F_\ell)}[u \circ f - u \circ g] \leq 0$ for all $\ell = 1, \ldots, L$. Let $(p^k)_{k \geq 1}$ be the perturbation of μ constructed in Lemma 5, with π as in Lemma 4: then

$$\mathrm{E}_{p^k} u \circ f = \sum_{\ell=1}^{L} \frac{k^{-\pi(\ell)}}{K_\pi^k} \mathrm{E}_{P_\mu(F_\ell)} u \circ f \leq \sum_{\ell=1}^{L} \frac{k^{-\pi(\ell)}}{K_\pi^k} \mathrm{E}_{P_\mu(F_\ell)} u \circ g = \mathrm{E}_{p^k} u \circ g,$$

for all k, contradiction. Thus, (i) holds.

Now consider (ii) and let $\omega \in \Omega$ be such that $u(f(\omega)) < u(g(\omega))$. Let $\mathscr{L}_\omega = \{\ell \in \{1, \ldots, L\} : \omega \in C_\mu(F_\ell)\}$. Then $\mathscr{L}_\omega \neq \emptyset$, because $\omega \in \Omega = C_\mu(\Omega)$ and by assumption there is $\ell \in \{1, \ldots, L\}$ with $C_\mu(F_\ell) = C_\mu(\Omega)$. By contradiction, assume that $\mathrm{E}_{P_\mu(F_\ell)}[u \circ f - u \circ g] \leq 0$ for all $\ell \in \mathscr{L}_\omega$.

Let $\ell^* = |\mathscr{L}_\omega|$, denote by δ_ω the Dirac measure concentrated on $\{\omega\}$, and let π be as in Lemma 4. Define $\rho : \{1, \ldots, L\} \to \mathbb{R}$ by letting $\rho(\ell) = \pi(\ell)$ if $\ell \in \mathscr{L}_\omega$, and $\rho(\ell) = \pi(\ell) + \Lambda$ otherwise, where $\Lambda = 2 + \max_{\ell \in \mathscr{L}_\omega} \pi(\ell)$.[9]

[9]That is, make the events F_ℓ for which $\omega \in C_\mu(F_\ell)$ the least implausible.

The function ρ is one-to-one: if $\ell \neq m$ and either $\ell, m \in \mathscr{L}_\omega$ or $\ell, m \notin \mathscr{L}_\omega$, then $\rho(\ell)-\rho(m) = \pi(\ell)-\pi(m) \neq 0$; if instead wlog $\ell \in \mathscr{L}_\omega$ and $m \notin \mathscr{L}_\omega$, then $\rho(\ell) \leq \max_{\ell' \in \mathscr{L}_\omega} \pi(\ell') < \Lambda < \rho(m)$. Moreover, suppose $C_\mu(F_\ell) \supseteq C_\mu(F_m)$. Again, if either $\ell, m \in \mathscr{L}_\omega$ or $\ell, m \notin \mathscr{L}_\omega$, then $\rho(m)-\rho(\ell) = \pi(m)-\pi(\ell) > 0$. If $\ell \in \mathscr{L}_\omega$ and $m \notin \mathscr{L}_\omega$, then it was just shown that $\rho(\ell) < \rho(m)$. Finally, it cannot be that $\ell \notin \mathscr{L}_\omega$ and $m \in \mathscr{L}_\omega$, because $\omega \in C_\mu(F_m) \subset C_\mu(F_\ell)$ implies $\ell \in \mathscr{L}_\omega$ as well.

Then the sequence $(p^k)_{k\geq 1}$ constructed given $F_1, \ldots, F_L$, $P_\mu(F_1), \ldots, P_\mu(F_L)$, and ρ in Lemma 5 is a perturbation of μ. Define a further sequence $(q^k)_{k\geq 1}$ by

$$q^k = \frac{1}{1+k^{-(\Lambda-1)}} p^k + \frac{k^{-(\Lambda-1)}}{1+k^{-(\Lambda-1)}} \delta_\omega,$$

where as usual δ_ω is the Dirac measure concentrated on ω. Then $(q^k)_{k\geq 1}$ is also a perturbation of μ. First, for every $F \in \mathscr{F}$, $p^k(F) > 0$ and so $q^k(F) > 0$ for all k. Second, fix $G \in \mathscr{F}$ and a measurable $E \subseteq G$; if $\omega \notin G$, $\delta_\omega(G) = \delta_\omega(E) = 0$, so $q^k(E|G) = p^k(E|G) \to \mu(E|G)$. Otherwise,

$$q^k(E|G) = \frac{p^k(E) + k^{-(\Lambda-1)} \delta_\omega(E)}{p^k(G) + k^{-(\Lambda-1)}} = \frac{p^k(E|G) + \frac{k^{-(\Lambda-1)}}{p^k(G)} \delta_\omega(E)}{1 + \frac{k^{-(\Lambda-1)}}{p^k(G)}}.$$

I claim that $\frac{k^{-(\Lambda-1)}}{p^k(G)} \to 0$, which implies that $q^k(E|G) \to \mu(E|G)$ because $p^k(E|G) \to \mu(E|G)$. Let ℓ be the unique index such that $C_\mu(G) = C_\mu(F_\ell)$. Since $\omega \in G$, $\ell \in \mathscr{L}_\omega$, which implies that $\rho(\ell) = \pi(\ell) \leq \max_{m \in \mathscr{L}_\omega} \pi(m) = \Lambda - 2$; thus,

$$p^k(G) = \frac{k^{-\rho(\ell)}}{K_\rho^k} P_\mu(F_\ell)(G) + \sum_{m \neq \ell : P_\mu(F_m)(G) > 0} \frac{k^{-\rho(m)}}{K_\rho^k} P_\mu(F_m)(G) \geq \frac{k^{-\rho(\ell)}}{K_\rho^k} P_\mu(F_\ell)(G)$$

which implies that

$$\frac{k^{-(\Lambda-1)}}{p^k(G)} \leq \frac{k^{-(\Lambda-1)}}{k^{-\rho(\ell)}} \cdot \frac{K_\rho^k}{P_\mu(F_\ell)(G)} \leq k^{-1} \cdot \frac{K_\rho^k}{P_\mu(F_\ell)(G)} \to 0$$

where the first inequality follows from the lower bound on $p^k(G)$ just derived, and the second from $\rho(\ell) \leq \Lambda - 2 < \Lambda - 1$; the limiting step follows from $K_\rho^k = \sum_\ell k^{-\rho(\ell)} \to 0$ and $k^{-1} \to 0$.

Finally,

$$
\begin{aligned}
\mathrm{E}_{q^k}[u\circ f-u\circ g]=&\frac{1}{1+k^{-(\Lambda-1)}}\mathrm{E}_{p^k}[u\circ f-u\circ g]+\frac{k^{-(\Lambda-1)}}{1+k^{-(\Lambda-1)}}[u(f(\omega))-u(g(\omega))]\\
=&\frac{1}{1+k^{-(\Lambda-1)}}\sum_{\ell\in\mathscr{L}_\omega}\frac{k^{-\rho(\ell)}}{K_\rho^k}\mathrm{E}_{P_\mu(F_\ell)}[u\circ f-u\circ g]\\
+&\frac{1}{1+k^{-(\Lambda-1)}}\sum_{\ell\notin\mathscr{L}_\omega}\frac{k^{-\rho(\ell)}}{K_\rho^k}\mathrm{E}_{P_\mu(F_\ell)}[u\circ f-u\circ g]+\frac{k^{-(\Lambda-1)}}{1+k^{-(\Lambda-1)}}[u(f(\omega))-u(g(\omega))]\\
\le&\frac{1}{1+k^{-(\Lambda-1)}}\sum_{\ell\notin\mathscr{L}_\omega}\frac{k^{-\rho(\ell)}}{K_\rho^k}\mathrm{E}_{P_\mu(F_\ell)}[u\circ f-u\circ g]+\frac{k^{-(\Lambda-1)}}{1+k^{-(\Lambda-1)}}[u(f(\omega))-u(g(\omega))]\\
=&\frac{k^{-(\Lambda-1)}}{1+k^{-(\Lambda-1)}}\left(\sum_{\ell\notin\mathscr{L}_\omega}\frac{k^{-[\rho(\ell)-(\Lambda-1)]}}{K_\rho^k}\mathrm{E}_{P_\mu(F_\ell)}[u\circ f-u\circ g]+[u(f(\omega))-u(g(\omega))]\right)
\end{aligned}
$$

where the inequality follows from the assumption that $\mathrm{E}_{P_\mu(F_\ell)}u\circ f\le \mathrm{E}_{P_\mu(F_\ell)}u\circ g$ for all $\ell\in\mathscr{L}_\omega$.

I claim that, eventually, the term in parentheses is negative. Since by assumption $u(f(\omega))<u(g(\omega))$, it is enough to show that, for every $\ell\notin\mathscr{L}_\omega$, $\frac{k^{-[\rho(\ell)-(\Lambda-1)]}}{K_\rho^k}\to 0$. To this end, recall that, for every $\ell\notin\mathscr{L}_\omega$, $\rho(\ell)=\pi(\ell)+\Lambda>\Lambda$ because π is strictly positive. Furthermore, let n be such that $C_\mu(F_n)=C_\mu(\Omega)=\Omega$, which must exist by assumption. There is no m such that $C_\mu(F_m)\supset C_\mu(F_n)=\Omega$, so $\pi(n)=\frac{n}{L+1}<1$. Furthermore, $\omega\in\Omega=C_\mu(F_n)$, so $n\in\mathscr{L}_\omega$, and hence $\rho(n)=\frac{n}{L+1}<1$ as well. Finally, by definition $K_\rho^k=\sum_\ell k^{-\rho(\ell)}\ge k^{-\rho(n)}$, so

$$\frac{k^{-[\rho(\ell)-(\Lambda-1)]}}{K_\rho^k}\le\frac{k^{-[1+(\rho(\ell)-\Lambda)]}}{k^{-\rho(n)}}=k^{-[1+(\rho(\ell)-\Lambda)-\rho(n)]}\le k^{-[1-\rho(n)]}\to 0.$$

Therefore, $(q^k)_{k\ge1}$ is a perturbation of μ for which it is not the case that $\mathrm{E}_{q^k}u\circ f>\mathrm{E}_{q^k}u\circ g$ eventually: contradiction. Thus, (ii) above must hold. ■

B Characterization: necessity

Fix a CCPS $\mu\in\Delta(\Sigma,\mathscr{F})$ and a non-constant, affine $u:X\to\mathbb{R}$; let $\succ$ on $\mathscr{A}=\mathscr{A}_0$ or $\mathscr{A}=\mathscr{A}_b$ be the structural preference represented by (u,μ). By Proposition 2, for every $F\in\mathscr{F}$ there exists a unique measure $P_\mu(F)$ that satisfies Eq. (7) for F.

Lemma 6 *For every $F \in \mathscr{F}$ and $x, y \in X$, $x \succ_F y$ iff $u(x) > u(y)$. Hence $x \succ_F y$ iff $x \succ y$.*

Proof: Fix $F \in \mathscr{F}$. Suppose that $u(x) > u(y)$ for $x, y \in X$. Let $(p^k)_{k\geq 1}$ be a perturbation of μ. Then $p^k(F) > 0$ for all k, which implies that, for all k, $\mathrm{E}_{p^k} u \circ [xFy] > \mathrm{E}_{p^k} u(y) = u(y)$. Hence, $xFy \succ y$, i.e., $x \succ_F y$. Conversely, if $x \succ_F y$, i.e., $xFy \succ y$, then eventually $\mathrm{E}_{p^k} u \circ [xFy] > \mathrm{E}_{p^k} u(y) = u(y)$; since $u \circ [xFy](\omega) = u(y)$ for $\omega \notin F$, it must be the case that $u(x) > u(y)$. To sum up, for all $x, y \in X$, $x \succ_F y$ iff $u(x) > u(y)$. ■

Lemma 7 *Consider $N \in \Sigma$ and $F \in \mathscr{F}$. Then N is implausible given F iff $\mu(N|F) = 0$; and it is implausible given $C_\mu(F)$ iff $P_\mu(F)(N) = 0$.*

Proof: I prove the first claim. To prove the second claim, replace F with $C_\mu(F)$ and $\mu(\cdot|F)$ with $P_\mu(F)(\cdot)$; in particular, recall that, by Corollary 2 part 1, if (p^k) is a perturbation of μ, then $p^k(C_\mu(F)) > 0$ for all k, and $\lim_k p^k(\cdot|C_\mu(F)) = P_\mu(F)$.

Suppose $\mu(N|F) = 0$, so $\mu(F \setminus N|F) = 1$. Consider $x, y, z \in X$ with $y \succ_F z$, so by Lemma 6, $u(y) > u(z)$. Let $(p^k)_{k\geq 1}$ be a perturbation of μ. Then $u(y) > u(z) = \mathrm{E}_{\mu(\cdot|F)} u \circ [xNz] = \mathrm{E}_{\mu(\cdot|F)} u \circ [x(F \cap N)z]$, and therefore eventually $u(y) > \mathrm{E}_{p^k(\cdot|F)} u \circ [x(F \cap N)z]$. Since $p^k(F) > 0$ for all k, and $[yFz](\omega) = z = [x(F \cap N)z](\omega)$ for $\omega \notin F$, $\mathrm{E}_{p^k} u \circ [yFz] > \mathrm{E}_{p^k} u \circ [x(F \cap N)z]$ eventually. Since (p^k) was arbitrary, $yFz \succ x(F \cap N)z$, so by Definition 3, $y \succ_F xNz$. Since $x, y, z \in X$ were arbitrary with $y \succ_F z$, N is implausible given F.

For the converse, it is enough to consider the case $N \subseteq F$: for general $N \in \Sigma$, $\mu(N|F) = \mu(N \cap F|F)$, and $N \cap F \subseteq F$.

Suppose that $\mu(N|F) > 0$. Since u is non-constant, there are $x, z \in X$ such that $u(x) > u(z)$ (so by Lemma 6 $x \succ_F z$ and $x \succ z$). Let $\alpha \in \big(0, \mu(N|F)\big)$ and $y = \alpha x + (1-\alpha)z$: thus, $x \succ_F y \succ_F z$. Furthermore,

$$u(y) = \alpha u(x) + (1-\alpha)u(z) < \mu(N|F)u(x) + [1-\mu(N|F)]u(z) = \mathrm{E}_{\mu(\cdot|F)} u \circ [x(N \cap F)z].$$

Hence, for any perturbation $(p^k)_{k\geq 1}$ of μ, eventually $u(y) < \mathrm{E}_{p^k(\cdot|F)} u\circ[x(N\cap F)z]$, and therefore, since $p^k(F) > 0$, also $\mathrm{E}_{p^k} u\circ[yFz] < \mathrm{E}_{p^k} u\circ[x(N\cap F)z]$. By contradiction, if N is implausible given F, then $y \succ_F xNz$; by Definition 3, this is equivalent to $yFz \succ xNz$. But then, for all large enough k, $\mathrm{E}_{p^k} u\circ[yFz] > \mathrm{E}_{p^k} u\circ[x(N\cap F)z]$, contradiction. Thus, N is plausible given F. ■

It follows that, for every $F \in \mathscr{F}$, $\mathrm{imp}(F) = \cup\{G \in \mathscr{F} : \mu(G|F) = 0\}$, and $\mathrm{imp}(C_\mu(F)) = \cup\{G \in \mathscr{F} : P_\mu(F)(G) = 0\}$. To simplify notation, let $\sigma(E) = E \setminus \mathrm{imp}(E)$ for any event $E \in \Sigma$.

The following observation follows from the fact that $\mathrm{imp}(C_\mu(F))$ is the union of finitely many events G with $P_\mu(F)(G) = 0$.

Observation 3 For $F \in \mathscr{F}$, $P_\mu(F)(\mathrm{imp}(C_\mu(F))) = 0$ and thus $P_\mu(F)(\sigma(C_\mu(F))) = 1$.

Lemma 8 *For every $F \in \mathscr{F}$, $\succ_{\sigma(F)}$ is an EU preference on $\mathscr{A}$, represented by u and $\mu(\cdot|F)$.*

Proof: Consider two acts $f, g \in \mathscr{A}$. It must be shown that $f \succ_{\sigma(F)} g$ iff $\mathrm{E}_{\mu(\cdot|F)}[u\circ f - u\circ g] > 0$. By Lemma 7, N is implausible given F iff $\mu(N|F) = 0$. Hence $F \setminus \sigma(F)$ is implausible given F, and $\mu(\sigma(F)|F) = 1$. Thus, it is enough to show that $f \succ_{\sigma(F)} g$ iff $\mathrm{E}_{\mu(\cdot|F)}[u\circ f\sigma(F)g - u\circ g] > 0$.

Suppose first $\mathrm{E}_{\mu(\cdot|F)}[u\circ f\sigma(F)g - u\circ g] > 0$. Then, for all perturbations $(p^k)_{k\geq 1}$ of μ, since $p^k(\cdot|F) \to \mu(\cdot|F)$, eventually $\mathrm{E}_{p^k(\cdot|F)}[u\circ f\sigma(F)g - u\circ g] > 0$ and $p^k(\sigma(F)|F) \to 1$; furthermore, since $p^k(F) > 0$ for all k, $p^k(\sigma(F)) = p^k(F)\cdot p^k(\sigma(F)|F)$ is eventually positive, so eventually $\mathrm{E}_{p^k}[u\circ f\sigma(F)g - u\circ g] > 0$. Hence, $f\sigma(F)g \succ g$, i.e., $f \succ_{\sigma(F)} g$.

Conversely, suppose that $f \succ_{\sigma(F)} g$, or $f\sigma(F)g \succ g$. To simplify the argument, use the characterization in Proposition 3 (which applies to $\mathscr{A} = \mathscr{A}_b$ as well). Since $f\sigma(F)g \succ g$, by Proposition 3 there is $G \in \mathscr{F}$ such that $\mathrm{E}_{P_\mu(G)}[u\circ f\sigma(F)g - u\circ g] > 0$. Since $[f\sigma(F)g](\omega) = g(\omega)$ for $\omega \notin \sigma(F)$, $P_\mu(G)(\sigma(F)) > 0$; therefore, there is $G' \in \cup_m \mathscr{C}^m(G)$ such that $P_\mu(G)(G'\cap\sigma(F)) > 0$, because by Eq. (7) $P_\mu(G)(\cup_m \mathscr{C}^m(G)) = 1$ and $\cup_m \mathscr{C}^m(G)$ is finite. Hence $G' \cap \sigma(F) \neq \emptyset$. Since $\sigma(F) = F \setminus \mathrm{imp}(F)$, G' cannot be implausible given F, because otherwise $G' \subseteq \mathrm{imp}(F)$. Hence

$\mu(G'|F) > 0$ by Lemma 7. Thus $F \in \mathscr{C}^1(G')$ and so, by Observation 2 part 1, $C_\mu(G') \supseteq C_\mu(F)$. By the same Observation, $G' \in \cup_m \mathscr{C}^m(G)$ implies $C_\mu(G) \supseteq C_\mu(G')$, so $C_\mu(G) \supseteq C_\mu(F)$. But $P_\mu(G)(\sigma(F)) > 0$ implies $P_\mu(G)(F) > 0$, and so, by Lemma 2 part 2, $C_\mu(F) \supseteq C_\mu(G)$. Thus, $C_\mu(G) = C_\mu(F)$ and so $P_\mu(G) = P_\mu(F)$, hence $\mathrm{E}_{P_\mu(F)}[u \circ f\sigma(F)g - u \circ g] > 0$. But then, since $P_\mu(F)(F) > 0$ and $[f\sigma(F)g](\omega) = g(\omega)$ for $\omega \notin F$, we must have $\mathrm{E}_{P_\mu(F)(\cdot|F)}[u \circ f\sigma(F)g - u \circ g] > 0$. Finally, $P_\mu(F)(\cdot|F) = \mu(\cdot|F)$ by Eq. (3), so $\mathrm{E}_{\mu(\cdot|F)}[u \circ f\sigma(F)g - u \circ g] > 0$. ■

It is now possible to **verify that Axioms 1–9b hold** on $\mathscr{A} = \mathscr{A}_0$ or $\mathscr{A} = \mathscr{A}_b$. The following simple restatement of Proposition 3 is convenient and will be used below.

Observation 4 Let $\succ^{u,\mu}$ be the structural preference on $\mathscr{A} = \mathscr{A}_0$ or $\mathscr{A} = \mathscr{A}_b$ represented by (u,μ). Then, for all $f, g \in \mathscr{A}$, $f \succ^{u,\mu} g$ iff (1) there is $F^* \in \mathscr{F}$ with $\int u \circ f \, dP_\mu(F^*) \neq \int u \circ g \, dP_\mu(F^*)$, and (2) for every $\omega \in \Omega$ with $u(f(\omega)) < u(g(\omega))$, there is $F \in \mathscr{F}$ such that $\omega \in C_\mu(F)$ and $\int u \circ f \, dP_\mu(F) > \int u \circ g \, dP_\mu(F)$.

Axioms 1 (Strict Partial Order), 2 (Prize Negative Transitivity), 3 (Independence) and Axiom 4 (Monotonicity) are immediate from Definition 2.

Axioms 5 (Non-degeneracy) and 6 (Prize continuity) follow from Lemma 6, because u is a non-constant, affine utility function.

For Axioms 7 and 8, Lemma 8 shows that, for every $F \in \mathscr{F}$, $\succ_{\sigma(F)}$ is an EU preference relation; hence, it is negatively transitive and Archimedean, so the Axioms hold.

Finally, by Remark 3 and Lemma 7, $C_\mu(F) = \cup\mathscr{F}(F)$; thus, by Remark 2, if $\mathscr{F}$ is laminar, then $\cup\mathscr{F}(F) \in \mathscr{F}$. Hence, even in the case of $\mathscr{F}$ laminar, it is enough to verify that $\succ$ satisfies Axiom 9b. Suppose that $f \succ g$. By Observation 4, there is $F \in \mathscr{F}$ with $\mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g] > 0$; furthermore, for any ω with $f(\omega) \prec g(\omega)$, there is one such F for which, in addition, $\omega \in C_\mu(F)$. Finally, for $x, y \in X$ such that $0 < u(y) - u(x) < \mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g]$, $\mathrm{E}_{P_\mu(F)}[u \circ (\frac{1}{2}f + \frac{1}{2}x) - u \circ (\frac{1}{2}g + \frac{1}{2}y)] > 0$. As shown above, $C_\mu(F) = \cup\mathscr{F}(F)$, and by Lemma 8 part 1, $\frac{1}{2}f + \frac{1}{2}x \succ_{C_\mu(F)} \frac{1}{2}g + \frac{1}{2}y$.

Thus Axiom 9b holds.

C Sufficiency: Preliminaries

Now consider a preference $\succ$ on $\mathscr{A} = \mathscr{A}_0$. Begin by proving Remark 1; the proof is essentially standard, but included for completeness.

Proof: Fix $f, g, k, k', E \in \mathscr{A}_0$ as in the Remark. By Independence (Axiom 3),

$$fEk \succ gEk \quad \Leftrightarrow \quad \frac{1}{2}fEk + \frac{1}{2}kEk' \succ \frac{1}{2}gEk + \frac{1}{2}kEk',$$

and similarly

$$fEk' \succ gEk' \quad \Leftrightarrow \quad \frac{1}{2}fEk' + \frac{1}{2}k \succ \frac{1}{2}gEk' + \frac{1}{2}k.$$

Now observe that $\frac{1}{2}fEk + \frac{1}{2}kEk' = \frac{1}{2}fEk' + \frac{1}{2}k$: in every state $\omega \in E$

$$\frac{1}{2}fEk(\omega) + \frac{1}{2}kEk'(\omega) = \frac{1}{2}f(\omega) + \frac{1}{2}k(\omega) = \frac{1}{2}fEk'(\omega) + \frac{1}{2}k(\omega),$$

and in every state $\omega \notin E$

$$\frac{1}{2}fEk(\omega) + \frac{1}{2}kEk'(\omega) = \frac{1}{2}k(\omega) + \frac{1}{2}k'(\omega) = \frac{1}{2}k'(\omega) + \frac{1}{2}k(\omega) = \frac{1}{2}fEk'(\omega) + \frac{1}{2}k(\omega).$$

Similarly $\frac{1}{2}gEk + \frac{1}{2}kEk' = \frac{1}{2}gEk' + \frac{1}{2}k$. The claim follows. ■

Next, I relate prior and conditional preferences over X. For $E \in \Sigma$, say that $\succ_E$ is **non-degenerate** if there exist $x', y' \in X$ with $x' \succ_E y'$.

Lemma 9 *Assume Axioms 1–4 and 6. Consider $E \in \Sigma$ such that $\succ_E$ is non-degenerate. Then, for every $x, y \in X$, $x \succ_E y$ if and only if $x \succ y$.*

Proof: Let x', y' be such that $x' \succ_E y'$. Fix $f \in \mathscr{A}_0$. Then $x \succ_E y$ iff $xEf \succ yEf$ for any $x, y \in X$, including x', y'.

Suppose $x \succ_E y$. If not $x \succ y$, then $xEf \succ yEf$ and not $x \succ y$, so Axiom 4 (Monotonicity) implies $xEf \succ xEf$, which contradicts Irreflexivity in Axiom 1. Thus, $x \succ y$.

Conversely, suppose $x \succ y$. By assumption $x'Ef \succ y'Ef$.

Case 1: not $x' \succ x$. Then Axiom 4 (Monotonicity) yields $xEf \succ y'Ef$. If also not $y \succ y'$, then the same Axiom yields $xEf \succ yEf$. Otherwise, $y \succ y'$. By Axiom 6 (Prize Continuity), there is $\alpha \in (0,1)$ such that $\alpha x + (1-\alpha)y' \succ y$. Then by Axiom 3 (Independence), $xEf \succ y'Ef$ implies $xEf = \alpha xEf + (1-\alpha)xEf \succ \alpha xEf + (1-\alpha)y'Ef = [\alpha x + (1-\alpha)y')]Ef$, and since Irreflexivity in Axiom 1 implies not $y \succ \alpha x + (1-\alpha)y'$, Axiom 4 implies $xEf \succ yEf$.

Case 2a: $x' \succ x$ and not $y \succ y'$. Then Axiom 4 (Monotonicity) yields $x'Ef \succ yEf$. Furthermore, by Axiom 6 (Prize Continuity), $x' \succ x$ and $x \succ y$ imply that there is $\alpha \in (0,1)$ such that $x \succ \alpha x' + (1-\alpha)y$. From Axiom 3, $x'Ef \succ yEf$ implies $[\alpha x' + (1-\alpha)y]Ef \succ yEf$. Irreflexivity implies not $\alpha x' + (1-\alpha)y \succ x$, so Axiom 4 finally implies $xEf \succ yEf$.

Case 2b: $x' \succ x$ and $y \succ y'$. By standard arguments, there is α such that not $\alpha x' + (1-\alpha)y' \succ y$ and not $y \succ \alpha x' + (1-\alpha)y'$.[10] Since $x'Ef \succ y'Ef$ by assumption, Axiom 3 (Independence) implies that $x'Ef = \alpha x'Ef + (1-\alpha)x'Ef \succ \alpha x'Ef + (1-\alpha)y'Ef = [\alpha x' + (1-\alpha)y']Ef$; since not $y \succ \alpha x' + (1-\alpha)y'$, Axiom 4 yields $x'Ef \succ yEf$. Similarly, by standard arguments there is $\beta \in (0,1)$ such that not $\beta x' + (1-\beta)y \succ x$ and not $x \succ \beta x' + (1-\beta)y$. By Axiom 3 (Independence), $x'Ef \succ yEf$ implies $[\beta x' + (1-\beta)y]Ef \succ yEf$, and since not $\beta x' + (1-\beta)y) \succ x$, Axiom 4 yields $xEf \succ yEf$. ■

[10]For every $\lambda, \gamma \in [0,1]$, Independence and $x' \succ y'$ imply that $\lambda x' + (1-\lambda)[\gamma x' + (1-\gamma)y'] \succ \lambda y' + (1-\lambda)[\gamma x' + (1-\gamma)y']$. If $1 \geq \alpha > \beta \geq 0$, then letting $\lambda = \alpha - \beta$ and $\gamma = \frac{\beta}{1-\alpha+\beta}$ yields $\alpha x' + (1-\alpha)y' \succ \beta x' + (1-\beta)y'$ (if $\alpha = 1$ and $\beta = 0$ then γ is not well-defined, but the conclusion still holds trivially). Now let $U = \{\beta : \beta x' + (1-\beta)y' \succ y\}$ and $L = \{y \succ \beta x' + (1-\beta)y'\}$. Clearly $1 \in U$ and $0 \in L$. If $\beta \in U$ and $1 \geq \beta' > \beta$, then $\beta' \in U$; similarly, $\alpha \in L$ and $\alpha' \in [0, \alpha)$ imply $\alpha' \in L$. By Irreflexivity, $U \cap L = \emptyset$. Let $\alpha = \inf U$. If $\alpha \in U$, then $\alpha x' + (1-\alpha)y' \succ y \succ y'$ and Axiom 6 (Prize Continuity) imply that there is $\lambda \in (0,1)$ with $\lambda[\alpha x' + (1-\alpha)y'] + (1-\lambda)y' \succ y$, so $\lambda\alpha \in U$, which contradicts the definition of α. If $\alpha \in L$, then $x' \succ y \succ \alpha x' + (1-\alpha)y'$ and Axiom 6 imply that there is $\lambda \in (0,1)$ with $y \succ \lambda x' + (1-\lambda)[\alpha x' + (1-\alpha)y']$, so $\lambda + (1-\lambda)\alpha \in L$; but $\lambda + (1-\lambda)\alpha > \alpha$, so there is $\gamma \in (\alpha, \lambda + (1-\lambda)\alpha)$ with $\gamma \in U$. But $\lambda + (1-\lambda)\alpha \in L$ and $\lambda + (1-\lambda)\alpha > \gamma$ imply $\gamma \in L$, contradiction. Thus, $\alpha \notin U \cup L$, as required.

Observation 5 (State Independence) Assume Axioms 1–4 and 6. If E is not Savage-null for $\succ$, then for all $x, y \in X$ and $f \in \mathscr{A}_0$, $x \succ y$ if and only if $xEf \succ yEf$.

Proof: If E is not Savage-null for $\succ$, there must be $f, g \in \mathscr{A}_0$ such that $f(\omega) = g(\omega)$ for $\omega \notin E$ and $f \succ g$. Let $x', y' \in X$ be such that not $f(\omega) \succ x'$ and not $y' \succ g(\omega)$ for all $\omega \in E$. Then Axiom 4, invoked twice, implies that $x'Ef \succ y'Eg = y'Ef$. By Definition 3, $x' \succ_E y'$. Hence Lemma 9 implies that $x \succ y$ iff $x \succ_E y$; by Definition 3, this is equivalent to the claim. ■

Observation 6 (Null Complement) Fix an event $E \in \Sigma$ and acts $f, g \in \mathscr{A}_0$. If $f(\omega) = g(\omega)$ for all $\omega \in E$, then for every $h \in \mathscr{A}_0$, $f \succ_E h$ (resp. $h \succ_E f$) iff $g \succ_E h$ (resp. $h \succ_E g$).

(This implies that $\Omega \setminus E$ is Savage-null for $\succ_E$, but is a strictly stronger statement. The conclusion can also be derived from Axiom 4, but it really is just a consequence of Definition 3.)

Proof: Fix $k \in \mathscr{A}_0$. If $f(\omega) = g(\omega)$ for all $\omega \in E$, then $fEk = gEk$. Then $f \succ_E h$ iff $fEk \succ hEk$, hence iff $gEk \succ hEk$, i.e., iff $g \succ_E h$; the other statement is proved similarly. ■

Remark 4 (Conditionally implausible events) *For all $E, N \in \Sigma$, N is implausible given E if and only if, for all $x, y, z \in X$ with $y \succ_E z$, $y \succ_E x(E \cap N)z$.*

Proof: Follows directly from Definition 3, since $x(E \cap N)z = [xNz]Ez$. ■

Next, I obtain a von Neumann-Morgenstern representation for (conditional) preferences over constant acts. Notice that the representation extends to preferences conditional upon events that do not belong to $\mathscr{F}$, but are supersets of some $F \in \mathscr{F}$.

Lemma 10 *Assume Axioms 1–6. Then there exists a cardinally unique, non-constant, affine function* $u : X \to \mathbb{R}$ *such that, for all* $x, y \in X$*, and all* $E \in \Sigma$ *such that* $\succ_E$ *is non-degenerate,* $x \succ_E y$ *iff* $u(x) > u(y)$*. In particular, this is the case for all* E *such that* $E \supseteq F$ *for some* $F \in \mathscr{F}$*.*

Proof: Under the assumed axioms (in particular, taking $F = \Omega$ in Axioms 5 and 6), the restriction of $\succ$ to constant acts satisfies the von Neumann and Morgenstern (1947) axioms (see in particular Fishburn, 1970, Theorem 8.4); hence, there exists a cardinally unique, affine, non-constant $u : X \to \mathbb{R}$ such that $x \succ y$ iff $u(x) > u(y)$.

By Lemma 9, if $\succ_E$ is non-trivial, then $x \succ_E y$ iff $x \succ y$, and therefore iff $u(x) > u(y)$.

Now consider $E \in \Sigma$ and $F \in \mathscr{F}$ with $E \supseteq F$. By Axiom 5 (Non-degeneracy), there are $x', y' \in X$ with $x' \succ_F y'$. By Lemma 9, $x' \succ y'$. By Definition 3, $x' F y' \succ y'$. By Irreflexivity (Axiom 1), not $y' \succ x'$, so Axiom 4 (Monotonicity) implies $x' E y' \succ y'$. Hence, by Definition 3, $x' \succ_E y'$, i.e., $\succ_E$ is non-degenerate. Thus u represents $\succ_E$ on X as well. ■

Corollary 3 *For every* $E \in \Sigma$ *such that* $F \subseteq E$ *for some* $F \in \mathscr{F}$*,* $\Omega \setminus E$ *is implausible given* E*.*

Proof: For all $x, z \in X$, $x(\Omega \setminus E)z = zEx$. Hence, $\Omega \setminus E$ is implausible given E iff $y \succ_E zEx$ for all $x, y, z \in X$ with $y \succ z$. By Observation 6, this is equivalent to $y \succ_E z$ for all $y, z \in X$ with $y \succ z$. By Lemma 10, since $F \subseteq E$ for some $F \in \mathscr{F}$, $y \succ z$ always implies $y \succ_E z$. ■

The last step of the proof of Lemma 10 also shows:

Corollary 4 *If* $E, F \in \Sigma$*,* $\succ_F$ *is non-degenerate, and* $E \supset F$*, then* $\succ_E$ *is non-degenerate. In particular, if* $F \in \mathscr{F}$ *and Axiom 5 holds, then* $\succ_E$ *is non-degenerate.*

Observation 7 Assume Axioms 1–4 and 6. Then, for every $E \in \Sigma$ such that $\succ_E$ is non-degenerate, $\succ_E$ satisfies the following conditional versions of Axioms 1–4:

Strict Partial Order $\succ_E$ is irreflexive and transitive

Negative Transitivity for Prizes for all $x, y, z \in X$, not $y \succ_E x$ and not $z \succ_E y$ imply not $z \succ_E x$

Independence for all $f, g, h \in \mathscr{A}_0$, $f \succ_E g$ iff $\alpha f + (1-\alpha)h \succ_E \alpha g + (1-\alpha)h$

Monotonicity For all $f, g, h \in \mathscr{A}_0$, if for all $\omega \in E$ not $g(\omega) \succ f(\omega)$ [resp. not $f(\omega) \succ g(\omega)$], then $g \succ_E h$ implies $f \succ_E h$ [resp. $h \succ_E g$ implies $h \succ_E f$]

(Recall also that, under the assumed axioms, $\succ$ and $\succ_E$ coincide on X by Lemma 9.)

Proof: Under the noted axioms, Lemma 9 implies that $\succ$ and $\succ_E$ coincide on X. Thus, Negative Transitivity for Prizes is immediate. Strict Partial Order and Independence follow from Definition 3 and,respectively, Axioms 1 and 3 for $\succ$. For Monotonicity, suppose for all $\omega \in E$ not $g(\omega) \succ f(\omega)$. By Axiom 1, for any $k \in \mathscr{A}_0$ and $\omega \in \Omega \setminus E$, not $k(\omega) \succ k(\omega)$, so for all $\omega \in \Omega$ not $gEk(\omega) \succ fEk(\omega)$. Hence by Axiom 4, $gEk \succ hEk$ implies $fEk \succ hEk$, and so $g \succ_E h$ implies $f \succ_E h$, as claimed. The other implication is proved similarly. ■

I now provide an equivalent condition for implausibility. The restriction to $f, g \in \mathscr{A}_0$, rather than $\mathscr{A}_b$, is necessary: consider the case of EU preferences represented by a finitely, but not countably additive probability.

Lemma 11 *Assume Axioms 1–6. Fix events $F, N \in \Sigma$ and assume that $\succ_F$ is non-degenerate. Then N is implausible given F iff, for all $f, g \in \mathscr{A}_0$, if $f(\omega) \succ g(\omega)$ for all $\omega \in F \setminus N$, then $f \succ_F g$.*

Proof: Under the stated assumptions, $\succ$ and $\succ_F$ have a common EU representation on X by Lemma 10, with utility u. For sufficiency, consider x, y, z with $y \succ_F z$, so $y \succ z$. Then, for every $\omega \in F \setminus N$, $y(\omega) = y \succ z = x(F \cap N)z(\omega)$, so by Remark 4, N is implausible given F.

For necessity, consider $f, g \in \mathscr{A}_0$ such that $f(\omega) \succ g(\omega)$ for all $\omega \in F \setminus N$. Let $y \in X$ be such that $u(y) = \frac{1}{2}\min_{\omega \in F} u(f(\omega)) + \frac{1}{2}\max_{\omega \in F} u(f(\omega))$, and let $\bar{f} \in \mathscr{A}_0$ be such that $u(\bar{f}(\omega)) = 2u(y) - u(f(\omega))$ for all $\omega \in F$ and, for definiteness, $\bar{f}(\omega) = f(\omega)$ for $\omega \notin F$; such an act exists because, for every $\omega \in F$, $2u(y) - u(f(\omega)) = \min_{\omega' \in F} u(f(\omega')) + \max_{\omega' \in F} u(f(\omega')) - u(f(\omega)) \in \left[\min_{\omega \in F} u(f(\omega)), \max_{\omega \in F} u(f(\omega))\right]$. Therefore, for every $\omega \in F$, $u(y) = \frac{1}{2}u(f(\omega)) + \frac{1}{2}u(\bar{f}(\omega))$. By Independence (Axiom 3 and Observation 7), $f \succ_F g$ iff $\frac{1}{2}f + \frac{1}{2}\bar{f} \succ_F \frac{1}{2}g + \frac{1}{2}\bar{f}$. Since u represents

$\succ$ on X, for every $\omega \in F$, not $y \succ \frac{1}{2}f(\omega)+\frac{1}{2}\bar{f}(\omega)$, so by Axiom 4 and Observation 7 (Monotonicity), $y \succ_F \frac{1}{2}g+\frac{1}{2}\bar{f}$ implies $\frac{1}{2}f+\frac{1}{2}\bar{f} \succ_F \frac{1}{2}g+\frac{1}{2}\bar{f}$, and therefore, by Independence, $f \succ_F g$. Hence, it is enough to show that $y \succ_F \frac{1}{2}g+\frac{1}{2}\bar{f}$.

Let $x, z \in X$ be such that $u(x) = \max_{\omega\in F\cap N} \frac{1}{2}u(g(\omega))+\frac{1}{2}u(\bar{f}(\omega))$ and $u(z) = \max_{\omega\in F\setminus N} \frac{1}{2}u(g(\omega))+\frac{1}{2}u(\bar{f}(\omega))$; if $N=\emptyset$, choose x arbitrarily. Note that $\frac{1}{2}u(g(\omega))+\frac{1}{2}u(\bar{f}(\omega)) = \frac{1}{2}u(g(\omega))+u(y)-\frac{1}{2}u(f(\omega)) = u(y)-\frac{1}{2}[u(f(\omega))-u(g(\omega))]$; since $f(\omega) \succ g(\omega)$ for all $\omega \in F\setminus N$, and u represents $\succ$ on X, $\frac{1}{2}u(g(\omega))+\frac{1}{2}u(\bar{f}(\omega)) < u(y)$ for every $\omega \in F\setminus N$, so in particular $y \succ z$ (recall that f and g are simple acts). Hence, since N is implausible given F, $y \succ_F x(F\cap N)z$ by Remark 4. But since u represents $\succ$ on X, for every $\omega \in F$, not $[\frac{1}{2}g+\frac{1}{2}\bar{f}](\omega) \succ x(F\cap N)z(\omega)$. Hence, by Axiom 4 and Observation 7 (Monotonicity), $y \succ_F \frac{1}{2}g+\frac{1}{2}\bar{f}$. ■

Lemma 12 (Properties of implausible events) *Assume Axioms 1–6. Fix $F, G, N, M \in \Sigma$ such that $\succ_F$ and $\succ_G$ are non-degenerate.*

1. *If N is implausible given F and $M \subset N$, then M is implausible given F. Thus, if M is plausible given F, so is N.*
2. *If $N \subseteq F$ is implausible given F, and $F \subseteq G$, then N is implausible given G. Thus, if N is plausible given G, it is plausible given F.*
3. *If N and M are both implausible given F, then so is $N \cup M$.*
4. *If N is implausible given G, $G \supset F \supset N$, and $G \setminus F$ is implausible given G, then N is also implausible given F.*

Notation: in the proof of part 4, as well as in the proofs of Lemmata 14 and 15 in the next section, it is necessary to consider composites of three acts. For $E_1, E_2 \in \Sigma$ such that $E_1 \cap E_2 = \emptyset$ and $f_1, f_2, f_3 \in \mathscr{A}_0$, let $f_1E_1f_2E_2f_3$ denote the act g such that $g(\omega) = f_1(\omega)$ for $\omega \in E_1$, $g(\omega) = f_2(\omega)$ for $\omega \in E_2$, and $g(\omega) = f_3(\omega)$ for $\omega \notin E_1 \cup E_2$.

Proof: Since $\succ_F$ and $\succ_G$ are non-degenerate, they agree with $\succ$ on X, and are represented by the utility function u, per Lemma 10. Throughout this proof, acts are assumed to belong to

$\mathscr{A}_0$, so Lemma 11 applies.

1: if $f(\omega) \succ g(\omega)$ for all $\omega \in F \setminus M$, then a fortiori this holds for all $\omega \in F \setminus N \subset F \setminus M$; since N is implausible given F, $f \succ_F g$ by Lemma 11. Since this is the case for all $f, g \in \mathscr{A}_0$, M is implausible given F by Lemma 11.

2: if $f(\omega) \succ g(\omega)$ for all $\omega \in G \setminus N$, then a fortiori this holds for all $\omega \in F \setminus N$. Hence, $f \succ_F g$ by Lemma 11 because N is implausible given F. By the definition of conditional preferences, in particular $fFg \succ g$. Since $f(\omega) \succ g(\omega)$ for all $\omega \in G \setminus F$, not $g(\omega) \succ f(\omega)$ for such ω because Axiom 1 implies that $\succ$ is asymmetric; and for $\omega \in F \cup (\Omega \setminus G)$, $fFg(\omega) = fGg(\omega)$, which implies that not $fFg(\omega) \succ fGg(\omega)$ because $\succ$ is irreflexive by Axiom 1. To sum up, for all ω, not $fFg(\omega) \succ fGg(\omega)$. Then, by Axiom 4 (Monotonicity) $fGg \succ g$. But then, by the definition of conditional preferences, $f \succ_G g$. Since f, g were arbitrary with $f(\omega) \succ g(\omega)$ for $\omega \in G \setminus N$, N is implausible for G by Lemma 11.

3: Fix F, N, M as in the statement. It is enough to consider the case $N \cap M = \emptyset$: this is because (i) for arbitrary implausible events N, M, $N \cup M = (N \setminus M) \cup (N \cap M) \cup (M \setminus N)$, and the sets on the rhs are all implausible by part 1; and (ii) if the union of two disjoint implausible events is implausible, by induction so is the union of finitely many (in particular, three) disjoint implausible events.

It is without loss of generality to assume that $[-1, 1] \subseteq u(X)$; also, let $z_0 \in X$ be such that $u(z_0) = 0$. For arbitrary $x, y, z \in X$, there exists $\alpha \in (0, 1)$ such that $u \circ [\alpha x + (1-\alpha) z_0] = \alpha u(x) \in [-\frac{1}{2}, \frac{1}{2}]$, and similarly $u \circ [\alpha y + (1-\alpha) z_0], u \circ [\alpha z + (1-\alpha) z_0] \in [-\frac{1}{2}, \frac{1}{2}]$. Then, by Independence (Axiom 3), $y \succ z$ iff $\alpha y + (1-\alpha) z_0 \succ \alpha z + (1-\alpha) z_0$, and $y \succ_F xNz$ iff $\alpha y + (1-\alpha) z_0 \succ_F [\alpha x + (1-\alpha) z_0] N [\alpha z + (1-\alpha) z_0]$. Thus, it is enough to show that $M \cup N$ satisfies the condition in Definition 4 for $x, y, z \in X$ such that $u(x), u(y), u(z) \in [-\frac{1}{2}, \frac{1}{2}]$.

Furthermore, for any such tuple $x, y, z \in X$, there is $\bar{z} \in X$ such that $u(\bar{z}) = -u(z)$, so that $u \circ [\frac{1}{2} z + \frac{1}{2} \bar{z}] = 0 = u(z_0)$, so neither $\frac{1}{2} z + \frac{1}{2} \bar{z} \succ z_0$ nor $z_0 \succ \frac{1}{2} z + \frac{1}{2} \bar{z}$. By Independence (Axiom 3), $y \succ z$ iff $\frac{1}{2} y + \frac{1}{2} \bar{z} \succ \frac{1}{2} z + \frac{1}{2} \bar{z}$, hence (from the EU representation of $\succ$ on X) iff $\frac{1}{2} y + \frac{1}{2} \bar{z} \succ z_0$.

Independence (Axiom 3 and Observation 7) also implies that $y \succ_F xNz$ iff $\frac{1}{2}y + \frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x + \frac{1}{2}\bar{z})N(\frac{1}{2}z + \frac{1}{2}\bar{z})$. Since not $\frac{1}{2}z + \frac{1}{2}\bar{z} \succ z_0$ and not $z_0 \succ \frac{1}{2}z + \frac{1}{2}\bar{z}$, Monotonicity (Axiom 4 and Observation 7) imply that, if $\frac{1}{2}y + \frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x + \frac{1}{2}\bar{z})N(\frac{1}{2}z + \frac{1}{2}\bar{z})$, then $\frac{1}{2}y + \frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x + \frac{1}{2}\bar{z})Nz_0$, and conversely, if $\frac{1}{2}y + \frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x + \frac{1}{2}\bar{z})Nz_0$, then $\frac{1}{2}y + \frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x + \frac{1}{2}\bar{z})N(\frac{1}{2}z + \frac{1}{2}\bar{z})$. Therefore, $y \succ_F xNz$ iff $\frac{1}{2}y + \frac{1}{2}\bar{z} \succ_F (\frac{1}{2}x + \frac{1}{2}\bar{z})Nz_0$. Clearly, $u \circ [\frac{1}{2}y + \frac{1}{2}\bar{z}], u \circ [\frac{1}{2}x + \frac{1}{2}\bar{z}] \in [-\frac{1}{2}, \frac{1}{2}]$. Thus, it is enough to show that, for all $x, y \in X$ with $u(x), u(y) \in [-\frac{1}{2}, \frac{1}{2}]$, $y \succ z_0$ implies $y \succ_F x[M \cup N]z_0$.

Fix such $x, y \in X$. Suppose that $y \succ z_0$. Since $u(x) \in [-\frac{1}{2}, \frac{1}{2}] \subset [-1, 1] \subseteq u(X)$, there is $x' \in X$ such that $u(x') = 2u(x)$. Since both N and M are implausible given F, $y \succ z_0$ implies both $y \succ_F x'Nz_0$ and $y \succ_F x'Mz_0$. Therefore, since N and M are disjoint, by Independence (Axiom 3 and Observation 7)[11], $y \succ_F (\frac{1}{2}x' + \frac{1}{2}z_0)(N \cup M)z_0$. But $u \circ [\frac{1}{2}x' + \frac{1}{2}z_0] = \frac{1}{2}u(x') + \frac{1}{2}u(z_0) = \frac{1}{2} \cdot 2u(x) = u(x)$, so by the EU representation of $\succ$ on X, not $x \succ \frac{1}{2}x' + \frac{1}{2}z_0$. Therefore, by Monotonicity (Axiom 4 and Observation 7), $y \succ_F x(N \cup M)z_0$. This proves the claim.

4: fix $x, y, z \in X$ with $y \succ z$. Since N and $G \setminus F$ are disjoint and both implausible given G, by Part 3 of this Lemma so is $N \cup (G \setminus F)$. Therefore, by definition, $y \succ_G x[N \cup (G \setminus F)]z$.

Suppose that not $y \succ x$; then, by Axiom 4 and Observation 7 (Monotonicity), $y \succ_G x[N \cup (G \setminus F)]z$ implies $y \succ_G xNy(G \setminus F)z$. By the definition of conditional preferences, $yGz \succ xNy(G \setminus F)z$. Since both acts assign the prize y to states in $G \setminus F$, and the prize z to states not in G, the definition of conditional preferences implies $y \succ_F xNz$. Thus, N is implausible given F.

If instead $y \succ x$, then $y \succ_F x$ as well. There are two subcases. If $x \succ_F z$, then not $z \succ_F x$ by Irreflexivity, and so Axiom 4 (Monotonicity) implies that $y \succ_F xNz$. If instead not $x \succ_F z$, then $y \succ z$ implies $y \succ_F z$ as well, and Axiom 4 again imply $y \succ_F xNz$. This completes the proof. ■

Corollary 5 *For any $F, N \in \Sigma$, if $\succ_F$ is non-degenerate and N is implausible given F, then $F \setminus N$*

[11] Suppose that $f \succ g$ and $f' \succ g'$. Then Independence implies $\frac{1}{2}f + \frac{1}{2}f' \succ \frac{1}{2}g + \frac{1}{2}f'$ and $\frac{1}{2}f' + \frac{1}{2}g \succ \frac{1}{2}g' + \frac{1}{2}g$. Thus, by Transitivity, $\frac{1}{2}f + \frac{1}{2}f' \succ \frac{1}{2}g + \frac{1}{2}g'$.

is plausible given F.

Proof: If N is implausible for F, so is $F \cap N$ by Part 1. Suppose $F \setminus N$ is also implausible for F. Then $F = (F \cap N) \cup (F \setminus N)$ is implausible given F by Part 3. Choose $x, y, z \in X$ with $y \succ z$ and, in particular, $x \succ y$; by Definition 4, $y \succ_F xFz$. By the definition of conditional preferences, $yFz \succ (xFz)Fz = xFz$, i.e., again by Definition 3, $y \succ_F x$. But by Axiom 5 and Lemma 9, $x \succ_F y$, which contradicts the fact that $\succ_F$ is irreflexive (Axiom 1 and Observation 7). ■

As in the proof of necessity, it is convenient to set $\sigma(E) = E \setminus \mathrm{imp}(E)$ for every $E \in \Sigma$, where $\mathrm{imp}(\cdot)$ is as in Definition 4. Note that $E \setminus \sigma(E) = E \cap \mathrm{imp}(E)$.

Corollary 6 *For any $E \in \Sigma$, if $\succ_E$ is non-degenerate, then* $\mathrm{imp}(E)$ *and* $E \setminus \sigma(E)$ *are implausible given E, and* $\sigma(E)$ *is plausible given E.*

Proof: By part 3 of the Lemma, $\mathrm{imp}(E)$ is implausible given E. Since $E \setminus \sigma(E) = E \cap \mathrm{imp}(E) \subseteq \mathrm{imp}(E)$, by part 1 of the Lemma it is also implausible given E. Finally, since $\mathrm{imp}(E)$ is implausible given E, by Corollary 5, $\sigma(E) = E \setminus \mathrm{imp}(E)$ is plausible given E. ■

D Sufficiency: main argument

Lemma 13 *Fix non-empty $E, F \in \Sigma$ and $p, q \in \Delta(\Sigma)$ such that $p(E) = q(F) = 1$, $p(E \cap F) > 0$, $q(E \cap F) > 0$, and $p(\cdot|E \cap F) = q(\cdot|E \cap F)$. Then:*

1. *there is a unique $r \in \Delta(\Sigma)$ such that $r(E \cup F) = 1$, $r(E) > 0$, $r(F) > 0$, $r(\cdot|E) = p$, and $r(\cdot|F) = q$.*

2. for every simple, Σ-measurable $\phi : \Omega \to \mathbb{R}$, the function $\phi' : \Omega \to \mathbb{R}$ such that

$$\phi'(\omega) = \begin{cases} 0 & \omega \in \Omega \setminus F \\ \frac{\mathrm{E}_p \phi}{p(E \cap F)} & \omega \in E \cap F \\ \phi(\omega) & \omega \in F \setminus E \end{cases}$$

satisfies $\mathrm{E}_p \phi' = \mathrm{E}_p \phi$ *and* $\mathrm{E}_r \phi = r(F) \cdot \mathrm{E}_q \phi'$*.*

Proof: (1): let $\rho \equiv \frac{p(E\cap F)}{q(E\cap F)}$ and define r by

$$r(G) = \frac{p(G \cap E) + \rho \cdot q(G \cap [F \setminus E])}{1 + \rho \cdot q(F \setminus E)} \quad \forall G \in \Sigma. \tag{12}$$

Since $p(E) = 1$,

$$r(E \cup F) = 1, \quad r(E) = \frac{1}{1 + \rho q(F \setminus E)} > 0, \quad \text{and} \quad r(F) = \frac{p(E \cap F) + \rho \cdot q(F \setminus E))}{1 + \rho q(F \setminus E)} > 0.$$

Since p, q are non-negative and additive[12], so is r: thus, $r \in \Delta(\Sigma)$. Moreover, for measurable $G \subseteq E$, $r(G|E) = p(G)$. In particular, this is true for measurable $G \subseteq E \cap F$; since for such G by assumption $p(G|E \cap F) = q(G|E \cap F)$, it follows that

$$r(G|F) = \frac{r(G)}{r(F)} = \frac{r(G)}{r(E)} \cdot \frac{r(E)}{r(F)} = p(G) \cdot \frac{r(E)}{r(F)} = p(G) \cdot \frac{1}{p(E \cap F) + \rho q(F \setminus E)} =$$
$$= \frac{p(G)}{p(E \cap F)} \cdot \frac{1}{1 + \frac{q(F \setminus E)}{q(E \cap F)}} = \frac{q(G)}{q(E \cap F)} \cdot \frac{q(E \cap F)}{q(E \cap F) + q(F \setminus E)} = q(G)$$

for $G \subseteq E \cap F$. Finally, consider $G \subseteq F \setminus E$. We have

$$r(G|F) = \frac{\rho \cdot q(G)}{p(E \cap F) + \rho \cdot q(F \setminus E)} = \frac{\rho \cdot q(G)}{\rho \cdot q(E \cap F) + \rho \cdot q(F \setminus E)} = q(G),$$

as required.

Now let $r' \in \Delta(\Sigma)$ be such that $r'(E \cup F) = 1$, $r'(E) > 0$, $r'(F) > 0$, $r'(\cdot|E) = p$, and $r'(\cdot|F) = q$. Then

$$\frac{r'(E \setminus F)}{r'(E \cap F)} = \frac{r'(E \setminus F|E)}{r'(E \cap F|E)} = \frac{p(E \setminus F)}{p(E \cap F)}$$

[12]Similarly, if p and q are countably additive, so is r (cf. Section 5).

and similarly $\frac{r'(F\setminus E)}{r'(E\cap F)} = \frac{q(F\setminus E)}{q(E\cap F)}$. Therefore,

$$1 = r'(E\cap F) + r'(E\setminus F) + r'(F\setminus E) = r'(E\cap F)\left(1 + \frac{p(E\setminus F)}{p(E\cap F)} + \frac{q(F\setminus E)}{q(E\cap F)}\right).$$

Thus, the value of $r'(E\cap F)$ is determined by p and q, and hence must coincide with $r(E\cap F)$ [this can also be verified directly by taking $G = E\cap F$ in Eq. (12)]. Hence, also $r'(E\setminus F) = r(E\setminus F)$ and $r'(F\setminus E) = r(F\setminus E)$. Thus $r'(E) = r(E)$ and $r'(F) = r(F)$. If $G\in\Sigma$ satisfies $G\subseteq E$, then $r'(G) = r'(G|E)\cdot r'(E) = p(G)\cdot r(E) = r(G|E)\cdot r(E) = r(G)$; similarly, if $G\subseteq F$, the $r'(G) = r(G)$. Finally, for arbitrary $G\in\Sigma$, $r'(G) = r'(G\cap[E\cup F]) = r'(G\cap E) + r'(G\cap[F\setminus E]) = r(G\cap E) + r(G\cap [F\setminus E]) = r(G)$.

(2): by direct calculation,

$$\mathrm{E}_p\phi' = p(E\setminus F)\cdot 0 + p(E\cap F)\cdot\frac{E_p\phi}{p(E\cap F)} = E_p\phi.$$

Moreover, $q = r(\cdot|F)$, so $r(F)\cdot q(G) = r(F)\cdot r(G|F) = r(G)$ for all measurable $G\subseteq F$, so

$$r(F)\cdot E_q\phi' = r(F)\cdot q(E\cap F)\cdot\frac{E_p\phi}{p(E\cap F)} + r(F)\cdot\int_{F\setminus E}\phi\,dq = r(E\cap F)\cdot\frac{E_p\phi}{p(E\cap F)} + \int_{F\setminus E}\phi\,dr$$

and finally, since $p = r(\cdot|E)$, and in particular $p(E\cap F) = r(E\cap F|E) = \frac{r(E\cap F)}{r(E)}$, the rhs equals

$$r(E)\cdot E_p\phi + \int_{F\setminus E}\phi\,dr = r(E)\int\phi\,dr(\cdot|E) + \int_{F\setminus E}\phi\,dr = \int_E\phi\,dr + \int_{F\setminus E}\phi\,dr = \mathrm{E}_r\phi.$$

■

Lemma 14 *Assume Axioms 1–6. Let u be as in Lemma 10. If there are $p,q\in\Delta(\Sigma)$ with $p(E) = q(F) = 1$, $p(E\cap F) > 0$, and $q(E\cap F) > 0$ such that*

$$\mathrm{E}_p u\circ f > \mathrm{E}_p u\circ g \Rightarrow f \succ_E g \tag{13}$$

$$\mathrm{E}_q u\circ f > \mathrm{E}_q u\circ g \Rightarrow f \succ_F g \tag{14}$$

then $p(\cdot|E\cap F)=q(\cdot|E\cap F)$*, and*

$$\mathrm{E}_r\, u\circ f > \mathrm{E}_r\, u\circ g \Rightarrow f \succ_{E\cup F} g, \tag{15}$$

where r *is as in Lemma 13.*

The intuition behind the proof is as follows. The first part of the argument shows that p and q agree on $E\cap F$, so Lemma 13 applies. The second part shows that r represents $\succ_{E\cup F}$ on $\mathscr{A}_0$. This leverages the fact that, per Lemma 13, the expected utility of an act h with respect to r is proportional to the expected utility of a modified act h' with respect to q. Specifically, h' coincides with h on $F\setminus E$, equals $x(E\cap F)z$ on E, and z at all other states, where x,z are such that $u(z)=0$ and $u(x)=\mathrm{E}_p\, u\circ h/p(E\cap F)$. However, since Eqs. (13) and (14) only apply to strict inequalities, a further perturbation of the act h' is required; footnote 13 elaborates.

Proof: Since u is non-constant, $\succ_E$ and $\succ_F$ are non-degenerate; by Corollary 4, so is $\succ_{E\cup F}$.

Consider $D\in\Sigma$ such that $D\subseteq E\cap F$. Fix $x,y\in X$ with $u(x)>u(y)$, so $x\succ_E y$ and $x\succ_F y$. Fix $\alpha\in[0,1]$. Suppose that $p(D|E\cap F)>\alpha$, or $p(D)>P(E\cap F)\alpha$; then $u(y)+p(D)[u(x)-u(y)]>u(y)+p(E\cap F)\alpha[u(x)-u(y)]$ because $u(x)>u(y)$. Equivalently, $p(D)u(x)+[1-p(D)]u(y)>p(E\cap F)[\alpha u(x)+(1-\alpha)u(y)]+[1-p(E\cap F)]u(y)$. By Eq. (13), this implies $xDy\succ_E \{\alpha u(x)+(1-\alpha)u(y)\}(E\cap F)y$. By Definition 3 (which is well-posed by Remark 1), since $D\subseteq E\cap F\subseteq E$, in particular $xDy\succ\{\alpha u(x)+(1-\alpha)u(y)\}(E\cap F)y$. Similarly, $p(D|E\cap F)<\alpha$ implies $xDy\prec\{\alpha u(x)+(1-\alpha)u(y)\}(E\cap F)y$. By a symmetric argument, using Eq. (14) and the fact that $D\subseteq E\cap F\subseteq F$, $q(D|E\cap F)>\alpha$ implies $xDy\succ\{\alpha u(x)+(1-\alpha)u(y)\}(E\cap F)y$ and $q(D|E\cap F)<\alpha$ implies $xDy\prec\{\alpha u(x)+(1-\alpha)u(y)\}(E\cap F)y$.

Suppose that $q(D|E\cap F)>p(D|E\cap F)$, and fix $\alpha\in\big(p(D|E\cap F),q(D|E\cap F)\big)$. Since $q(D|E\cap F)>\alpha$, $xDy\succ\{\alpha u(x)+(1-\alpha)u(y)\}(E\cap F)y$; but since $p(D|E\cap F)<\alpha$, $xDy\prec\{\alpha u(x)+(1-\alpha)u(y)\}(E\cap F)y$, which contradicts Irreflexivity in Axiom 1. Similarly, $q(D|E\cap F)<p(D|E\cap F)$ also leads to a contradiction. Hence, $q(D|E\cap F)=p(D|E\cap F)$.

Now assume wlog that $u(X)\supseteq[-1,1]$. Consider first acts $f,g\in\mathscr{A}_0$ such that $u(f(\omega))/p(E\cap F)\in(-1,1)$ and $u(g(\omega))/p(E\cap F)\in(-1,1)$ for every ω. Suppose that $\mathrm{E}_r\, u\circ f>\mathrm{E}_r\, u\circ g$. Then, since f, g take on finitely many values, there exists $\epsilon > 0$ such that

$$\frac{u(f(\omega))-\epsilon}{p(E\cap F)} \in (-1,1)\ \forall\omega, \quad \frac{u(g(\omega))+\epsilon}{p(E\cap F)} \in (-1,1)\ \forall\omega, \quad \text{and} \quad \mathrm{E}_r\, u\circ f - \epsilon > \mathrm{E}_r\, u\circ g + \epsilon. \tag{16}$$

Therefore, there exist $x, y \in X$ such that $u(x) = [\mathrm{E}_p\, u\circ f - \epsilon]/p(E\cap F)$ and $u(y) = [\mathrm{E}_p\, u\circ g + \epsilon]/p(E\cap F)$; also let $z \in X$ be such that $u(z) = 0$. Moreover, since $u(f(\omega))-\epsilon \in \big(-p(E\cap F), p(E\cap F)\big) \subseteq (-1,1)$ and $u(g(\omega))+\epsilon \in \big(-p(E\cap F), p(E\cap F)\big) \subseteq (-1,1)$, there exist $f^{-\epsilon}, g^{+\epsilon} \in \mathscr{A}_0$ such that $u(f^{-\epsilon}(\omega)) = u(f(\omega))-\epsilon$ and $u(g^{+\epsilon}(\omega)) = u(g(\omega))+\epsilon$ for all ω. Then $u(x) = \mathrm{E}_p\, u\circ f^{-\epsilon}/p(E\cap F)$ and $u(y) = \mathrm{E}_p\, u\circ g^{+\epsilon}/p(E\cap F)$; furthermore, $\mathrm{E}_r\, u\circ f^{-\epsilon} = \mathrm{E}_r\, u\circ f - \epsilon > \mathrm{E}_r\, u\circ g + \epsilon = \mathrm{E}_r\, u\circ g^{+\epsilon}$.

Consider the acts $f^{-\epsilon}[F\setminus E]x[E\cap F]z$ and $g^{+\epsilon}[F\setminus E]y[E\cap F]z$. By part 2 of Lemma 13,

$$\mathrm{E}_q\, u\circ\big(f^{-\epsilon}[F\setminus E]x[E\cap F]z\big) = \frac{\mathrm{E}_r\, u\circ f^{-\epsilon}}{r(F)} > \frac{\mathrm{E}_r\, u\circ g^{+\epsilon}}{r(F)} = \mathrm{E}_q\, u\circ\big(g^{+\epsilon}[F\setminus E]y[E\cap F]z\big),$$

so Eq. (14) implies that

$$f^{-\epsilon}[F\setminus E]x[E\cap F]z \succ_F g^{+\epsilon}[F\setminus E]y[E\cap F]z.$$

By Definition 3, taking z as the act on the complement of F,

$$f^{-\epsilon}[F\setminus E]x[E\cap F]z \succ g^{+\epsilon}[F\setminus E]y[E\cap F]z;$$

and applying Definition 3 again, taking z as the act on the complement of $E\cup F$, these preference rankings imply

$$f^{-\epsilon}[F\setminus E]x[E\cap F]z \succ_{E\cup F} g^{+\epsilon}[F\setminus E]y[E\cap F]z. \tag{17}$$

By the definition of $f^{-\epsilon}$ and x, $\mathrm{E}_p\, u\circ f > \mathrm{E}_p\, u\circ f^{-\epsilon} = u(x)p(E\cap F) = \mathrm{E}_p\, u\circ x[E\cap F]z$. By Eq. (13), $f \succ_E x[E\cap F]z$.[13] Similarly, $y[E\cap F]z \succ_E g$. By Definition 3, taking $f^{-\epsilon}[F\setminus E]z$ and, respectively, $g^{+\epsilon}[F\setminus E]z$ as the act on the complement of E,

$$f^{-\epsilon}[F\setminus E]fEz \succ f^{-\epsilon}[F\setminus E]x[E\cap F]z \quad \text{and} \quad g^{+\epsilon}[F\setminus E]y[E\cap F]z \succ g^{+\epsilon}[F\setminus E]gEz;$$

[13] Perturbing the utility profile of f and g by ϵ is crucial for this step. If $\epsilon = 0$, so $f^{-\epsilon} = f$ and therefore $u(x) = \mathrm{E}_p\, u\circ f/p(E\cap F)$, then $\mathrm{E}_p\, u\circ f = \mathrm{E}_p\, u\circ x[E\cap F]z$, and Eq. (13) does not apply.

and again by Definition 3, taking z as the act on the complement of $E \cup F$,

$$f^{-\epsilon}[F \setminus E]f \succ_{E\cup F} f^{-\epsilon}[F \setminus E]x[E \cap F]z \quad \text{and} \quad g^{+\epsilon}[F \setminus E]y[E \cap F]z \succ_{E\cup F} g^{+\epsilon}[F \setminus E]g. \quad (18)$$

Combining Eqs. (17) and (18), by Transitivity (Observation 7)

$$f^{-\epsilon}[F \setminus E]f \succ_{E\cup F} g^{+\epsilon}[F \setminus E]g.$$

Finally, for every $\omega \in F \setminus E$, $u\big(f^{-\epsilon}[F \setminus E]f\big)(\omega) = u(f(\omega)) - \epsilon < u(f(\omega))$ and $u\big(g^{+\epsilon}[F \setminus E]g\big)(\omega) = u(g(\omega)) + \epsilon > u(g(\omega))$; and for all $\omega \notin F \setminus E$, $u\big(f^{-\epsilon}[F \setminus E]f\big)(\omega) = u(f(\omega))$ and $u\big(g^{+\epsilon}[F \setminus E]g\big)(\omega) = u(g(\omega))$. Thus, since u represents $\succ$ on X, for all ω, not $\big(f^{-\epsilon}[F \setminus E]f\big)(\omega) \succ f(\omega)$ and not $g(\omega) \succ \big(g^{+\epsilon}[F \setminus E]g\big)(\omega)$. Hence, by Monotonicity (Observation 7), $f \succ_{E\cup F} g$.

Finally, consider arbitrary $f, g \in \mathscr{A}_0$, and suppose that $\mathrm{E}_r\, u \circ f > \mathrm{E}_r\, u \circ g$. By finiteness there are $z', z'' \in X$ such that $u(z') \geq u(f(\omega)) \geq u(z'')$ and $u(z') \geq u(g(\omega)) \geq u(z'')$ for all ω. Let $\alpha \in (0,1)$ be such that $\alpha u(z')/p(E \cap F), \alpha u(z'')/p(E \cap F) \in (-1,1)$, and consider the acts $f' = \alpha f + (1-\alpha)z$ and $g' = \alpha g + (1-\alpha)z$, where again $u(z) = 0$. Then $u(f'(\omega))/p(E \cap F)), u(g'(\omega))/p(E \cap F) \in (-1,1)$ for all ω, and furthermore $\mathrm{E}_r\, u \circ f' = \alpha \mathrm{E}_r\, u \circ f > \alpha \mathrm{E}_r\, u \circ g = \mathrm{E}_r\, u \circ g'$, so by the argument just given, $f' \succ_{E\cup F} g'$, i.e., $\alpha f + (1-\alpha)z \succ_{E\cup F} \alpha g + (1-\alpha)z$. By Independence (Observation 7), this implies $f \succ_{E\cup F} g$. ■

Lemma 15 *Assume Axioms 1–6. Let u be as in Lemma 10. If there are $p, q \in \Delta(\Sigma)$ with $p(E) = q(F) = 1$, $p(E \cap F) = 0$, and $q(E \cap F) > 0$ such that*

$$\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g \Rightarrow f \succ_E g \quad (19)$$

$$\mathrm{E}_q\, u \circ f > \mathrm{E}_q\, u \circ g \Leftrightarrow f \succ_F g \quad (20)$$

then $F \setminus E$ is implausible given $E \cup F$, and

$$\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g \Rightarrow f \succ_{E\cup F} g. \quad (21)$$

This is the counterpart to Lemma 14 when $p(E \cap F) = 0$. In this case, it is sufficient to show that F is implausible given $E \cup F$; this in turn is used to show that p satisfies Eq. (21).

Proof: Since u is non-constant, $\succ_E$ and $\succ_F$ are non-degenerate; by Corollary 4, so is $\succ_{E \cup F}$.

Consider $x, y, z \in X$ with $y \succ z$. Assume wlog that $u(z) = 0$. Since $p(E \cap F) = 0$, $\mathrm{E}_p\, u \circ x[E \cap F]z = u(z) < u(y)$, so by Eq. (19) $y \succ_E x[E \cap F]z$. Since this holds for all x, y, z with $y \succ z$, $E \cap F$ is implausible given E.

Now suppose first that $\sup u(X) > \frac{1-q(E\cap F)}{q(E\cap F)} u(x)$. Then by definition there is $\bar{x} \in X$ such that $u(\bar{x}) > \frac{1-q(E\cap F)}{q(E\cap F)} u(x)$. By the argument just given, in particular $y \succ_E \bar{x}[E \cap F]z$. By Definition 3 taking $y[F \setminus E]z$ as the act on the complement of E,

$$y[E \cup F]z \succ \bar{x}[E \cap F]y[F \setminus E]z.$$

Since $y \succ z$, Asymmetry (implied by Axiom 1) yields not $z \succ y$, and furthermore not $\bar{x} \succ \bar{x}$ and not $z \succ z$ by Irreflexivity (Axiom 1), so for every ω, not $\big(\bar{x}[E \cap F]z\big)(\omega) \succ \big(\bar{x}[E \cap F]y[F \setminus E]z\big)(\omega)$. Then, by Monotonicity (Axiom 4),

$$y[E \cup F]z \succ \bar{x}[E \cap F]z$$

and therefore, by Definition 3, taking z as the act on the complement of $E \cup F$,

$$y \succ_{E \cup F} \bar{x}[E \cap F]z.$$

By the choice of $\bar{x}$, $\mathrm{E}_q\, u \circ \bar{x}[E \cap F]z = q(E \cap F)u(\bar{x}) > q(E \cap F) \cdot \frac{1-q(E\cap F)}{q(E\cap F)} u(x) = [1 - q(E \cap F)u(x) = \mathrm{E}_q\, u \circ x[F \setminus E]z$, so by Eq. (20),

$$\bar{x}[E \cap F]z \succ_F x[F \setminus E]z.$$

Applying Definition 3 with z as the act on the complement of F,

$$\bar{x}[E \cap F]z \succ x[F \setminus E]z$$

and applying it again with z as the act on the complement of $E \cup F$,

$$\bar{x}[E \cap F]z \succ_{E \cup F} x[F \setminus E]z.$$

Thus, by Transitivity (Observation 7), $y \succ_{E\cup F} x[F \setminus E]z$.

Finally, if $\frac{1-q(E\cap F)}{q(E\cap F)} u(x) \geq \sup u(X)$, note that $\sup u(X) \geq u(y) > u(z) = 0$. For α sufficiently small, $x' = \alpha x + (1-\alpha)z$ and $y' = \alpha y + (1-\alpha)z$ satisfy $y' \succ z$ by Independence and $\sup u(X) > \frac{1-q(E\cap F)}{q(E\cap F)} u(x')$. By the result just shown, $y' \succ_{E\cup F} x'[F\setminus E]z$, i.e., $\alpha y + (1-\alpha)z \succ_{E\cup F} \alpha\big(x[F\setminus E]z\big) + (1-\alpha)z$. By Independence (Observation 7), $y \succ_{E\cup F} x[F \setminus E]z$.

Since $x, y, z \in X$ were arbitrary prizes with $y \succ z$, $F \setminus E$ is implausible given $E \cup F$. Furthermore, $E \cap F$ is implausible given E, and hence also given $E \cup F$ by Lemma 12 part 2. Therefore, $F = (E \cap F) \cup (F \setminus E)$ is implausible given $E \cup F$ by part 3 of the same Lemma.

Now consider $f, g \in \mathscr{A}_0$ with $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$. Let $x, y \in X$ be such that $\mathrm{E}_p\, u \circ f > u(x) > u(y) > \mathrm{E}_p\, u \circ g$. Since u represents $\succ$ on X, $x \succ y$. Since $F \setminus E$ is implausible given $E \cup F$, by Lemma 11

$$f[F \setminus E]x \succ_{E\cup F} g[F \setminus E]y.$$

Finally, since $\mathrm{E}_p\, u \circ f > u(x)$, by Eq. (19) $f \succ_E x$. With f as the act on the complement of E in Definition 3, $f \succ xEf$, and again by taking f as the act on the complement of $E \cup F$, $f \succ_{E\cup F} xEf$. Finally, for all $\omega \in E \cup F$, $(xEf)(\omega) = (f[F \setminus E]x)(\omega)$, so by Observation 6 $f \succ_{E\cup F} f[F \setminus E]x$. Similarly, $g[F \setminus E]y \succ_{E\cup F} g$. By Transitivity (Observation 7), $f \succ_{E\cup F} g$. ■

Remark 5 *Fix $E \in \Sigma$ such that $\succ_E$ is irreflexive, a non-constant, affine $u : X \to \mathbb{R}$, and $p, q \in \Delta(\Sigma)$ with $p(E) = 1$. If, for all $f, g \in \mathscr{A}_0$, both $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g \;\Rightarrow\; f \succ_E g$ and $\mathrm{E}_q\, u \circ f > \mathrm{E}_q\, u \circ g \;\Rightarrow\; f \succ_E g$, then $p = q$.*

Proof: Consider $D \in \Sigma$ with $D \subseteq E$. Fix $x, y \in X$ with $u(x) > u(y)$; wlog assume $u(x) = 1$ and $u(y) = 0$. Suppose $\alpha \in \big(p(D), 1\big]$. Then $\mathrm{E}_p\, u \circ xDy = p(D) < \alpha = u(\alpha x + (1-\alpha)y)$, so $xDy \prec_E \alpha x + (1-\alpha)y$. If $\mathrm{E}_q\, u \circ xDy > u(\alpha x + (1-\alpha)y)$, then $xDy \succ_E \alpha x + (1-\alpha)y$, which violates Irreflexivity; thus, $q(D) = \mathrm{E}_q\, u \circ xDy \leq u(\alpha x + (1-\alpha)y) = \alpha$. Similarly, $\alpha \in \big[0, p(D)\big)$ implies $q(D) \geq \alpha$. Now suppose $p(D) < q(D)$ and choose $\alpha \in \big(p(D), q(D)\big)$: then $\alpha > p(D)$ implies

$\alpha \geq q(D)$, contradiction. Similarly, if $p(D) > q(D)$, choose $\alpha \in \big(q(D), p(D)\big)$: then $\alpha < p(D)$ implies $\alpha \leq q(D)$, contradiction. Thus, $p(D) = q(D)$. ■

Lemma 16 (Basic conditional EU representation) *Assume Axioms 1–8. Fix $F \in \mathscr{F}$. Then:*

1. *$\succ_F$ and $\succ_{\sigma(F)}$ are non-degenerate.*
2. *$\succ_{\sigma(F)}$ admits an EU representation (u, p), with a unique $p \in \Delta(\Sigma)$ that satisfies $p(\sigma(F)) = 1$, and $u : X \to \mathbb{R}$ as in Lemma 10.*
3. *for every $f, g \in \mathscr{A}_0$, $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$ implies $f \succ_F g$*
4. *an event $N \in \Sigma$ is implausible given F if and only if $p(N) = 0$.*

Proof: 1: $\succ_F$ is non-degenerate by Corollary 4. Moreover, by Corollary 6, $F \setminus \sigma(F)$ is implausible given F, and $x\sigma(F)y(\omega) = x \succ y$ for all $\omega \in F \setminus [F \setminus \sigma(F)] = \sigma(F)$, so $x\sigma(F)y \succ_F y$ by Lemma 11. By the definition of conditional preferences, in particular $x\sigma(F)y \succ y$, and so, again by Definition 3, $x \succ_{\sigma(F)} y$. Thus, $\succ_{\sigma(F)}$ is also non-degenerate.

2: The relation $\succ_{\sigma(F)}$ is irreflexive by Axiom 1 and Observation 7, and negatively transitive by Axiom 7. It satisfies Independence by Axiom 3 and Observation 7; and it satisfies the Archimedean property by Axiom 8. As was just noted, $x \succ_{\sigma(F)} y$, so the preference is non-degenerate. Finally, Observation 5 shows that it also satisfies State Independence. Hence, by Theorem 13.3 in Fishburn (1970), $\succ_{\sigma(F)}$ admits an EU representation (v, p). Furthermore, since $\succ_{\sigma(F)}$ and $\succ$ agree on X, v is cardinally equivalent to the utility u in Lemma 10, so one can take $v = u$.

3: let $x, y \in X$ be such that $\mathrm{E}_p\, u \circ f > u(x) > u(y) > \mathrm{E}_p\, u \circ g$. By part 2, $f \succ_{\sigma(F)} x \succ_{\sigma(F)} y \succ_{\sigma(F)} g$, and furthermore $x \succ y$ because u also represents $\succ$ on X. Since $F \setminus \sigma(F)$ is implausible given F by Corollary 6, by Lemma 11 $x \succ y$ implies that $x\sigma(F)f \succ_F y\sigma(F)g$. Furthermore, by the definition of conditional preferences, $f \succ_{\sigma(F)} x$ iff $f \succ x\sigma(F)f$, i.e., iff $f \succ_F x\sigma(F)f$. Similarly, $y \succ_{\sigma(F)} g$ iff $y\sigma(F)g \succ_F g$. Thus, by Transitivity (Axiom 1 and Observation 7), $f \succ_F g$.

4: fix $N \in \Sigma$. Suppose that $p(N) = 0$ and fix $x, y, z \in X$ with $x \succ_F y$, so $u(y) > u(z)$; this is possible because u is non-constant. Then $u(y) > u(z) = \mathrm{E}_p\, u \circ xNz$, so $y \succ_F xNz$ by part 3 of this Lemma. Hence N is implausible for F.

Conversely, suppose $p(N) > 0$. Pick $x, z \in X$ such that $x \succ_F z$, and consider $y_\epsilon = \epsilon x + (1-\epsilon)z$, where $\epsilon \in (0,1)$. Then, since u is affine, $x \succ_F y_\epsilon \succ_F z$ for all such ϵ; furthermore, $u(y_\epsilon) = \epsilon u(x) + (1-\epsilon)u(z)$. On the other hand, $\mathrm{E}_p\, u \circ xNz = p(N)u(x) + [1-p(N)]u(z)$, so for $\epsilon < p(N)$, $u(y_\epsilon) < \mathrm{E}_p\, u \circ xNz$, and by part 3 of this Lemma, $y_\epsilon \prec_F xNz$. By Irreflexivity in Axiom 1 and Observation 7, this implies that $y_\epsilon \succ_F xNz$ cannot also hold: thus, N is not implausible given F. ■

Now define the array $\mu = (\mu(\cdot|F))_{F \in \mathscr{F}}$ by letting $\mu(\cdot|F) = p_F$ for every $F \in \mathscr{F}$, where p_F is the probability delivered by Lemma 16 for F. For every $F \in \mathscr{F}$, let $C^m_\mu(F)$, $\mathscr{C}^m_\mu(F)$, and $C_\mu(F)$ be as in Section A. I first establish a result parallel to Remark 3, which relates directly to Definition 5:

Corollary 7 *For every $F \in \mathscr{F}$, $\mathscr{F}(F) = \mathscr{C}_\mu(F)$.*

Proof: By Definition 5, $\mathscr{F}(F)$ is the smallest subset $\mathscr{C}$ of $\mathscr{F}$ such that (i) $F \in \mathscr{C}$, and (ii) for all $G \in \mathscr{F}$, if $\cup\mathscr{C}$ is plausible given G, then $G \in \mathscr{C}$. Since $\mu(\cdot|G) = p_G$, where p_G is the probabilty delivered by Lemma 16 for G, so that $\cup\mathscr{C}$ is plausible given G iff $\mu(\cup\mathscr{C}|G) > 0$ by part 4 of that Lemma, (ii) above is equivalent to (ii) in Remark 3. By the latter Remark, $\mathscr{C} = \mathscr{C}_\mu(F)$, so $\mathscr{F}(F) = \mathscr{C}_\mu(F)$. ■

Finally, I can show that μ is a CCPS by leveraging Proposition 2.

Lemma 17 *Assume Axioms 1–8. Consider an ordered list $F_1, \dots, F_L \in \mathscr{F}$ such that, for every $\ell = 2, \dots, L$, $\mu(\cup_{m=1}^{\ell-1} F_m | F_\ell) > 0$. Then there exists a unique $p \in \Delta(\Sigma)$ such that*

1. *$p(F_1) > 0$, $p(\cup_\ell F_\ell) = 1$, and for all $\ell = 1, \dots, L$ and $E \subseteq F_\ell$, $p(E) = \mu(E|F_\ell)p(F_\ell)$.*

2. *for every* $f, g \in \mathscr{A}_0$, $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$ *implies* $f \succ_{\cup_{\ell=1}^L F_\ell} g$*, where* $u : X \to \mathbb{R}$ *is the function that represents* $\succ$ *on* X *per Lemma 10.*
3. *an event* $N \in \Sigma$ *is implausible given* $\cup_{\ell=1}^L F_\ell$ *if and only if* $p(N) = 0$*.*
4. $p(\sigma(\cup_{\ell=1}^L F_\ell)) = 1$*; moreover, for every* $f, g \in \mathscr{A}_0$, $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$ *implies* $f \succ_{\sigma(\cup_{\ell=1}^L F_\ell)} g$*.*
5. $\succ_{\cup_{\ell=1}^L F_\ell}$ *is non-degenerate.*

Proof: By Remark 5, if a $p \in \Delta(\Sigma)$ with the property in part 2 exists, it is unique.

1 and 2: I argue by induction on L.

For $L = 1$, part 1 holds trivially by setting $p = \mu(\cdot|F_1)$, and part 2 follows from Lemma 16 parts 2 and 3. Thus, assume the claim holds for some $L \geq 1$, let $p^L \in \Delta(\Sigma)$ be the probability satisfying parts 1 and 2 of this Lemma, and assume that $F_{L+1} \in \mathscr{F}$ is such that $\mu(\cup_{\ell=1}^L F_\ell | F_{L+1}) > 0$.

If also $p^L(F_{L+1}) > 0$, then taking $E = \cup_{\ell=1}^L F_\ell$, $F = F_{L+1}$, $p = p^L$ and $q = \mu(\cdot|F_{L+1})$ in Lemma 14 yields a new probability $p^{L+1} \in \Delta(\Sigma)$ such that $\mathrm{E}_{p^{L+1}}\, u \circ f > \mathrm{E}_{p^{L+1}}\, u \circ g$ implies $f \succ_{\cup_{\ell=1}^{L+1} F_\ell} g$: that is, part 2 of this Lemma holds. Furthermore, since p^{L+1} is the probability delivered by Lemma 13, $p^{L+1}(\cup_{\ell=1}^{L+1} F_\ell) = 1$, $p^{L+1}(F_{L+1}) > 0$ and $p^{L+1}(\cdot|F_{L+1}) = \mu(\cdot|F_{L+1})$, so for all measurable $E \subseteq F_{L+1}$, $p^{L+1}(E) = \mu(E|F_{L+1}) p^{L+1}(F_{L+1})$. Similarly, Lemma 13 also implies that $p^{L+1}(\cup_{\ell=1}^L F_\ell) > 0$ and $p^{L+1}(\cdot| \cup_{\ell=1}^L F_\ell) = p^L$; thus, by the inductive hypthesis, for all $\ell = 1, \ldots, L$ and measurable $E \subseteq F_\ell$, $p^{L+1}(E) = p^L(E) p^{L+1}(\cup_{m=1}^L F_m) = \mu(E|F_\ell) p^L(F_\ell) p^{L+1}(\cup_{m=1}^L F_m) = \mu(E|F_\ell) p^{L+1}(F_\ell | \cup_{m=1}^L F_m) p^{L+1}(\cup_{m=1}^L F_m) = \mu(E|F_\ell) p^{L+1}(F_\ell)$. Finally, $p^{L+1}(F_1) = p^L(F_1) p^{L+1}(\cup_{m=1}^L F_m) > 0$. Thus, part 1 of this Lemma holds for $L+1$.

If instead $p^L(F_{L+1}) = 0$, then, taking $E = \cup_{\ell=1}^L F_\ell$, $F = F_{L+1}$, $p = p^L$ and $q = \mu(\cdot|F_{L+1})$, Lemma 15 implies that $p^{L+1} \equiv p^L$ is such that $\mathrm{E}_{p^{L+1}}\, u \circ f > \mathrm{E}_{p^{L+1}}\, u \circ g$ implies $f \succ_{\cup_{\ell=1}^{L+1} F_\ell} g$. Furthermore, $p^{L+1}(\cup_{\ell=1}^{L+1} F_\ell) = p^L(\cup_{\ell=1}^{L+1} F_\ell) \geq p^L(\cup_{\ell=1}^L F_\ell) = 1$ and $p^{L+1}(F_1) = p^L(F_1) > 0$. For $\ell = 1, \ldots, L$ and measurable $E \subseteq F_\ell$, by the induction hypothesis $p^{L+1}(E) = p^L(E) = \mu(E|F_\ell) p^L(F_\ell) = \mu(E|F_\ell) p^{L+1}(F_\ell)$; and for measurable $E \subseteq F_{L+1}$, since $p^L(F_{L+1}) = 0$ by assumption, $p^{L+1}(E) = p^L(E) = 0$ and $\mu(E|F_{L+1}) p^{L+1}(F_{L+1}) = \mu(E|F_{L+1}) p^L(F_{L+1}) = 0$. Thus, again, part 1 of this Lemma holds for $L+1$.

Henceforth consider an arbitrary $L \geq 1$. To streamline notation, let $F = \cup_{\ell=1}^L F_\ell$ in the remainder of the proof.

3: fix $N \in \Sigma$. Suppose that $p(N) = 0$ and fix $x, y, z \in X$ with $y \succ_F z$, so $u(y) > u(z)$; this is possible because u is non-constant. Then $u(y) > u(z) = \mathrm{E}_p\, u \circ xNz$, so $y \succ_F xNz$ by part 2 of this Lemma. Hence N is implausible for F.

Conversely, suppose $p(N) > 0$. Pick $x, z \in X$ such that $x \succ_F z$, and consider $y_\epsilon = \epsilon x + (1-\epsilon)z$, where $\epsilon \in (0,1)$. Then, since u is affine, $x \succ_F y_\epsilon \succ_F z$ for all such ϵ; furthermore, $u(y_\epsilon) = \epsilon u(x) + (1-\epsilon)u(z)$. On the other hand, $\mathrm{E}_p\, u \circ xNz = p(N)u(x) + [1-p(N)]u(z)$, so for $\epsilon < p(N)$, $u(y_\epsilon) < \mathrm{E}_p\, u \circ xNz$, and by part 2 of this Lemma, $y_\epsilon \prec_F xNz$. By Irreflexivity in Axiom 1 and Observation 7, this implies that $y_\epsilon \succ_F xNz$ cannot also hold: thus, N is not implausible given F.

4: by part 2 and the fact that u is non-degenerate by Lemma 10, $x \succ_F y$ for some $x, y \in X$, so $\succ_F$ is non-degenerate and Corollary 6 implies that $F \setminus \sigma(F)$ is implausible given F. Then $p(F \setminus \sigma(F)) = 0$ by part 3, and so $p(\sigma(F)) = p(F) = 1$. Hence, if $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$, then $\mathrm{E}_p\, u \circ f\sigma(F)z = \mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g = \mathrm{E}_p\, u \circ g\sigma(F)z$ for any $z \in X$. By part 2, $f\sigma(F)z \succ_F g\sigma(F)z$. Taking z as the act on the complement of F in Definition 3, $f\sigma(F)z = [f\sigma(F)z]Fz \succ [g\sigma(F)z]Fz = g\sigma(F)z$. Thus, again taking z as the act on the complement of $\sigma(F)$ in Definition 3, $f \succ_{\sigma(F)} g$.

5: this follows from part 2 because u is non-constant. ■

Corollary 8 *For every $F \in \mathscr{F}$, there is $p \in \Delta(\Sigma)$ that satisfies Eq. (7), and furthermore*

1. *for every $f, g \in \mathscr{A}_0$, $\mathrm{E}_p\, u \circ f > \mathrm{E}_p\, u \circ g$ implies $f \succ_{C_\mu(F)} g$, where $u : X \to \mathbb{R}$ is the function that represents $\succ$ on X per Lemma 10.*
2. *an event $N \in \Sigma$ is implausible given $C_\mu(F)$ if and only if $p(N) = 0$.*
3. *$\succ_{C_\mu(F)}$ is non-degenerate.*

In particular, μ is a CCPS.

Proof: Fix $F \in \mathscr{F}$ and let M be as in Observation 1. Remark 2 yields an ordered sequence

$F_1,\dots,F_L$ such that $\mathscr{C}_\mu(F)=\mathscr{C}^M_\mu(F)=\{F_1,\dots,F_L\}$ for which the conditions in the Lemma hold. The probability p delivered by the Lemma satisfies $0<p(F_1)=p(F)$, $p(\cup_\ell F_\ell)=p(C_\mu(F))=1$, and $p(E)=\mu(E|F_\ell)p(F_\ell)$ for all ℓ and all measurable $E\subseteq F_\ell$. To complete the proof, suppose $G\subseteq C_\mu(F)$: then $\mu(C^M_\mu(F)|G)=\mu(C_\mu(F)|G)>0$, so $G\in\mathscr{C}^{M+1}_\mu(F)=\mathscr{C}^M_\mu(F)$ and so $G=F_\ell$ for some $\ell=1,\dots,L$. It follows that p satisfies Eq. (7), as required.

Properties 1–3 follow by applying the Lemma to the ordered list $F_1,\dots,F_L$. Finally, since p as above exists for every $F\in\mathscr{F}$, Proposition 2 implies that μ is a CCPS. ■

Henceforth, denote by $P_\mu(F)$ the unique probability delivered by Lemma 17.

It is now possible to prove sufficiency (i.e. the second claim) in Theorem 1. As in the statement of the Theorem, let $\succ^{u,\mu}$ be the structural preference relation represented by μ and u.

Fix $f,g\in\mathscr{A}_0$ and assume that $f\succ^{u,\mu}g$. By Proposition 3, there are $F_1,\dots,F_M\in\mathscr{F}$ such that $\mathrm{E}_{P_\mu(F_m)}[u\circ f-u\circ g]>0$ for $m=1,\dots,M$, and $u(f(\omega))\geq u(g(\omega))$ for all $\omega\notin\cup_m C_\mu(F_m)$. It is wlog to assume that there are no $\ell,m\in\{1,\dots,M\}$ with $C_\mu(F_\ell)\supsetneq C_\mu(F_m)$: if one such pair exists, the event F_m can be disregarded.

For every $m=1,\dots,M$, let $D_m=C_\mu(F_m)\setminus\cup_{\ell=1}^{m-1}C_\mu(F_\ell)$. Then $D_1,\dots,D_M$ is a partition of $B\equiv\cup_m C_\mu(F_m)$. Furthermore, $P_\mu(F_m)(D_m)=1$ for all m: if there is m with $P_\mu(F_m)(D_m)<1$, then there must be $\ell<m$ with $P_\mu(F_m)(C_\mu(F_\ell))>0$; by Lemma 2 part 1, this implies $C_\mu(F_\ell)\supsetneq C_\mu(F_m)$, contradiction. Therefore, $\mathrm{E}_{P_\mu(F_m)}1_{D_m}u\circ f=\mathrm{E}_{P_\mu(F_m)}u\circ f>\mathrm{E}_{P_\mu(F_m)}u\circ g=\mathrm{E}_{P_\mu(F_m)}1_{D_m}u\circ g$ for all m. Wlog, assume that there is $z\in X$ with $u(z)=0$: then, also $\mathrm{E}_{P_\mu(F_m)}u\circ fD_mz>\mathrm{E}_{P_\mu(F_m)}u\circ gD_mz$ for all m; hence, by Lemma 17 part 2, $fD_mz\succ_{C_\mu(F_m)}gD_mz$ for all m. By Definition 3, taking z as the act on the complement of $C_\mu(F_m)$, $fD_mz\succ gD_mz$ for all m.

Now apply Definition 3 repeatedly:

$$
\begin{aligned}
&f \succ_{D_1} g \Rightarrow f D_1 g \succ g D_1 g = g,\\
&f \succ_{D_2} g \Rightarrow f D_2(f D_1 g) \succ g D_2(f D_1 g) \Rightarrow f(D_1 \cup D_2) g \succ f D_1 g,\\
&\qquad \dots\\
&f \succ_{D_m} g \Rightarrow f D_m[f(\cup_{\ell=1}^{m-1} D_\ell) g] \succ g D_m[f(\cup_{\ell=1}^{m-1} D_\ell) g] \Rightarrow f(\cup_{\ell=1}^{m} D_\ell) g \succ f(\cup_{\ell=1}^{m-1} D_\ell) g\\
&\qquad \dots\\
&f \succ_{D_M} g \Rightarrow f D_M[f(\cup_{\ell=1}^{M-1} D_\ell) g] \succ g D_M[f(\cup_{\ell=1}^{M-1} D_\ell) g] \Rightarrow f(\cup_{\ell=1}^{M} D_\ell) g \succ f(\cup_{\ell=1}^{M-1} D_\ell) g
\end{aligned}
$$

Therefore, by Transitivity (Axiom 1), $f(\cup_m D_m)g \succ g$.

Finally, recall that for all $\omega \notin \cup_m C_\mu(F_m) = \cup_m D_m$, $u(f(\omega)) \geq u(g(\omega))$, so not $g(\omega) \succ f(\omega)$. Then, by Monotonicity (Axiom 4), $f \succ g$.

As for Theorem 2, Appendix B shows that 2 $\Rightarrow$ 1; the argument just given for Theorem 1 shows that 1 implies that $f \succ^{u,\mu} g \Rightarrow f \succ g$. Thus, it remains to be show that 1 also implies that $f \succ g \Rightarrow f \succ^{u,\mu} g$.

We first show this under the assumption that $\succ$ satisfies Axiom 9b, with no restriction on $\mathscr{F}$. We then show that, if $\mathscr{F}$ is laminar, then Axiom 9a implies Axiom 9b, which completes the proof.

$$
\mathrm{E}_{P_\mu(F)} u \circ \left[\left(\frac{1}{2} f + \frac{1}{2} x\right) C g\right] = \frac{1}{2} \mathrm{E}_{P_\mu(F)} u \circ (f C g) + \frac{1}{2} u(x) < \frac{1}{2} \mathrm{E}_{P_\mu(F)} u \circ g + \frac{1}{2} u(y) = \mathrm{E}_{P_\mu(C)} u \circ \left[\left(\frac{1}{2} g + \frac{1}{2} y\right) C g\right].
$$

Fix $f, g \in \mathscr{A}_0$ and assume that $f \succ g$. By contradiction, suppose that not $f \succ^{u,\mu} g$. There are two cases. The first is that, for all ω, not $f(\omega) \prec g(\omega)$, so $u(f(\omega)) \geq u(g(\omega))$. By Axiom 9b, there must be $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ x$ such that $\frac{1}{2} f + \frac{1}{2} x \succ_{\cup \mathscr{F}(F)} \frac{1}{2} g + \frac{1}{2} y$. By Corollary 7, $\cup \mathscr{F}(F) = C_\mu(F)$. Finally, by Proposition 3, not $f \succ^{u,\mu} g$ implies that $\mathrm{E}_{P_\mu(F)(\cdot)}[u \circ f - u \circ g] = 0$.

In the second case, $u(f(\omega)) < u(g(\omega))$, i.e., $f(\omega) \prec g(\omega)$, for some ω. Axiom 9b implies that there is $F \in \mathscr{F}$ such that $\omega \in \cup \mathscr{F}(F)$, and $x, y \in X$ with $y \succ x$, such that $\frac{1}{2} f + \frac{1}{2} x \succ_{\cup \mathscr{F}(F)} \frac{1}{2} g + \frac{1}{2} y$.

Again, by Corollary 7, $\cup\mathscr{F}(F) = C_\mu(F)$; and by Proposition 3, not $f \succ^{u,\mu} g$ implies that, in particular, $\mathrm{E}_{P_\mu(F)(\cdot)}[u\circ f - u\circ g] \leq 0$, because $\omega \in \cup\mathscr{F}(F) \subseteq C_\mu(F)$.

To sum up, in either case, there exist $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ x$, $\frac{1}{2}f + \frac{1}{2}x \succ_{\cup\mathscr{F}(F)} \frac{1}{2}g + \frac{1}{2}y$, and $\mathrm{E}_{P_\mu(F)(\cdot)}[u\circ f - u\circ g] \leq 0$. But since $u(x) < u(y)$,

$$\mathrm{E}_{P_\mu(F)} u\circ\left(\frac{1}{2}f + \frac{1}{2}x\right) = \frac{1}{2}\mathrm{E}_{P_\mu(F)} u\circ f + \frac{1}{2}u(x) < \frac{1}{2}\mathrm{E}_{P_\mu(F)} u\circ g + \frac{1}{2}u(y) = \mathrm{E}_{P_\mu(F)} u\circ\left(\frac{1}{2}g + \frac{1}{2}y\right).$$

Hence, by Corollary 8 part 1, $\frac{1}{2}f + \frac{1}{2}x \prec_{C_\mu(F)} \frac{1}{2}g + \frac{1}{2}y$, or equivalently, since $C_\mu(F) = \cup\mathscr{F}(F)$, $\frac{1}{2}f + \frac{1}{2}x \prec_{\cup\mathscr{F}(F)} \frac{1}{2}g + \frac{1}{2}y$. This contradicts the fact that $\succ$ (and hence $\succ_{\cup\mathscr{F}(F)}$) is asymmetric.

Thus, $f \succ^{u,\mu} g$. This completes the proof of Theorem 2 for the case of $\mathscr{F}$ general.

If $\mathscr{F}$ is laminar, the following Lemma proves the remaining claim in Theorem 2.

Lemma 18 *Assume that $\succ$ satisfies Axioms 1–8 and $\mathscr{F}$ is laminar. Then $\succ$ satisfies Axiom 9a if and only if it satisfies Axiom 9b.*

Proof: By Theorem 1 part 2, $\succ^{u,\mu} \subseteq \succ$, where $\succ^{u,\mu}$ is the structural preference represented by (u,μ), and N is plausible for $\succ$ given G iff $\mu(N|G) > 0$.

By Remark 3, $\mathscr{F}(F) = \mathscr{C}_\mu(F)$, and by Remark 2, since $\mathscr{F}$ is laminar, $C_\mu(F) \in \mathscr{F}$. It follows immediately that Axiom 9b implies Axiom 9a.

For the converse, I first claim that, since $\succ$ satisfies Axioms 1–8, it satisfies Axiom 9a iff Property P1 holds, and Axiom 9b iff Property P2 holds, where these properties are as follows:

Property P1: for every $f, g \in \mathscr{A}$: if $f \succ g$, there is $F \in \mathscr{F}$ such that $\mathrm{E}_{\mu(\cdot|C_\mu(F))}[u\circ f - u\circ g] > 0$; furthermore, if ω is such that $u(f(\omega)) < u(g(\omega))$, we can choose F so that $\omega \in C_\mu(F)$.

Property P2: for every $f, g \in \mathscr{A}$: if $f \succ g$, there is $F \in \mathscr{F}$ such that $\mathrm{E}_{\mu(\cdot|F)}[u\circ f - u\circ g] > 0$; furthermore, if ω is such that $u(f(\omega)) < u(g(\omega))$, we can choose F so that $\omega \in F$.

To see this, suppose that there are $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ x$ such that $\frac{1}{2}f + \frac{1}{2}x \succ_{C_\mu(F)} \frac{1}{2}g + \frac{1}{2}y$. If $\mathrm{E}_{P_\mu(F)}[u\circ f - u\circ g] \leq 0$, then $\mathrm{E}_{P_\mu(F)}[u\circ\left(\frac{1}{2}f + \frac{1}{2}x\right) - u\circ\left(\frac{1}{2}g + \frac{1}{2}y\right)] < 0$, which by the definition of conditional preferences, Proposition 3 and Theorem 1 part (ii) implies $\frac{1}{2}f + \frac{1}{2}x \prec_{C_\mu(F)}$

$\frac{1}{2}g+\frac{1}{2}y$, contradiction. Thus, $\mathrm{E}_{P_\mu(F)}[u\circ f-u\circ g]>0$. Conversely, if $\mathrm{E}_{P_\mu(F)}[u\circ f-u\circ g]>0$, then there are $x,y\in X$ with $y\succ x$ but $u(y)-u(x)$ sufficiently small so that $\mathrm{E}_{P_\mu(F)}[u\circ\left(\frac{1}{2}f+\frac{1}{2}x\right)-u\circ\left(\frac{1}{2}g+\frac{1}{2}y\right)]>0$, so again, by the definition of conditional preferences, Proposition 3 and Theorem 1 part (ii), $\frac{1}{2}f+\frac{1}{2}x\succ_{C_\mu(F)}\frac{1}{2}g+\frac{1}{2}y$. Hence, Axiom 9b is equivalent to Property P1. Repating the argument replacing $C_\mu(F)$ with F yields the equivalence of Axiom 9a and Property P2.

It is thus enough to show that P2 implies P1. Suppose that $f\succ g$, so by P2, there is $F\in\mathscr{F}$ such that $\mathrm{E}_{\mu(\cdot|F)}[u\circ f-u\circ g]>0$. Let $G_1=F$. Inductively, for $k\geq 1$, suppose $G_k\in\mathscr{F}$ and $\mathrm{E}_{\mu(\cdot|G_k)}[u\circ f-u\circ g]>0$. By construction, $P_\mu(G_k)(G_k)>0$; since $\mathscr{F}$ is laminar, equivalently $\mu(G_k|C_\mu(G_k))>0$.

Let $W=\{\omega\in C_\mu(G_k):f(\omega)\prec g(\omega)\}$. By P2, for each $\omega\in W$ there is $F_\omega\in\mathscr{F}$ such that $\omega\in F_\omega$ and $\mathrm{E}_{\mu(\cdot|F_\omega)}[u\circ f-u\circ g]>0$. Furthermore, since $\mathscr{F}$ is finite, there are $F_1,\dots,F_N\in\mathscr{F}$ such that $F_\omega\in\{F_1,\dots,F_N\}$ for each $\omega\in W$. In addition, let $F_0=G_k$. Since $\mathscr{F}$ is laminar,if $F_n\cap F_m\neq\emptyset$, then wlog $F_n\supseteq F_m$ and F_m, so $W\subseteq\cup[\{F_0,\dots,F_N\}\setminus\{F_m\}]$ and $G_k\subseteq F_\ell$ for some ℓ: that is, F_m can be dropped. Since $\mathscr{F}$ is finite, repeating this process finitely many times yields a collection $F_0,\dots,F_M$ of disjoint events; wlog continue to assume that $G_k\subseteq F_0$.

Suppose first that $F_n\subseteq C_\mu(G_k)$ for each $n=1,\dots,M$ (this is automatically true for $n=0$). Then

$$\begin{aligned}\mathrm{E}_{\mu(\cdot|C_\mu(G_k))}[u\circ f-u\circ g]=&\mu(F_0|C_\mu(G_k))\mathrm{E}_{\mu(\cdot|F_0)}[u\circ f-u\circ g]+\sum_{n=1}^{M}\mu(F_n|C_\mu(G_k))\mathrm{E}_{\mu(\cdot|F_n)}[u\circ f-u\circ g]\\&+\int_{C_\mu(G_k)\setminus\cup_{n=0}^{M}F_n}(u\circ f-u\circ g)d\mu(\cdot|C_\mu(G_k))>0\end{aligned}$$

because $\mu(F_0|C_\mu(G_k))\geq\mu(G_k|C_\mu(G_k))>0$. Since, as shown above, $C_\mu(G_k)=\cup\mathscr{F}(G_k)$, P1 holds.

If instead there is n such that $F_n\supset C_\mu(G_k)\supseteq G_k$, then let $G_{k+1}=F_n$: then $\mathrm{E}_{\mu(\cdot|G_{k+1})}[u\circ f-u\circ g]>0$ and $G_{k+1}\supset G_k\supseteq F$, and the preceding construction be repeated.

To sum up, either $C_\mu(G_k)=\cup\mathscr{F}(G_k)$ for some $k\geq 1$, or else one obtains an infinite sequence $(G_k)_{k\geq 1}$ such that $G_k\in\mathscr{F}$ and $G_{k+1}\supset G_k$ for each $k\geq 1$. This contradicts the finiteness of $\mathscr{F}$. This completes the proof. ■

Finally, I prove Proposition 1, on the extension of structural preferences from $\mathscr{A}_0$ to $\mathscr{A}_b$.

Lemma 19 *Fix $\phi, \psi \in B(\Sigma)$ such that $\phi(\Omega) \cup \psi(\Omega) \subseteq [\alpha, \beta]$. There exist sequences $(\phi^n)_{n \geq 1}$, $(\psi^n)_{n \geq 1}$ in $B_0(\Sigma)$ with $\phi^n(\Omega) \cup \psi^n(\Omega) \subseteq [\alpha, \beta]$ for every $n \geq 1$ such that (i) $\psi^n \to \psi$ uniformly and $\psi^n(\omega) \geq \psi(\omega)$ for every $n \geq 1$ and $\omega \in \Omega$, (ii) $\phi^n \to \phi$ uniformly and $\phi^n(\omega) - \psi^n(\omega) \leq \phi(\omega) - \psi(\omega)$ for every $\omega \in \Omega$ and $n \geq 1$, and (iii) for every ω, $\phi(\omega) \geq \psi(\omega)$ implies $\phi^n(\omega) \geq \psi^n(\omega)$ for every $n \geq 1$.*

Proof: Let $\delta = \beta - \alpha$ and, for every $n \geq 1$, define $\phi^n, \psi^n \in \mathscr{A}$ by

$$\psi^n(\omega) = \begin{cases} \alpha + \frac{1}{2^n}\delta & \alpha \leq \psi(\omega) \leq \alpha + \frac{1}{2^n}\delta \\ \alpha + \frac{m+1}{2^n}\delta & \alpha + \frac{m}{2^n}\delta < \psi(\omega) \leq \alpha + \frac{m+1}{2^n}\delta,\ 0 < m \leq 2^n - 1 \end{cases}$$

and

$$\phi^n(\omega) = \begin{cases} \alpha + \frac{m}{2^n}\delta & \phi(\omega) < \psi(\omega),\ \alpha + \frac{m}{2^n}\delta \leq \phi(\omega) < \alpha + \frac{m+1}{2^n}\delta,\ 0 \leq m < 2^n - 1 \\ \alpha + \frac{2^n-1}{2^n}\delta & \phi(\omega) < \psi(\omega),\ \alpha + \frac{2^n-1}{2^n}\delta \leq \phi(\omega) \leq \alpha + \frac{2^n}{2^n}\delta \\ \max\left(\alpha + \frac{m}{2^n}\delta, \psi^n(\omega)\right) & \phi(\omega) \geq \psi(\omega),\ \alpha + \frac{m}{2^n}\delta \leq \phi(\omega) < \alpha + \frac{m+1}{2^n}\delta,\ 0 \leq m < 2^n - 1 \\ \max\left(\alpha + \frac{2^n-1}{2^n}\delta, \psi^n(\omega)\right) & \phi(\omega) \geq \psi(\omega),\ \alpha + \frac{2^n-1}{2^n}\delta \leq \phi(\omega) \leq \alpha + \frac{2^n}{2^n}\delta. \end{cases}$$

By construction, $\phi^n(\Omega) \cup \psi^n(\Omega) \subseteq [\alpha, \beta]$.

For any $n \geq 1$ and ω, by construction $\psi^n(\omega) \geq \psi(\omega)$; furthermore, $|\psi^n(\omega) - \psi(\omega)| \leq \delta 2^{-n} \to 0$ for every ω. Thus, (i) holds.

Now consider $n \geq 1$, ω, and m such that $\alpha + \frac{m}{2^n}\delta \leq \phi(\omega) < \alpha + \frac{m+1}{2^n}\delta$ or (if $m = 2^n - 1$) $\alpha + \frac{m}{2^n}\delta = \alpha + \frac{2^n-1}{2^n} \leq \phi(\omega) \leq \alpha + \frac{2^n}{2^n}\delta = \alpha + \frac{m+1}{2^n}\delta$. If in addition either $\phi(\omega) < \psi(\omega)$, or $\phi(\omega) \geq \psi(\omega)$ and $\psi^n(\omega) \leq \alpha + \frac{m}{2^n}\delta$, then $\phi^n(\omega) = \alpha + \frac{m}{2^n}\delta \leq \phi(\omega)$, so $\phi^n(\omega) - \psi^n(\omega) \leq \phi(\omega) - \psi(\omega)$; furthermore, $|\phi^n(\omega) - \phi(\omega)| \leq \delta 2^{-n}$. If instead $\phi(\omega) \geq \psi(\omega)$ and $\psi^n(\omega) > \alpha + \frac{m}{2^n}\delta$, then $\phi^n(\omega) = \psi^n(\omega)$, so that $\phi^n(\omega) - \psi^n(\omega) = 0 \leq \phi(\omega) - \psi(\omega)$; furthermore, since $\phi(\omega) \geq \psi(\omega)$ and $\phi(\omega) \leq \alpha + \frac{m+1}{2^n}\delta$, also $\psi(\omega) \le \alpha + \frac{m+1}{2^n}\delta$, so by definition $\psi^n(\omega) = \alpha + \frac{m+1}{2^n}\delta$, and therefore $|\phi^n(\omega) - \phi(\omega)| = |\alpha + \frac{m+1}{2^n}\delta - \phi(\omega)| \le \delta 2^{-n}$. Thus, (ii) holds.

To sum up, for every ω, since $|\phi^n(\omega) - \phi(\omega)| \le \delta 2^{-n}$, so $\phi^n \to \phi$ uniformly, and in addition, by construction, $\phi^n(\omega) - \psi^n(\omega) \le \phi(\omega) - \psi(\omega)$ for every ω, so (ii) holds, and $\phi^n(\omega) \ge \psi^n(\omega)$ for every ω such that $\phi(\omega) \ge \psi(\omega)$, so (iii) holds. ■

Proof of Proposition 1: The proof of necessity in Appendix B applies for preferences $\succ$ on either $\mathscr{A}_0$ or $\mathscr{A}_b$; specifically, Lemma 8, Observation 4, and the verification that Axioms 1–8 and 9b hold invoke either Definition 2 or Proposition 3, which apply to a preference $\succ$ on $\mathscr{A}_b$ as well.

Now suppose $\succ'$ coincides with $\succ$ on $\mathscr{A}_0$, and satisfies Axioms 1–8 and 9b on $\mathscr{A}_b$.[14] Fix $f, g \in \mathscr{A}_b$ such that $f \succ g$. It must be shown that then $f \succ' g$.

To prove this, suppose first that there are $f', g' \in \mathscr{A}_b$ such that

$$f' \succ' g' \qquad \text{and} \qquad \forall \omega, \quad \frac{1}{2}f'(\omega) + \frac{1}{2}g(\omega) \not\prec' \frac{1}{2}f(\omega) + \frac{1}{2}g'(\omega). \tag{22}$$

I claim that then $f \succ' g$. From $f' \succ' g'$ and Axiom 3 (Independence), $\frac{1}{2}f' + \frac{1}{2}f \succ' \frac{1}{2}g' + \frac{1}{2}f$. From the second condition in Eq. (22) and Axiom 4 (Monotonicity), $\frac{1}{2}f' + \frac{1}{2}f \succ' \frac{1}{2}f' + \frac{1}{2}g$. Then, from Axiom 3 (Independence), $f \succ' g$, as claimed. Thus, to complete the proof, it is enough to construct $f', g' \in \mathscr{A}_b$ with the properties in Eq. (22).

Since $f, g \in \mathscr{A}_b$, there are $x, y \in X$ such that $x \not\succ f(\omega) \not\succ y$ and $x \not\succ g(\omega) \not\succ y$ for all ω [there are prizes that satisfy these conditions separately for f and g; take x to be the weakly worse lower bound, and y the weakly better upper bound]. Let $\alpha = u(x)$, $\beta = u(y)$, and $\delta = \beta - \alpha$. Let $(\phi^n)_{n\ge1}$ and $(\psi^n)_{n\ge1}$ be the sequences delivered by Lemma 19 for $\phi = u \circ f$ and $\psi = u \circ g$, where $u \circ f(\Omega) \cup u \circ g(\Omega) \subseteq [\alpha, \beta]$. Since $\alpha, \beta \in u(X)$, for every $n \ge 1$ there exist $f^n, g^n \in \mathscr{A}_0$ with $u \circ f^n = \phi^n$ and $u \circ g^n = \psi^n$.

[14] Actually, the argument only requires Axioms 3 and 4. Analogously, Proposition 4.1 in Gilboa and Schmeidler (1989) only requires that the extending preference satisfy Monotonicity on bounded acts.

By Proposition 3, $f \succ g$ implies that there are $F_1, \ldots, F_L \in \mathscr{F}$ such that $\mathrm{E}_{P_\mu(F_\ell)}[u \circ f - u \circ g] > 0$ for all ℓ, and $u(f(\omega)) \geq u(g(\omega))$ for all $\omega \notin \cup_\ell C_\mu(F_\ell)$. Since $u \circ f^n \to u \circ f$ and $u \circ g^n \to u \circ g$ in $B(\Sigma)$, for n sufficiently large, $\mathrm{E}_{P_\mu(F_\ell)}[u \circ f^n - u \circ g^n] > 0$ for all ℓ. Furthermore, Lemma 19 ensures that, for all $\omega \notin \cup_\ell C_\mu(F_\ell)$, $u(f^n(\omega)) \geq u(g^n(\omega))$ because, for such ω, $u(f(\omega)) \geq u(g(\omega))$. Thus, for n large, by Proposition 3, $f^n \succ g^n$, hence $f^n \succ' g^n$, and one can take $f' = f^n$, $g' = g^n$ for any such n in Eq. (22).

It remains to be show that $f \succ' g$ implies $f \succ g$. Since $\succ$ is represented by (u, μ) on $\mathscr{A}_b$, this requires adapting the last step in the proof of Sufficiency to the case $\mathscr{A} = \mathscr{A}_b$. The argument relies on Lemma 19. Thus, suppose that $f \succ' g$ and, by contradiction, assum that $f \not\succ g$.

Since $\succ$ and $\succ'$ agree on $\mathscr{A}_0$, and the Definition 4 depends solely on preferences over $\mathscr{A}_0$, given $E \in \Sigma$ and $F \in \mathscr{F}$, E is (im)plausible given F for $\succ$ iff it is (im)plausible given F for $\succ'$. This implies that the sets $\mathscr{F}(F)$, for $F \in \mathscr{F}$, are the same whether they are defined via $\succ$ or $\succ'$. Furthermore, (u, μ) trivially represents $\succ'$ on $\mathscr{A}_0$, because $\succ'$ agrees with $\succ$ there. Thus, we can refer to the sets $C_\mu(F)$ and probabilities $P_\mu(F)$ for $F \in \mathscr{F}$ for both $\succ$ and $\succ'$.

There are two cases. First, suppose $u(f(\omega)) \geq u(g(\omega))$ for all ω. Since $\succ'$ satisfies Axiom 9b, there must be $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ' x$, or equivalently, $y \succ x$, such that $\frac{1}{2}f + \frac{1}{2}x \succ'_{\cup\mathscr{F}(F)} \frac{1}{2}g + \frac{1}{2}y$. Furthermore, by Corollary 7, $\cup\mathscr{F}(F) = C_\mu(F)$. Finally by Proposition 3, $f \not\succ g$ implies that, in particular, $\mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g] = 0$.

In the second case, there is ω such that $u(f(\omega)) < u(g(\omega))$. Again, by Axiom 9b, there must be $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ' x$, or equivalently, $y \succ x$, such that $\omega \in \mathscr{F}(F)$ and $\frac{1}{2}f + \frac{1}{2}x \succ'_{\cup\mathscr{F}(F)} \frac{1}{2}g + \frac{1}{2}y$. Furthermore, by Corollary 7, $\cup\mathscr{F}(F) = C_\mu(F)$. Finally by Proposition 3, $f \not\succ g$ implies that, in particular, $\mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g] \leq 0$, because $\omega \in \cup\mathscr{F}(F) \subseteq C_\mu(F)$.

To sum up, in both cases, there are $F \in \mathscr{F}$ and $x, y \in X$ with $y \succ x$ such that $\hat{f} \equiv \frac{1}{2}f + \frac{1}{2}x \succ'_{\cup\mathscr{F}(F)} \frac{1}{2}g + \frac{1}{2}y \equiv \hat{g}$ and $\mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g] \leq 0$; furthermore, $\cup\mathscr{F}(F) = C_\mu(F)$. Therefore, $\mathrm{E}_{P_\mu(F)}[u \circ \hat{f} - u \circ \hat{g}] = \frac{1}{2}\mathrm{E}_{P_\mu(F)}[u \circ f - u \circ g] + \frac{1}{2}[u(x) - u(y)] < 0$, and so also $\mathrm{E}_{P_\mu(F)}[u \circ \hat{f} C_\mu(F) \hat{g} - u \circ \hat{g}] < 0$, because $P_\mu(F)(C_\mu(F)) = 1$.

Now invoke Lemma 19 with $\phi = u \circ \hat{f} C_\mu(F) \hat{g}$ and $\psi = u \circ \hat{g}$ to obtain sequences $(f^n)_{n \geq 1}$ and

$(g^n)_{n\geq 1}$ in $\mathscr{A}_0$ such that $u\circ f^n \to u\circ \hat{f}\, C_\mu(F)\hat{g}$, $u\circ g^n \to u\circ \hat{g}$, $u\circ g^n - u\circ f^n \leq u\circ \hat{g} - u\circ \hat{f}\, C_\mu(F)\hat{g}$, and $u(g^n(\omega)) \geq u(f^n(\omega))$ for every ω such that $u(\hat{g}(\omega)) \geq (u\circ \hat{f}\, C_\mu(F)\hat{g})(\omega)$. Since $\mathrm{E}_{P_\mu(F)}[u\circ \hat{f}\, C_\mu(F)\hat{g} - u\circ \hat{g}] < 0$, $\mathrm{E}_{P_\mu(F)}[u\circ f^n - u\circ g^n] < 0$ for all n large. Furthermore, for $\omega \notin C_\mu(F)$, $(u\circ \hat{f}\, C_\mu(F)\hat{g})(\omega) = u(\hat{g}(\omega))$, so $u(g^n(\omega)) - u(f^n(\omega)) \leq u(\hat{g}(\omega)) - (u\circ \hat{f}\, C_\mu(F)\hat{g})(\omega) = 0$ for such ω; but also, $u(\hat{g}(\omega)) = (u\circ \hat{f}\, C_\mu(F)\hat{g})(\omega)$ implies $u(\hat{g}(\omega)) \geq (u\circ \hat{f}\, C_\mu(F)\hat{g})(\omega)$, and so $u(g^n(\omega)) \geq u(f^n(\omega))$, so in fact $u(f^n(\omega)) = u(g^n(\omega))$ for $\omega \notin C_\mu(F)$. Therefore, letting $\tilde{f}^n = f^n C_\mu(F) g^n$ for all n, the sequences $(\tilde{f}^n)_{n\geq 1}$ and $(g^n)_{n\geq 1}$ have the same properties as $(f^n)_{n\geq 1}$ and $(g^n)_{n\geq 1}$, and in addition $\tilde{f}^n(\omega) = g^n(\omega)$ for $\omega \notin C_\mu(F)$.

By Corollary 8 part 1, $\tilde{f}^n \prec_{C_\mu(F)} g^n$ for n large. Since $\tilde{f}^n(\omega) = g^n(\omega)$ for all $\omega \notin C_\mu(F)$, $\tilde{f}^n \prec g^n$ for such n. Since $\succ$ and $\succ'$ agree on $\mathscr{A}_0$, $\tilde{f}^n \prec' g^n$ for such n And since $u(\tilde{g}^n(\omega)) - u(\tilde{f}^n(\omega)) \leq u(\hat{g}(\omega)) - (u\circ \hat{f}\, C_\mu(F)\hat{g})(\omega)$ for every n and ω, Eq. (22) implies that $\hat{f}\, C_\mu(F)\hat{g} \prec' \hat{g}$, i.e., $\frac{1}{2}f + \frac{1}{2}x = \hat{f} \prec_{C_\mu(F)} \hat{g} = \frac{1}{2}g + \frac{1}{2}y$, or equivalently $\frac{1}{2}f + \frac{1}{2}x = \hat{f} \prec_{\cup\mathscr{F}(F)} \hat{g} = \frac{1}{2}g + \frac{1}{2}y$ because $C_\mu(F) = \cup\mathscr{F}(F)$, contradicting Asymmetry. ■

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