A Behavioral Characterization of Plausible Priors : Online Appendix

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March 6, 2005

Abstract

This Online Appendix contains (i) additional results and discussion related to the notion of plausible priors, and (ii) proofs that have been omitted from the main text in the interest of brevity. In particular, it presents two constructions that deliver a continuum of behaviorally equivalent representations for arbitrary MEU and Bewley preferences.

*An earlier draft of this paper was circulated under the title "Expected Utility with Many Unique Priors." I would like to thank Michèle Cohen, Eddie Dekel, Paolo Ghirardato, Faruk Gul, Alessandro Lizzeri, Fabio Maccheroni, Wolfgang Pesendorfer, and Pietro Veronesi, as well as several RUD 2002 participants, for helpful discussion. All errors are my own.

6 Online Appendix

This Online Appendix contains (i) additional results and discussion related to Axiom 6 and the notion of plausible priors (Sec. 6.2–6.5), and (ii) proofs that have been omitted from the main text in the interest of brevity (Sec. 6.6). See the paper for references to theorems, etc., as well as citations; also the numbering of sections and subsections continues the numbering in the paper.

6.1 Observations on Robust Mixture Neutrality

6.1.1 Alternative Formulation of Axiom 6 and the role of subsequences

Axiom 6 requires that every uniformly convergent sequence $f_k \to f$ admit a sub-sequence of mutually robustly mixture-meutral acuts. In light of the foregoing discussion, the following alternative formulation might seem appropriate: *for all sequences $f_k \to f$, there exists K such that $f_k \approx f$ for all $k \geq K$.*

This alternative formulation is problematic whenever, intuitively, the limiting act f lies in the intersection of sets of acts that the decision-maker evaluates employing different priors: that is, whenever $f \in C_n \cap C_m$ for $n \neq m$, where C_n and C_m are elements of the proper covering whose existence is asserted in Theorem 2.6. This may be seen even in the simple setting of Ex. 1. Let $f = (0, \frac{1}{2}, \frac{1}{2})$ and, for all $k \geq 1$, let

$$f_k = \left(0, \frac{1}{2} + \frac{(-1)^k}{k+1}, \frac{1}{2}\right).$$

It is clear that $f_k \to f$. However, while $f_k \simeq f$ for all k, there is no K such that $f_k \approx f$ for all $k \geq K$.On the other hand, it is clear that $f_k \approx f_\ell$ whenever k and ℓ are both even or both odd. Thus, by employing subsequences, the issues highlighted here are circumvented, while still capturing the intuition that acts that are uniformly close in preference do not provide hedging opportunities.

Observe that the simpler, alternative formulation must be discarded mainly because *robust* mixture neutrality must be employed to correctly reflect the absence of hedging opportunities for non-MEU preferences: the issues highlighted here do not arise if simple mixture neutrality is employed instead.

6.2 Alternative "Multiple-Priors" Representations of MEU and Bewley Preferences

The Introduction provides an example of a MEU preference (reproduced in Example 1) that also admits an α-MEU representation, characterized by a larger set of priors. The main message conveyed by the example is that, while, for any preference relation $\succeq$ satisfying the Gilboa-Schmeidler [14] axioms, there exists a unique (weak*-closed, convex) set of priors that yields a MEU representation of $\succeq$, there may be other sets of priors that yield different representations of the same preferences.

This section illustrates that such alternative representations, involving different sets of priors, can *always* be found for MEU preferences. To this end, I provide two constructions. Consider a MEU preference represented by a set $\mathcal{P}$ of priors. The first construction delivers α-MEU representations of the same preferences, characterized by strictly *larger* sets $\mathcal{Q}_\alpha$ of priors. The second construction delivers another representation of the same preferences, which employs strictly *smaller* sets of priors; in particular, if $\mathcal{P}$ is "symmetric" (defined below), the second representation is a generalized, "hyper-pessimistic" α-MEU rule, where the parameter α is allowed to take arbitrary positive values.

These results have implications for incomplete preferences that admit Bewley-type representations. To clarify, suppose $f \succeq g$ if and only if $\int u \circ f \, dp \geq \int u \circ g \, dp$ for all $p \in \mathcal{P}$, where $\mathcal{P}$ is a weak*-closed, convex set of priors. This condition is clearly equivalent to

$$\min_{p \in \mathcal{P}} \int [u \circ f - u \circ g] \, dp \geq 0.$$

Thus, both constructions described below can be immediately adapted to provide alternative representations of Bewley preferences. For instance, if S is finite and the set $\mathcal{P}$ satisfies the conditions of Proposition 6.1 below, for $\alpha < 1$ sufficiently large, there exists a set of priors $\mathcal{Q}_\alpha \supsetneq \mathcal{P}$ such that $f \succeq g$ if and only if

$$\alpha \min_{q \in \mathcal{Q}_\alpha} \int [u \circ f - u \circ g] \, dq + (1 - \alpha) \max_{q \in \mathcal{Q}_\alpha} \int [u \circ f - u \circ g] \geq 0.$$

If $\mathcal{P}$ is symmetric and S is arbitrary, an analogous representation holds, and $\mathcal{Q}_\alpha$ can be made arbitrarily small.

The first construction generalizes the intuition behind the example in the Introduction. For simplicity, I assume that S is finite (and non-singleton), so $\Sigma = 2^S$ and $B(S, \Sigma) = \mathbb{R}^S$. Denote by $\Delta(S)$ the set of probability distributions over S.

The following proposition states that all MEU preferences characterized by a set $\mathcal{P}$ strictly positive priors admit a continuum of equivalent α-MEU preferences, each characterized by a strictly larger collection $\mathcal{Q}_\alpha$ of probabilities. If the set $\mathcal{P}$ is *symmetric*, in the sense that there exists $p_0 \in \mathcal{P}$ such that $p \in \mathcal{P} \Rightarrow p_0 - (p - p_0) \in \mathcal{P}$, the result is straightforward: the appropriate construction is indicated in M. Siniscalchi, "Vector-Adjusted Expected Utility," Princeton Economic Theory Working Paper 01S3, 2001, Sec. 4.2.1; incidentally, this is the case in the Ellsberg example, where p_0 is the uniform distribution. Here, I show that the symmetry condition can be dispensed with, although the construction of the appropriate set $\mathcal{Q}_\alpha$ is somewhat less obvious.

Proposition 6.1 *Consider the functional $I : \mathbb{R}^S \to \mathbb{R}$ defined by*

$$I(a) = \min_{p \in \mathcal{P}} \int a \; dp$$

where $\mathcal{P} \subset \Delta(S)$. If, for some $\varepsilon > 0$, $\min_{s \in S, p \in \mathcal{P}} p(s) \geq \varepsilon$, then there exists a nonempty interval $(\bar{\alpha}, 1]$ such that, for all $\alpha \in (\bar{\alpha}, 1]$, there exists a set $\mathcal{Q}_\alpha \supset \mathcal{P}$ of priors such that, for all $a \in \mathbb{R}^S$,

$$I(a) = \alpha \min_{q \in \mathcal{Q}_\alpha} \int a \; dq + (1 - \alpha) \max_{q \in \mathcal{Q}_\alpha} \int a \; dq. \tag{15}$$

Proof. Note first that $\epsilon \leq \frac{1}{2}$, because S contains at least 2 points; thus, $1 - \epsilon \geq \frac{1}{2}$. I first claim that, for any $\alpha \in (1 - \epsilon, 1]$, and for any $p, p' \in \mathcal{P}$, there exist $q, q' \in \Delta(S)$ such that $\alpha q + (1 - \alpha) q' = p$ and $(1 - \alpha) q + \alpha q' = p'$. To see this, let $q = \frac{\alpha p - (1-\alpha) p'}{2\alpha - 1}$ and $q' = \frac{\alpha p' - (1-\alpha) p}{2\alpha - 1}$. Then $q, q' \in \mathbb{R}^S$ satisfy $\alpha q + (1 - \alpha) q' = p$, $(1 - \alpha) q + \alpha q' = p'$, and $\sum_s q(s) = \sum_s q'(s) = 1$. Furthermore, for any $s \in S$, $q(s) = \frac{\alpha p(s) - (1-\alpha) p'(s)}{2\alpha - 1} > \frac{(1-\epsilon)\epsilon - \epsilon p'(s)}{2\alpha - 1} = \frac{\epsilon[1 - \epsilon - p'(s)]}{2\alpha - 1} \geq 0$: the first inequality follows from $\alpha > 1 - \epsilon$ and $p(s) \geq \epsilon$, whereas the second follows from the assumption that there exists $s' \in S \setminus \{s\}$, so $1 - p'(s) \geq p'(s') \geq \epsilon$. Similar calculations show that $q'(s) > 0$; hence, $q, q' \in \Delta(S)$, as claimed.

Now, for any $\alpha \in (1 - \epsilon, 1]$, consider the set $\mathcal{Q}_\alpha \subset \mathbb{R}^S$ defined by

$$\mathcal{Q}_\alpha = \bigcup \left\{ \{q, q'\} : \sum_s q(s) = \sum_s q'(s) = 1, \; \alpha q + (1 - \alpha) q' \in \mathcal{P}, (1 - \alpha) q + \alpha q' \in \mathcal{P} \right\}.$$

The preceding claim implies that $\mathcal{Q}_\alpha \subset \Delta(S)$, and clearly $\mathcal{P} \subset \mathcal{Q}_\alpha$, $\mathcal{P} = \mathcal{Q}_1$ (let $q = q' = p$ for any $p \in \mathcal{P}$). Furthermore, $\mathcal{Q}_\alpha$ is convex: to see this, suppose that $q, \bar{q} \in \mathcal{Q}_\alpha$, so that there exist $q', \bar{q}' \in \mathcal{Q}_\alpha$ such that $\alpha q + (1 - \alpha) q' \in \mathcal{P}, (1 - \alpha) q + \alpha q' \in \mathcal{P}$ and also $\alpha \bar{q} + (1 - \alpha) \bar{q}' \in \mathcal{P}, (1 - \alpha) \bar{q} + \alpha \bar{q}' \in \mathcal{P}$. Then, for any $\lambda \in [0, 1]$,

$$\alpha[\lambda q + (1 - \lambda) \bar{q}] + (1 - \alpha)[\lambda q' + (1 - \lambda) \bar{q}'] = \lambda[\alpha q + (1 - \alpha) q'] + (1 - \lambda)[\alpha \bar{q} + (1 - \alpha) \bar{q}'] \in \mathcal{P}$$

and similarly

$$(1 - \alpha)[\lambda q + (1 - \lambda) \bar{q}] + \alpha[\lambda q' + (1 - \lambda) \bar{q}'] = \lambda[(1 - \alpha) q + \alpha q'] + (1 - \lambda)[(1 - \alpha) \bar{q} + \alpha \bar{q}'] \in \mathcal{P},$$

which imply that $\lambda q + (1-\lambda)\bar{q}, \lambda q' + (1-\lambda)\bar{q}' \in \mathcal{Q}_\alpha$, as required. Finally, the set $\mathcal{Q}_\alpha$ is closed: suppose $\{q_n\} \subset \mathcal{Q}_\alpha$ has limit $q \in \mathbb{R}^S$; for every n, there is $q'_n \in \mathcal{Q}_\alpha$ such that $\alpha q_n + (1-\alpha)q'_n, (1-\alpha)q_n + \alpha q'_n \in \mathcal{P}$. Possibly by considering a subsequence, $q'_n \to q' \in \mathbb{R}^S$; it is then clear that $q, q' \in \mathcal{Q}_\alpha$.

It remains to be shown that the representation in (15) holds, in particular for $\alpha \in (1-\epsilon, 1)$. To this end, fix $a \in \mathbb{R}^S$, and consider $q_a \in \arg\min_{q\in\mathcal{Q}_\alpha} \int a\, dq$. Then there exists $Q_a \in \mathcal{Q}_\alpha$ such that $p_a \equiv \alpha q_a + (1-\alpha)Q_a \in \mathcal{P}$ and $P_a \equiv (1-\alpha)q_a + \alpha Q_a \in \mathcal{P}$. I claim first that $p_a \in \arg\min_{p\in\mathcal{P}} \int a\ dp$. Suppose that $\int a\ dp < \int a\ dp_a$ for some $p \in \mathcal{P}$. By the initial claim, there exists $q, q' \in \Delta(S)$ such that $\alpha q + (1-\alpha)q' = p$ and $(1-\alpha)q + \alpha q' = P_\alpha$. Note that $p = \lambda q + (1-\lambda)P_\alpha$ and $p_a = \lambda q_a + (1-\lambda)P_a$, where $\lambda = \frac{2\alpha-1}{\alpha} \in (0,1)$; thus, $\int a\ dp < \int a\ dp_a$ implies that $\int a\ dq < \int a\ dq_a$, which contradicts the choice of q_a and proves the claim.

Next, I claim that $P_a \in \arg\max_{p\in\mathcal{P}} \int a\ dp$. Again, suppose that there exists $P \in \mathcal{P}$ such that $\int a\ dP > \int a\ dP_a$, and let $q, q' \in \Delta(S)$ such that $\alpha q + (1-\alpha)q' = p_a$ and $(1-\alpha)q + \alpha q' = P$. Now $p_a = \lambda q + (1-\lambda)P = \lambda q_a + (1-\lambda)P_a$, where $\lambda = \frac{2\alpha-1}{\alpha}$ as above. Since $\int a\ dP > \int a\ dP_a$, it must be the case that $\int a\ dq < \int a\ dq_a$, which again contradicts the choice of q_a.

Finally, I claim that $Q_a \in \arg\max_{q\in\mathcal{Q}_\alpha} \int a\ dq$. To see this, pick an element $Q \in \arg\max_{q\in\mathcal{Q}_\alpha} \int a\ dq$. Then it is also the case that $Q \in \arg\min_{q\in\mathcal{Q}_\alpha} \int (-a)\ dq$, so the previous argument implies that there exists q such that $\alpha Q + (1-\alpha)q \in \arg\min_{p\in\mathcal{P}} \int (-a)\ dp = \arg\max_{p\in\mathcal{P}} \int a\ dp$ and $(1-\alpha)Q + \alpha q \in \arg\max_{p\in\mathcal{P}} \int (-a)\ dp = \arg\min_{p\in\mathcal{P}} \int a\ dp$. Therefore, $\int a\ d[\alpha Q + (1-\alpha)q] = \int a\ d[(1-\alpha)q_a + \alpha Q_a]$ and $\int a\ d[(1-\alpha)Q + \alpha a] = \int a\ d[\alpha q_a + (1-\alpha)Q_a]$, which implies that $\int a\ dQ_a = \int a\ dQ$ and $\int a\ dq_a = \int a\ dq$, as required.

Therefore,

$$\begin{aligned}
\alpha \min_{q\in\mathcal{Q}_\alpha} \int a\ dq + (1-\alpha) \max_{q\in\mathcal{Q}_\alpha} \int a\ dq &= \alpha \int a\ dq_a + (1-\alpha) \int a\ dQ_a = \\
&= \int a\ d[\alpha q_a + (1-\alpha)Q_a] = \\
&= \int a\ dp_a = \\
&= \min_{p\in\mathcal{P}} \int a\ dp = \\
&= I(a).
\end{aligned}$$

■

The second construction applies to arbitrary state spaces, and is simple to describe. For symmetric sets of priors, the representation one obtains has the functional form of the α-

MEU model, but α is allowed to take arbitrary non-negative values; it might be (informally!) interpreted as suggesting a form of "hyper-pessimism". For arbitrary sets of priors, it is a linear combination of a minimum EU term with a reference expectation. The construction is interesting mainly because it yields a *strictly smaller* set of priors than the one appearing in the MEU representation. Indeed, this set can be made arbitrarily small.

Thus, assume S is a (finite or infinite) set, and Σ an algebra of subsets of S. As above, let $I : B(S,\Sigma) \to \mathbb{R}$ be defined by $I(a) = \min_{p\in\mathcal{P}} \int a\,dP$ for a weak*-closed, convex set $\mathcal{P}$ of probability charges on (S,Σ). Fix an arbitrary point $p_0 \in \mathcal{P}$, and an arbitrary $\alpha \in (0,1)$; let

$$\mathcal{Q}_\alpha = \{\alpha p + (1-\alpha)p_0 \;:\; p \in \mathcal{P}\}.$$

Clearly, $\mathcal{Q}_\alpha$ is weak*-closed and convex; furthermore, $\mathcal{Q}_\alpha$ is a strict subset of $\mathcal{P}$. Then:

$$\begin{aligned}
&\frac{1}{\alpha}\min_{q\in\mathcal{Q}_\alpha}\int a\,dq + \left(1-\frac{1}{\alpha}\right)\int a\,dp_0 = \\
&= \frac{1}{\alpha}\min_{p\in\mathcal{P}}\int a\,d[\alpha p + (1-\alpha)p_0] + \left(1-\frac{1}{\alpha}\right)\int a\,dp_0 = \\
&= \min_{p\in\mathcal{P}}\int a\,dp + \frac{1-\alpha}{\alpha}\int a\,dp_0 + \left(1-\frac{1}{\alpha}\right)\int a\,dp_0 = \\
&= \min_{p\in\mathcal{P}}\int a\,dp = I(a).
\end{aligned}$$

The essential point is that, since $p_0 \in \mathcal{Q}_\alpha$, it is still the case that the representation one obtains is based on a "multiple-priors" rule, in the sense that the collection of integrals $\{\int a\,dq\}_{q\in\mathcal{Q}_\alpha}$ fully determines the evaluation of a.

If the set $\mathcal{P}$ is symmetric around p_0, then so is $\mathcal{Q}_\alpha$, and it can be shown that $\min_{q\in\mathcal{Q}_\alpha}\int a\,dq + \max_{q\in\mathcal{Q}_\alpha}\int a\,dq = 2\int a\,dp_0$. In this case, the above representation reduces to a generalized β-MEU rule, with $\mathcal{Q}_\alpha$ as set of priors and $\beta = \frac{1}{2} + \frac{1}{2\alpha}$.

6.3 Probabilistic Sophistication

As noted in the Introduction, an alternative definition of "plausible prior" might require that preferences over acts in some subset C of L be consistent with some probabilistically sophisticated non-EU preference functional.

While this is an interesting extension, certain difficulties must be addressed. Recall that the intuitive interpretation of multiple-prior decision models suggests that individuals might react to ambiguity by evaluating different acts using different priors; it does not suggest that their *risk attitudes* should also change, depending on the act being evaluated. Indeed, holding risk attitudes fixed strengthens the intuitive connection between ambiguity and multiplicity of priors.[36]

Hence, in order to properly extend this intuition while allowing for non-EU risk attitudes, a two-step procedure seems necessary: given a preference relation $\succeq$, first elicit a suitable Machina-Schmeidler preference functional $\mathcal{V}$ over probability distributions on outcomes; then, deem a probability charge P a $\mathcal{V}$-plausible prior iff there is a set $C \subset L$ such that preferences over C are represented by associating with any (simple) act f in C the number $\mathcal{V}(P \circ f^{-1})$,[37] and P is the only probability charge with this property.

If actual objective lotteries are available, as in the original Anscombe-Aumann framework, the functional $\mathcal{V}$ can be derived from preferences over constant acts (cf. Machina and Schmeidler [25]); this would at least make it possible to define $\mathcal{V}$-plausible priors, as above. But in a fully subjective decision framework, even this preliminary step is likely to prove challenging. [Again, recall that objective lotteries do *not* play an essential role in the present paper.] In either case, ensuring uniqueness of the plausible priors seems non-trivial.

6.4 Conditional Evaluations

The discussion in Section 2.2.3 involves the notion of "conditional evaluation" of an act given a set E. In particular, the intuition for the key behavioral assumption of this paper, Axiom 6, invokes the assumption that conditional evaluations satisfy Axioms 1–5. Prior-by-prior updating of MEU preferences clearly satisfies this condition; more generally, the discussion following Theorem 3.2 implies that this is the case for prior-by-prior updating of general plausible-priors preferences. This is also the case for h-Bayesian updating à la Gilboa and Schmeidler [15].

One caveat to that discussion is that Bayesian updating, as well as h-Bayesian updating, is well-defined for acts that are suitably "non-null". For prior-by-prior updating, the relevant notion is Def. 3.1; for all MEU and all plausible-priors preferences, this implies that $P(E) > 0$ for all probability charges P in the representation.

Now say that an event E is *Savage-null* if, for all $f, g \in L$, $f(s) \sim g(s)$ for all $s \notin E$ implies $f \sim g$. Clearly, a Savage-null event is "not non-null" ("null" for short), but the

[36]On the other hand, contemplating changes in risk preferences is an intriguing possibility; Chew and Sagi ("Small Worlds: Modelling Attitudes towards Sources of Uncertainty," mimeo, October 2003) have recently suggested that it may lead to a novel interpretation of the Ellsberg paradox and related phenomena. But since the focus of the present paper is on the "traditional" view of such behavior patterns, it seems important to hold risk preferences fixed.

[37]More explicitly, if f delivers the outcomes $y_1, \ldots, y_I$, $P \circ f^{-1}$ is the distribution that assigns probability $P(f^{-1}(y_i))$ to each y_i, for $i = 1, \ldots, I$.

converse is not true. If a preference has a MEU or a plausible-priors representation, and E is Savage-null, then $P(E) = 0$ for all relevant priors P.

It is intuitively clear that hedging opportunities w.r.to Savage-null events are irrelevant: these events simply play no role in the decision-maker's evaluation of acts. On the other hand, events that are not Savage-null are, by definition, relevant for *some* comparisons of acts; therefore, hedging with respect to such events may be behaviorally significant.

The objective of this subsection is to show that the characterization of Bayesian updating in Section 3.1 can actually be extended to events that are null, but not Savage-null. In particular, the resulting notion of conditional evaluation of acts is continuous; hence, the discussion in Section 2.2.3 applies to hedging relative to such events as well.

As a final introductory note, while the analysis in this subsection focuses on Bayesian updating, it is conceivable that similar constructions may be carried out for other updating rules, under suitable conditions.

Section 3.1 characterizes Bayesian updating with respect to non-null events by means of two axioms, Consequentialism (Axiom 7) and Dynamic Constant-act Consistency (Axiom 8); incidentally, as noted in Footnote 14, while the analysis therein focuses on plausible-priors preferences, the same characterization result applies to MEU preferences that do not satisfy the plausible-priors axioms.

As the proof of Theorem 3.2 shows, conditional preferences that satisfy Axioms 7 and 8 can equivalently be described as follows: f is preferred to f' conditional upon E if and only if $y \succeq y'$, where

$$y \sim fEy, \quad y' \sim f'Ey'.$$

In other words, conditional preferences are characterized by the above *fixpoint conditions*, which may be viewed as a *definition of conditional preferences* given E, as well as of *conditional evaluations* of the acts involved. Notice that the functional counterpart of the above fixpoint condition for an arbitrary $a \in B(S, \Sigma)$ is

$$x = I(aEx), \quad \text{or equivalently} \quad I(1_E[a - x]) = 0,$$

where $x \in \mathbb{R}$. For any unconditional preference that satisfies Axioms 1–5 (including all MEU, α-MEU, and plausible-prior preferences), Lemma 5.19 shows that, if E is non-null, then the equation $I(1_E[a - c]) = 0$ has a *unique* solution: in other words, conditional preferences are well-defined with respect to non-null events. Furthermore, this unique solution is monotonic, normalized, and constant-linear, hence continuous, as a function of a: this means that conditional preferences defined using the above fixpoint conditions will satisfy Axioms 1–5, as required. *This subsection argues that, for many preferences of interest, this fixpoint* *characterization of prior-by-prior Bayesian updating can be generalized to all non-Savage-null events*. Specifically, it will be shown that

1. for MEU (and, similarly, 0-MEU) preferences, a simple modification of the fixpoint condition allows one to obtain a characterization of prior-by-prior updating with respect to non-Savage-null, but possibly null events;

2. for α-MEU preferences, with $\alpha \in (0,1)$, there is a unique solution to the fixpoint condition even if the event E is null (but not Savage-null).

3. finally, for plausible-prior preferences, if the conditioning event is null, but not Savage-null, it is possible to construct a selection from the correspondence associating with each act the set of solutions to the fixpoint condition, so as to define a condit ional preference that satisfies Axioms 1–5.

In all three cases, the corresponding notion of conditional evaluation satisfies all the properties referred to in Section 2.2.3, including the Consequentialism axiom invoked in Ex. 3.

For notational simplicity, as in the proof of Theorem 2.6, $P(a) = \int a\,dP$ for any $a \in B(S,\Sigma)$ and probability charge P on (S,Σ).

1. Consider a set $\mathcal{Q}$ of probability charges on (S,Σ), and an event $E \subset \Sigma$ such that $Q^*(E) > 0$ for some $Q^* \in \mathcal{Q}$. For a MEU (or 0-MEU, i.e. "maxmax") preference $\succeq$ with priors $\mathcal{Q}$, it is possible to define preferences conditional upon E, denoted $\succeq_E$, simply by letting $\mathcal{Q}_E = \text{cl}\,\{Q(\cdot|E) : Q \in \mathcal{Q}, Q(E) > 0\}$ (where the closure is taken w.r.to the weak* topology on $ba(S,\Sigma)$) and considering the MEU decision rule characterized by the set $\mathcal{Q}_E$ of posterior probabilities. Similarly, conditional evaluations of functions $a \in B(S,\Sigma)$ are readily defined by $x(a,E) = \min_{Q\in\mathcal{Q}_E} \int a\,dQ$.

Suppose that $Q^0(E) = 0$ for some $Q^0 \in \mathcal{Q}$. I claim that then

$$x(a,E) = \sup\{x : I(1_E(a-x)) = 0\} \quad \text{and} \quad \inf\{x : I(1_E(a-x)) = 0\} = -\infty$$

(for maxmax preferences, replace "sup" and "inf" with "inf" and "sup", and "$-\infty$" with "$+\infty$"). Note first that the proof of Lemma 5.19 shows that $\{x : I(1_E(a-x)) = 0\}$ is an interval. Furthermore, this set is bounded above, because for $x = \sup_{s\in E} a(s) + 1$, since $Q^*(E) > 0$, $I(1_E(a-x)) \le Q^*(1_E(a-x)) = -Q^*(E) + Q^*(1_E(a - \sup_{s\in E} a)) \le -Q^*(E) < 0$. Finally, since $Q^0(E) = 0$, for any $M > 0$, if $x = \inf_{s\in E} a(s) - M$ then $1_E(a-x) \ge M > 0$; hence, $I(1_E(a-x)) = Q^0(1_E(a-x)) = 0$, so there are arbitrarily small solutions to the equation $I(1_E(a-x))$. Thus, by continuity, $\{x : I(1_E(a-x)) = 0\} = (-\infty, x]$, for some

$x < \infty$. This proves the second claim, and it remains to be shown that $x = x(a, E)$, as defined above.

To see this, note that, for $Q \in \mathcal{Q}$ such that $Q(E) > 0$, $0 = I(1_E(a-x)) \leq Q(1_E(a-x)) = Q(1_E a) - Q(E)x$, so $x \leq Q(a|E)$. If $\{Q_k\} \subset \mathcal{Q}$, $Q_k(E) > 0$ for all k, and $Q_k(\cdot|E) \to Q_E$ in the weak* topology, then $Q_E(a) = \lim_k Q_k(a|E) \geq x$. Therefore, $x \leq \min_{Q \in \mathcal{Q}_E} Q(a) = x(a, E)$. Moreover, if, for some $\epsilon > 0$, $x + \epsilon \leq Q(a|E)$ for all $Q \in \mathcal{Q}$ with $Q(E) > 0$, then $I(1_E(a - (x + \epsilon))) = Q^0(1_E(a - (x + \epsilon))) = 0$, which contradicts the definition of x. Thus, $x = \inf_{Q \in \mathcal{Q}:Q(E)>0} Q(a|E) \geq \inf_{Q_E \in \mathcal{Q}_E} Q_E(a) = \min_{Q_E \in \mathcal{Q}_E} Q_E(a)$. Hence $x = x(a, E)$.

2. Let $\mathcal{Q}$, Q^*, Q^0 be as above. Fix $\alpha \in (0, 1)$ and consider an α-MEU preference with priors $\mathcal{Q}$. Note that Q minimizes $\bar{Q}(aEx)$ iff it minimizes $\bar{Q}(1_E(aEx))$ for $\bar{Q} \in \mathcal{Q}$, because $1_E(aEx) = aEx - 1_S x$. Now suppose $x = I(aEx)$ and $x' = I(aEx')$, with $x' \geq x$, and denote by q, Q respectively the minimizers and maximizers of $\bar{Q}(aEx)$; similarly, denote by q', Q' the minimizers and maximizers of $\bar{Q}(aEx')$. Then

$$\begin{aligned} x &= \alpha q(aEx) + (1 - \alpha)Q(aEx) \\ x' &= \alpha q'(aEx') + (1 - \alpha)Q'(aEx') \end{aligned}$$

or equivalently

$$\begin{aligned} 0 &= \alpha q(1_E(a - x)) + (1 - \alpha)Q(1_E(a - x)) \\ 0 &= \alpha q'(1_E(a - x')) + (1 - \alpha)Q'(1_E(a - x')). \end{aligned}$$

Now suppose that $q(1_E(a - x)) > q'(1_E(a - x'))$. Then $Q(1_E(a - x)) < Q'(1_E(a - x')) \leq Q'(1_E(a - x))$, which contradicts the fact that Q maximizes $\bar{Q}(1_E(a - x))$. Suppose instead $q(1_E(a - x)) < q'(1_E(a - x'))$; now $q(1_E(a - x')) \leq q(1_E(a - x)) < q'(1_E(a - x'))$, which contradicts the fact that q' minimizes $\bar{Q}(1_E(a - x'))$. Thus $q(1_E(a - x)) = q'(1_E(a - x')) \leq q(1_E(a - x'))$, i.e. $q(1_E a) - q(E)x \leq q(1_E a) - q(E)x'$, i.e. $(x' - x)q(E) \leq 0$; hence, $q(E) = 0$, which implies $Q(1_E(a - x)) = 0$.

Thus, $q(aEx) = Q(aEx) = x$; now observe that, if $\bar{Q}(E) > 0$ and $\bar{Q}(a|E) < x$, then $\bar{Q}(aEx) < x = q(aEx)$, a contradiction; similarly, if $\bar{Q}(E) > 0$ and $\bar{Q}(a|E) > x$, then $\bar{Q}(aEx) > x = Q(aEx)$, also a contradiction. Hence, $\bar{Q}(a|E) = x$ for all $\bar{Q} \in \mathcal{Q}$ with $\bar{Q}(E) > 0$; in particular, $Q^*(a|E) = x$.

Moreover, $q(E) = 0$ also implies that $q'(1_E(a - x') = q(1_E(a - x)) = 0$, and hence also $Q'(1_E(a - x')) = 0$, so $q'(aEx') = Q'(aEx') = x'$. By the argument just given, $Q^*(a|E) = x'$. But this contradicts the fact that $x' > x$.

3. Turning to plausible-priors preferences, denote the relevant probabilities by $P_1, \dots, P_N$, and fix a non-Savage-null event $E \in \Sigma$.

Observe that $I(aEx) = x$ iff $I((\lambda a)E(\lambda x)) = \lambda x$ for all $\lambda > 0$; hence, it is wlog to restrict attention to $a \in B(S,\Sigma)$ such that $\|a\| \leq 1$. For the purposes of this proof, it is useful to restrict $x(a,E)$ to the unit ball of $B(S,\Sigma)$, and redefine it as $x(a,E) = \{x \in [-1,1] : I(aEx) = x\}$; this avoids notational complications that arise e.g. with MEU preferences, for which $x(a,E)$ may be unbounded.

Note first that, for all preferences satisfying Axioms 1–5, the correspondence $a \mapsto x(a,E) \subset \mathbb{R}$ is closed- and convex-valued, as well as upper hemicontinuous. To see this, note that if $I(aEx_k) = x_k \in [-1,1]$ for all k and $x_k \to x$, then $x \in [-1,,1]$ and $I(aEx) = \lim_k I(aEx_k) = \lim_k x_k = x$; also, if $x > x'$ and $I(1_E(a-x)) = I(1_E(a-x')) = 0$, then for all $x'' \in [x',x]$, $1_E(a-x) \leq 1_E(a-x'') \leq 1_E(a-x')$, so by monotonicity of I, $I(1_E(a-x'')) = 0$. Finally, if $a_k \to a$, $x_k \to x$, and $I(a_kEx_k) = x_k$ for all k, by continuity of I, $I(aEx) = \lim_k I(a_kEx_k) = \lim_k x_k = x$ [because $a_kEx_k \to aEx$ in norm]. Thus, $x \in x(a,E)$; since X is compact-valued, it is upper hemicontinuous.

Next, for plausible-priors preferences, $x(\cdot,\cdot)$ is also *lower*-hemicontinuous. To see this, suppose that $a_k \to a$, but there is $x^* \in x(a,E)$ such that, for all subsequences $\{k(\ell)\}$, $\{x_{k(\ell)}\} \subset [-1,1]$ and $x_{k(\ell)} \in x(a_{k(\ell)},E)$ for all ℓ implies $x_{k(\ell)} \not\to x^*$.

Let $\bar{x}(a,E) = \sup\{x : \exists\{k(\ell)\},\{x_{k(\ell)}\} \text{ s.t. } x_{k(\ell)} \in x(a_{k(\ell)},E), x_{k(\ell)} \to x\} \in x(a,E)$, and for definiteness, suppose $x^* > \bar{x}(a,E)$. Then, for any $\lambda \in (0,1]$, $\lambda x^* + (1-\lambda)\bar{x}(a,E) \notin x(a_k,E)$, except possibly for finitely many k's; otherwise, $\bar{x}(a,E)$ would not be the supremum of all points in $x(a,E)$ that are limits of points in $x(a_k,E)$.

Now let $x^1 = x^*$ and consider the sequence $\{a_kEx^1\} \to aEx^1$; there is n such that $a_{k_1(\ell)}Ex^1 \in C_n$ for a suitable subsequence $\{a_{k_1(\ell)}\}$, and since $a_{k_1(\ell)}Ex^1 \to aEx^1$, $aEx^1 \in C_n$ as well: for notational simplicity, suppose $n = 1$. Then $P_1(E) > 0$: otherwise, $I(a_{k_1(\ell)}Ex^1) = P_1(1_Ea_{k_1(\ell)}) + [1-P(E)]x^1 = x^1$, i.e. $x^1 \in x(a_{k_1(\ell)},E)$ for all ℓ. Now, arguing by induction, suppose that, for $m = 1,\dots,n$, the points $x^m = \frac{1}{m}x^* + (1-\frac{1}{m})\bar{x}(a,E)$ have been defined in such a way that $aEx^m \in C_m$, $aEx^m \notin C_{m'}$ for any $m' \in \{1,\dots n\} \setminus \{m\}$, and furthermore $P_m(E) > 0$ [as above, the assumption that the priors corresponding to each $x^1,\dots,x^n$ are $P_1,\dots,P_n$ is merely for notational simplicity]. To complete the inductive step, consider x^{n+1}. There is a subsequence $\{a_{k_{n+1}(\ell)}\}$ such that $a_{k_{n+1}(\ell)}Ex^{n+1} \in C_{n^*}$ for some $n^* \in \{1,\dots,N\}$; it must be the case that $P_{n^*}(E) > 0$, because otherwise $I(a_{k_{n+1}(\ell)}Ex^{n+1}) = P_{n^*}(a_{k_{n+1}(\ell)}Ex^{n+1}) = x^{n+1}$, which contradicts the fact that $x^{n+1} \notin x(a_{k_{n+1}(\ell)},E)$. The key observation is that, furthermore, $n^* \notin \{1,\dots,n\}$. To see this, suppose that $n^* = m \in \{1,\dots,n\}$; then $aEx^m \in C_m$ and also $aEx^{n+1} \in C_m$; since $x^m = I(aEx^m) = P_m(aEx^m)$ and $P_m(E) > 0$, $x^m = P_m(a|E)$; similarly, $x^{n+1} = I(aEx^{n+1}) = P_m(aEx^{n+1})$ and $P_m(E) > 0$ imply $x^{n+1} = P_m(a|E)$: since $x^m \neq x^{n+1}$, a contradiction is obtained. Thus, $n^* > n$, and for notational simplicity it can be assumed that $n^* = n+1$.

To summarize, if $x(\cdot,\cdot)$ is not lower hemicontinuous, then there exists a procedure that, for all $n \geq 1$, selects a point $x^n \in x(a,E)$ such that $aEx^n \in C_n$ and $aEx^n \notin C_m$ for any $m \neq n$. But this contradicts the fact that there are finitely many sets C_n.

Finally, consider the function $x^*(a,E) : B(S,\Sigma) \to \mathbb{R}$ defined by $a \mapsto \max x(a,E) \cap [\inf_E a(s), \sup_E a(s)]$. If $a = 1$, then $I(1\,E\,1) = 1$, so $x^*(1,E) = 1$, i.e. $x^*(\cdot,E)$ is normalized. If $a(s) \geq b(s)$ for all s and $\xi = x^*(b,E)$, then $I(aE\xi) \geq I(bE\xi) = \xi$, so $x^*(a,E) \geq x^*(b,E)$, i.e. $x^*(\cdot,E)$ is monotonic. Finally, it is clear that $x^*(\cdot,E)$ is also c-linear. Hence, if conditional preferences are defined via x^*, they satisfy Axioms 1–5, as needed.

A simple preference definition of x^* is as follows: y is the conditional evaluation of f iff (i) $y' \succeq f(s) \succeq y''$ for all s implies $y' \succeq y \succeq y''$; (ii) $y \sim fEy$; and (iii) if z satisfies (ii), then $y \succeq z$. [Of course, a dual definition, wherein "max" is replaced by "min" and condition (iii) is suitably modified, is also possible]. I have not investigated a modification of Axiom 8 that delivers (i)–(iii), although it should be possible to formulate one such axiom.

6.5 Alternative definitions of the set L of acts

Let $B_L(S,\Sigma) = \{\gamma u \circ f : \gamma \geq 0, f \in L\}$. Given the definition of L in the text, the assumption that Σ is sigma-algebra ensures that $B(S,\Sigma) = B_L(S,\Sigma)$. If instead Σ is an algebra, then $B_L(S,\Sigma$ is a (dense) proper subset of $B(S,\Sigma)$. However, none of the results depends on the fact that Σ is a sigma-algebra—except of course for Proposition 3.4, which explicitly assumes that it is.

To clarify (cf. the proof of Lemma 5.1), recall that, if a preference relation $\succeq$ satisfies Axioms 1–5) on the set L_0 of simple Σ-measurable acts, then it has a unique extension to the set L of Σ-measurable acts—and indeed, in case Σ is not a sigma-algebra, to the set of all maps $f : S \to Y$ such that $u \circ f \in B(S,\Sigma)$. The question is whether the unique extension of a preference that satisfies Axioms 1–5 *and* Axiom 6 on L_0 extends to a preference that also satisfies Axiom 6 on L. While I conjecture that this is the case, I have been unable to provide a proof. Thus, while it would be sufficient to impose Axioms 1–5 on preferences over L_0, in order to obtain a representation over a suitably larger set of acts, Axiom 6 must be required to hold for all acts in such set.

Now, if Σ is only an algebra, two approaches are possible. The first is to define L so as to ensures that $B_L(S,\Sigma) = B(S,\Sigma)$; in this case, all results in the paper hold as stated. This can be achieved as follows. First, define $\succeq$ as a relation on L_0, and assume that it satisfies Axioms 1–5 on that set. Then, say that a function $f \in Y^S$ is *($\Sigma, \succeq$)-continuous* iff, for all $y, y' \in Y$ such that $y \succ y'$, there exists a finite partition $F_1, \ldots, F_I$ of S such that, for every

$i = 1, \ldots, I$, $F_i \in \Sigma$ and, for all $s, s' \in F_i$,

$$\frac{1}{2}f(s) + \frac{1}{2}y' \preceq \frac{1}{2}f(s') + \frac{1}{2}y \quad \text{and} \quad \frac{1}{2}f(s') + \frac{1}{2}y' \preceq \frac{1}{2}f(s) + \frac{1}{2}y$$

(cf. Eq. (3) in the text). Then let L be the set of all $(\Sigma, \succeq_0)$-continuous acts, and consider the (unique) extension of $\succeq$ to L, also denoted $\succeq$ for simplicity. Finally, require that $\succeq$ satisfy Axiom 6 on L.

This achieves the intended result: the set $B_L(S, \Sigma)$ corresponding to L is the collection of all Σ-*continuous* functions $a : S \to \mathbb{R}$: that is, for every $\epsilon > 0$, there exists a finite partition $\{F_1, \ldots, F_I\} \subset \Sigma$ of S such that, for all i and $\epsilon > 0$, $s, s' \in F_i$ implies $|a(s) - a(s')| < \epsilon$. By Proposition 4.7.2 in K.P.S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges*, Academic Press, London, 1983, $B_L(S, \Sigma) = B(S, \Sigma)$. Thus, the analysis in the paper applies verbatim, and all results hold as stated.

An alternative approach is to be content with representation on a smaller set of acts. If Σ is an algebra and the definition of the set L of acts is as in the text, Theorem 2.6 (and also Theorem 3.2) remains true as stated, provided $B(S, \Sigma)$ is replaced by $B_L(S, \Sigma)$ in the definition of proper covering; in particular, the closure and interior of sets are taken w.r.to the relative topology. The proof given in the text is still valid, again provided the interior and closure of sets are suitably reinterpreted.

Indeed, the same is true if L is replaced with the set L_0 of simple Σ-measurable acts; in this case, $B(S, \Sigma)$ must be replaced with the set $B_0(S, \Sigma)$ of simple functions, i.e. linear combinations of indicator functions of sets in Σ. Furthermore, since $B_0(S, \Sigma)$ is also dense in $B(S, \Sigma)$, the same plausible priors are elicited by restricting attention to L_0 or L.

Thus, in any case, the analysis in the paper still applies. The formulation chosen for the main text allows for simpler definitions and statement of the results.

6.6 Main Results: Omitted Proofs

6.6.1 Proof of Lemma 5.1 (sketch)

The "if" part is obvious. For the converse, Gilboa and Schmeidler [14], Lemmata 3.1-3.4 establish the existence of a non-constant, affine u and a unique, normalized, monotonic, c-linear functional I that represents $\succeq$ on $L_0 \subset L$. It is easy to see that I, u actually represent $\succeq$ on the entire set L.[38] For the last claim, see e.g. the proof of Lemma 3.4 in [14].

[38] Suppose $I(u \circ f) > I(u \circ g)$ (the argument for the opposite strict inequality is symmetric). There exist sequences $\{f^n\}_{n \geq 1}, \{g^n\}_{n \geq 1} \subset L_0$ such that $u \circ f^n \uparrow u \circ f$ and $u \circ g^n \downarrow u \circ g$, so for n large, $I(u \circ f^n) > I(u \circ g^n)$;

6.6.2 Proof of Lemma 5.2

Begin with Part 1. In terms of the representation, $f_k \to f$ is equivalent to

$$\begin{aligned} \alpha u(f_k(s)) + (1-\alpha)u(y') &\le \alpha u(f(s)) + (1-\alpha)u(y) \\ \alpha u(f(s)) + (1-\alpha)u(y') &\le \alpha u(f_k(s)) + (1-\alpha)u(y) \end{aligned}$$

and rearranging terms yields

$$\begin{aligned} u(f_k(s)) - u(f(s)) &\le \tfrac{1-\alpha}{\alpha}[u(y) - u(y')], \\ u(f(s)) - u(f_k(s)) &\le \tfrac{1-\alpha}{\alpha}[u(y) - u(y')] \end{aligned}$$

which is equivalent to

$$|u(f_k(s) - u(f(s))| \le \frac{1-\alpha}{\alpha}[u(y) - u(y')];$$

in turn, this holds for all $s \in S$ if and only if

$$\|u \circ f_k - u \circ f\| \le \frac{1-\alpha}{\alpha}[u(y) - u(y')].$$

Now suppose that $f_k \to f$. Fix $\epsilon > 0$; by non-degeneracy, there exist y, y' such that $y \succ y'$; choose α so $\frac{1-\alpha}{\alpha}[u(y) - u(y')] = \frac{1}{2}\epsilon$, i.e. $\alpha = \frac{u(y)-u(y')}{\frac{1}{2}\epsilon + u(y) - u(y')} \in (0,1)$, so $f_k \to f$ implies that $\|u \circ f_k - u \circ f\| < \epsilon$ for k large; that is, $u \circ f_k \to u \circ f$ in $B(S, \Sigma)$. Conversely, suppose the latter assertion is true, and fix y, y', α as in the definition. Since $u \circ f_k \to u \circ f$ in norm, for k large, $\|u \circ f_k - u \circ f\| < \epsilon = \frac{1-\alpha}{\alpha}[u(y) - u(y')]$, which implies that $f_k \to f$.

Part 2 is trivial. For 3, let $\gamma \in (0,1)$ and note that, by c-linearity of I, $I(\gamma a + (1-\gamma)[\alpha b + \beta]) = I(\gamma a + (1-\gamma)\alpha b) + (1-\gamma)\beta = \Gamma I(\frac{\gamma}{\Gamma}a + \frac{(1-\gamma)\alpha}{\Gamma}b) + (1-\gamma)\beta = \gamma I(a) + (1-\gamma)\alpha I(b) + (1-\gamma)\beta = \gamma I(a) + (1-\gamma)I(\alpha b + \beta)$, where $\Gamma = \gamma + (1-\gamma)\alpha > 0$.

For 4, note that the claim is trivial for $\alpha = \beta = 0$ because I is c-linear. Assuming that one of α, β is positive, by Part 3 it is enough to consider the case $\beta = 1 - \alpha$. Let $\gamma \in [0,1]$; then $I(\gamma a + (1-\gamma)[\alpha a + (1-\alpha)b) = I([\gamma + (1-\gamma)\alpha)a + (1-\gamma)(1-\alpha)b) = [\gamma + (1-\gamma)\alpha]I(a) + (1-\gamma)(1-\alpha)I(b) = \gamma I(a) + (1-\gamma)[\alpha I(a) + (1-\alpha)I(b)] = \gamma I(a) + (1-\gamma)I(\alpha a + (1-\alpha)b)$, where the third equality follows from $\gamma + (1-\gamma)\alpha = 1 - (1-\gamma)(1-\alpha)$ and $a \simeq b$, and the fifth from $a \simeq b$.

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since I, u represent $\succeq$ on L_0, $f \succeq f^n \succ g^n \succeq g$. Next, suppose $I(u \circ f) = I(u \circ g)$. Arguing by contradiction, assume $f \succ g$ (the case $f \prec g$ is symmetric). By C-Independence and Weak Order, if $y_f \sim f$ and $y_g \sim g$, then $\frac{1}{2}f + \frac{1}{2}y_g \sim \frac{1}{2}g + \frac{1}{2}y_f$. By the preceding result, $I\left(\frac{1}{2}u \circ f + \frac{1}{2}u(y_g)\right) \sim I\left(\frac{1}{2}u \circ g + \frac{1}{2}u(y_f)\right)$. By C-Linearity and the assumption that $I(u \circ f) = I(u \circ g)$, $u(y_f) = u(y_g)$; since u represents $\succeq$ on L_c, this implies $y_f \sim y_g$, a contradiction.

For 5, note that, by the sup-norm continuity of I, for every $\alpha \in [0,1]$, $I(\alpha a + (1-\alpha)b) = \lim_k I(\alpha a_k + (1-\alpha)b_k) = \alpha \lim_k I(a_k) + (1-\alpha)\lim_k I(b_k) = \alpha I(a) + (1-\alpha)I(b)$.

For 6, suppose $u \circ f \approx u \circ g$, consider $\gamma \in [0,1]$, and $\{h_k\} \to f$. Then $\{u \circ h_k\} \to u \circ f$, so there exists K such that $k \geq K$ implies $u \circ h_k \simeq \gamma u \circ f + (1-\gamma)u \circ g = u \circ [\gamma f + (1-\gamma)g]$, hence $h_k \simeq \gamma f + (1-\gamma)g$ by Part 2, as required; similarly for $h_k \to g$. Next, suppose $f \approx g$, fix $\gamma \in [0,1]$, and consider $\{c_k\} \to u \circ f$; here we must consider two cases.

First, suppose f is degenerate, i.e. $u \circ f$ is constant. Since $\succeq$ is non-degenerate, there exist $y, y' \in Y$ such that $y \succ f \succ y'$, $y \sim f \succ y'$, or $y \succ f \sim y'$. In the first subcase, since $c_k \to u \circ f$, for k large it must be the case that $u(y) \geq c_k(s) \geq u(y')$ for all $s \in S$; but then, for k large, there exists $\{h_k\} \subset L$ such that $u \circ h_k = c_k$, and $f \approx g$ implies $h_k \simeq \gamma f + (1-\gamma)g$ for k large, hence $c_k \simeq \gamma u \circ f + (1-\gamma)u \circ g$ by Part 2 and the fact that u is affine. Next, consider the second subcase, i.e. $y \sim f \succ y'$ (the third subcase can be dealt with symmetrically). Since $c_k \to u \circ f$, and $u \circ f$ is the constant function that assigns $u(y)$ to each state s, it must be the case that $\sup c_k - \inf c_k \to 0$ [recall that sup and inf are monotonic and constant-linear, hence sup-norm continuous functionals on $B(S,\Sigma)$]; hence, for k large, $\sup c_k - \inf c_k < u(y) - u(y')$. For such large k, let $c'_k = c_k - \sup c_k + u(y)$. Then $u(y) \geq c'_k(s) \geq u(y')$ for all s, so there exists h_k such that $u \circ h_k = c'_k$; furthermore, $c'_k \to u \circ f$ [again because sup is continuous]. Therefore $f \approx g$ implies $h_k \simeq \gamma f + (1-\gamma)g$ for k large; by Part 2, $c'_k \simeq \gamma u \circ f + (1-\gamma)u \circ g$, and by Part 3, for every such k, $c_k \simeq \gamma u \circ f + (1-\gamma)u \circ g$, as required. This completes the analysis of this case.

Now suppose f is not degenerate; then $\sup u \circ f > \inf u \circ f$, so for k large also $\sup c_k > \inf c_k$, and for such k we can define

$$c'_k(s) = \inf u \circ f + [\sup u \circ f - \inf u \circ f]\frac{c_k(s) - \inf c_k}{\sup c_k - \inf c_k}.$$

Now observe that $\inf c'_k = \inf f$ and $\sup c'_k = \sup f$; hence, since there exist $y, y' \in Y$ such that $y \succeq f(s) \succeq y'$ for all s, it follows that also $u(y) \geq c'_k(s) \geq u(y')$, so there exists $h_k \in L$ such that $c'_k = u \circ h_k$. Then $f \approx g$ implies that, for k large, $h_k \simeq \gamma f + (1-\gamma)g$, Part 2 yields $c'_k \simeq \gamma u \circ f + (1-\gamma)u \circ g$, and Part 3 finally yields $c_k \simeq \gamma u \circ f + (1-\gamma)u \circ g$. This concludes the proof of this part.

For 7, simply consider the constant sequence given by $h_k = f$, and the constant sequence given by $h_k = g$.

For 8, suppose $a \approx b$, fix γ, and consider $\{c_k\} \to a$. Let $\Gamma = \gamma + (1-\gamma)\alpha > 0$, and consider $\gamma' = \frac{\gamma}{\Gamma} \in [0,1]$; then, for k large, $c_k \simeq \gamma' a + (1-\gamma')b = \frac{1}{\Gamma}[\gamma a + (1-\gamma)\alpha b]$; by Part 3, this implies $c_k \simeq \gamma a + (1-\gamma)\alpha b + (1-\gamma)\beta = \gamma a + (1-\gamma)[\alpha b + \beta]$. Now consider $c_k \to \alpha b + \beta$, and again fix γ and define Γ, γ' as above. Note that $c'_k = \frac{c_k - \beta}{\alpha} \to b$, so eventually

$c'_k \simeq \gamma' a + (1-\gamma')b = \frac{1}{\Gamma}[\gamma a + (1-\gamma)\alpha b]$. Apply Part 3 to get $c'_k \simeq \gamma a + (1-\gamma)[\alpha b + \beta]$, as above; then apply it again to get $c_k \simeq \gamma a + (1-\gamma)[\alpha b + \beta]$, as required. Hence, $a \approx \alpha b + \beta$.

For Part 9, consider $a, b, \{c_k\} \to a, \lambda \in (0,1)$. Fix $\gamma \in [0,1]$, let $b' = \gamma a + (1-\gamma)b$, and let c'_k be such that $\lambda c'_k + (1-\lambda)b = c_k$, i.e. $c'_k = \frac{c_k - (1-\lambda)b}{\lambda}$. Since $a \approx b$, there is K such that $c'_k \simeq b'$ for $k \geq K$. This implies that $b' \simeq \lambda c'_k + (1-\lambda)b' = c_k - (1-\lambda)b + (1-\lambda)b = c_k$ for such k. Thus, in particular, for every c and $\gamma' \in [0,1]$, taking $\gamma = \lambda\gamma'$ it follows that there is K such that $c_k \simeq \gamma[\lambda a + (1-\lambda)b] + (1-\gamma)b$ for $k \geq K$. Furthermore, for every γ and $\{c_k\} \to b$, since $a \approx b$, there is also K' such that $c_k \approx \gamma a + (1-\gamma)b$ for all $k \geq K$; again taking $\gamma = \lambda\gamma'$ we see that, for all $\gamma' \in [0,1]$, $c_k \simeq \gamma'[\lambda a + (1-\lambda)b] + (1-\gamma')b$. Hence $\lambda a + (1-\lambda)b \approx b$.

Finally, for Part 10, it is clear that the property in the statement of the Lemma implies $a \approx b$. Conversely, fix γ, and suppose that for all $\epsilon > 0$ there is c with $\|c - a\| < \epsilon$ and $c \not\simeq \gamma a + (1-\gamma)b$. Then, for all $k \geq 1$, there is c_k such that $\|c_k - a\| < \frac{1}{k}$ but $c_k \not\simeq \gamma a + (1-\gamma)b$. Since $c_k \to a$, this contradicts $a \approx b$.

6.6.3 Proof of necessity in Corollary 2.8 (α-MEU preferences)

Consider a plausible-prior preference $\succeq$, with proper covering $C_1, \ldots, C_N$ and priors $P_1, \ldots, P_N$. For brevity, for any $a \in B(S, \Sigma)$ and any probability charge Q, denote $\int a dQ$ simply by $Q(a)$; also, as in §5.1.2, for all $n = 1, \ldots, N$ and $a \in C_n$, let $I(a) = P_n(a)$.

Suppose that $\succeq$ is also an α-MEU preference, with $\alpha \neq \frac{1}{2}$: there exists a set $\mathcal{C}$ of probability charges on (S, Σ) such that $I(a) = \alpha \min_{Q \in \mathcal{C}} Q(a) + (1-\alpha)\max_{Q \in \mathcal{C}} Q(a)$. To further simplify notation, let $M(a) = \min_{Q \in \mathcal{C}} Q(a)$; then $I(a) = \alpha M(a) - (1-\alpha)M(-a)$.

For $n, m \in \{1, \ldots, N\}$, let $C_{n,m} = \{a \in C_n : -a \in C_m\}$. Note that $a \in C_{n,m}$ implies $\gamma a + \beta \in C_{n,m}$ for all $\gamma \geq 0$ and $\beta \in \mathbb{R}$. Then $a \in C_{n,m}$ implies $I(a) = P_n(a)$ and $I(-a) = P_m(-a)$; hence $\alpha M(a) - (1-\alpha)M(-a) = P_n(a)$ and $\alpha M(-a) - (1-\alpha)M(a) = P_m(-a)$, which implies that $M(a) = \frac{\alpha P_n(a) - (1-\alpha)P_m(a)}{2\alpha - 1}$ for all $a \in C_{n,m}$. This implies that M is affine on every $C_{n,m}$; since it is also positively homogeneous and monotonic, it has a unique positive extension $J_{n,m}$ to the linear subspace $C_{n,m} - C_{n,m}$ (cf. Lemma 5.11).

Since $\bigcup_{n=1}^{N} \bigcup_{m=n}^{N} C_{n,m} = B(S, \Sigma)$, there is a collection $\mathcal{M}$ of pairs (n, m) such that $N \geq m \geq n \geq 1$ and interior $C_{n,m} \neq \emptyset$; furthermore, $\bigcup_{(n,m) \in \mathcal{M}}$ cl interior $C_{n,m} = B(S, \Sigma)$ (cf. the argument in Lemma 5.12). For every such (n, m), $C_{n,m} - C_{n,m} = B(S, \Sigma)$, and the functional $J_{n,m}$ is characterized by a unique probability charge $Q_{n,m}$.

Finally, let $a \in$ interior $C_{n,m}$ and $b \in$ interior $C_{n',m'}$, where $(n,m), (n',m') \in \mathcal{M}$ and $(n,m) \neq (n',m')$. There exists $\gamma \in (0,1)$ such that $\gamma a + (1-\gamma)b \in C_{n',m'}$, so $M(\gamma a + (1-\gamma)b) = J_{n',m'}(\gamma a + (1-\gamma)b) = \gamma J_{n',m'}(a) + (1-\gamma)J_{n',m}(b') = \gamma J_{n',m'}(a) + (1-\gamma)M(b)$. On the other hand, $M(\gamma a + (1-\gamma)b) \geq \gamma M(a) + (1-\gamma)M(b)$, so $J_{n',m'}(a) \geq M(a) = J_{n,m}(a)$. By continuity, this is true whenever $a \in \text{cl interior } C_{n,m}$ and $b \in \text{cl interior } C_{n',m'}$. That is, for all $a \in B(S, \Sigma)$, $M(a) = \min_{(n,m)\in\mathcal{M}} J_{n,m}(a)$, and therefore, since the maxmin representation is unique, $\mathcal{C}$ must be the closed convex hull of the charges $\{Q_{n,m} : (n,m) \in \mathcal{M}\}$.

The above construction clearly does not apply to the case $\alpha = \frac{1}{2}$. The result is indeed false in this case: for instance, let $S = \{s_1, s_2, s_3\}$ and $\mathcal{Q} = \{q \in \Delta(S) : \sum_s (q(r) - \frac{1}{3})^2 \leq \varepsilon\}$, for $\varepsilon \in (0, \frac{1}{\sqrt{6}})$: the corresponding "$\frac{1}{2}$-MEU" preferences can easily be seen to be SEU, with a single, uniform prior on S; yet $\mathcal{Q}$ is not the convex hull of finitely many points.