Efficient Sorting in a Dynamic Adverse-Selection Model: Web Appendix

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This version, May 2004.

1 Equilibrium under selling

1.1 Equilibrium Under Selling with Unobservable Vintages

There are only "new" and "old" cars; prices are p_n and p_u. Types $\theta \in [\theta_0, \bar{\theta}]$ buy new cars and keep only quality q_0 (i.e. sell as soon as car depreciates). Types $\theta \in [\theta_{01}, \theta_0]$ buy new cars and keep qualities q_0 and q_1. Types $\theta \in [\theta_1, \theta_{01}]$ buy used cars, keep q_1, and sell q_2. It is convenient to denote masses of buyers as follows: $v_0 = 1 - F(\theta_0)$ is the mass of new car buyers who only keep q_0; $v_{01,0}$ (resp. $v_{01,1}$) is the mass of new car buyers who keep q_0 and q_1 and, in any given period (in steady state), happpen to own a quality-q_0 (resp. q_1); it must be the case that $v_{01,0}+v_{01,1} = F(\theta_{01})-F(\theta_0)$. Furthermore, let $v_{1,1}$ be the mass of types who buy used and happen to own a quality-q_1 car; finally, $v_{1,2}$ is the mass of buyers who buy used and happen to own a quality-q_2; it must be the case that $v_{1,1} + v_{1,2} = F(\theta_{01}) - F(\theta_1)$ and $1 - F(\theta_1) = Y$.

Now let φ denote the fraction of new cars that are bought by types who then only keep quality q_0. We have

$$\begin{aligned}
v_0 &= (1 - \gamma_0\Delta)v_0 + v_{1,2}\gamma_2\Delta\varphi \\
v_{01,0} &= (1 - \gamma_0\Delta)v_{01,0} + v_{1,2}\gamma_2\Delta(1 - \varphi) \\
v_{01,1} &= (1 - \gamma_1\Delta)v_{01,1} + \gamma_0\Delta v_{01,0} \\
v_{1,1} &= (1 - \gamma_1\Delta)v_{1,1} + \gamma_0\Delta v_0 \\
v_{1,2} &= (1 - \gamma_2\Delta)v_{1,2} + \gamma_1\Delta(v_{01,1} + v_{1,1}).
\end{aligned}$$

To clarify: in steady state, the total mass of types θ who buy new and keep q_0 equals the mass of such individuals whose car did not die in the previous period , plus the mass of such individuals whose cars died in the previous period and was replaced by a new car; in particular, the steady-state flow of replacement cars equals the mass of cars that were in the hands of buyers who buy

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used, happened to hold a car of quality q_2, and whose car died. An identical interpretation holds for $v_{01,0}$. The interpretation of $v_{01,1}$ is similar, but now the interpretation of the second term is different: the inflow of buyers into this category equals the mass of buyers who are also buying new and keeping q_0 and q_1, who had a car of quality q_0 in the previous period, which however depreciated. For $v_{1,1}$, the second term represents the mass of cars held by consumers who buy new and keep only q_0 (these are the only used cars that enter the market at quality level q_1). Finally, for $v_{1,2}$, the second term has an analogous interpretation; the first is more noteworthy. Recall that, in the equilibrium we are trying to construct, cars of quality q_2 that do not depreciate (hence, die) are immediately sold; however, until they depreciate, they remain part of the pool of used cars. Hence the first term.

Rearranging terms and noting that $(v_0 + v_{01,0})\gamma_0\Delta = (v_{01,1} + v_{1,1})\gamma_1\Delta = v_{1,2}\gamma_2\Delta$ and $(v_0 + v_{01,0}) + (v_{01,1} + v_1) + v_{1,2} = Y$, we get $v_{1,2}\gamma_2(\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}) = Y$ and therefore

$$\begin{aligned} v_0 + v_{01,0} &= \frac{\gamma_0^{-1}}{\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}} Y \equiv \lambda_0 Y \\ v_{01,1} + v_{1,1} &= \frac{\gamma_1^{-1}}{\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}} Y \equiv \lambda_1 Y \\ v_{1,2} &= \frac{\gamma_2^{-1}}{\gamma_0^{-1} + \gamma_1^{-1} + \gamma_2^{-1}} Y \equiv \lambda_2 Y. \end{aligned}$$

Finally, taking into account the way each quality is split among each group,

$$\begin{aligned} v_0 &= \lambda_0 Y \varphi \\ v_{01,0} &= \lambda_0 Y (1-\varphi) \\ v_{01,1} &= \lambda_1 Y (1-\varphi) \\ v_{1,1} &= \lambda_1 Y \varphi \\ v_{1,2} &= \lambda_2 Y. \end{aligned}$$

Note that *these quantities are independent of* Δ. Moreover, the fraction of quality-q_0 cars owned by buyers who keep q_0 and q_1 is $\frac{\lambda_0}{\lambda_0+\lambda_1} = \frac{\gamma_0^{-1}}{\gamma_0^{-1}+\gamma_1^{-1}} \equiv \varphi_0$, the fraction of quality-q_0 cars in the new market is $\frac{\lambda_0}{\lambda_0+\lambda_1(1-\varphi)} = \frac{\gamma_0^{-1}}{\gamma_0^{-1}+\gamma_1^{-1}(1-\varphi)} \equiv \lambda_0^n$, and the fraction of quality-q_1 cars in the used market is $\frac{\lambda_1\varphi}{\lambda_1\varphi+\lambda_2} = \varphi_1$. All these quantities are also independent of Δ.

Turn now to the value functions. Consider buyers who participate in the used-car market. Recall they must keep q_1 and sell q_2 immediately (because there are some quality-q_1 cars in the used-car market, and the price they get for their car equals the price they pay for another used car). We must determine the fraction of quality-q_1 cars that are supplied in every period. Types $\theta \in [\theta_0, \bar{\theta}]$ sell quality-q_1 cars, so the fresh supply of this quality equals $v_0\gamma_0\Delta = \lambda_0 Y\varphi\gamma_0\Delta$; on the other hand, the $v_{01,1}$ types $\theta \in [\theta_{01}, \theta_0]$ who held a quality-q_1 car in the previous period, which then depreciated, sell a mass $v_{01,1}\gamma_1\Delta = \lambda_1 Y(1-\varphi)\gamma_1\Delta$ of quality-q_2, cars. Furthermore, the $v_{1,2}$ types $\theta \in [\theta_1, \theta_{01}]$ who had a bad draw in the previous period, as well as the $v_{1,1}$ types in the same interval who had a quality-q_1 car in the previous period, which then depreciated, are also reselling their cars on the used market. This adds $v_{1,1}\gamma_1\Delta + v_{1,2}(1-\gamma_2\Delta) = \lambda_1 Y\varphi\gamma_1\Delta + \lambda_2 Y(1-\gamma_2\Delta)$ cars (note that we must make sure that the cars offered do not die). Hence, the fraction of quality-q_1 used cars offered each period in vintage 1 is

$$\begin{aligned} & \frac{\lambda_0 Y\varphi\gamma_0\Delta}{\lambda_0 Y\varphi\gamma_0\Delta + \lambda_1 Y(1-\varphi)\gamma_1\Delta + \lambda_1 Y\varphi\gamma_1\Delta + \lambda_2 Y(1-\gamma_2\Delta)} \\ = & \frac{\lambda_0\varphi\gamma_0\Delta}{\lambda_0\varphi\gamma_0\Delta + \lambda_1\gamma_1\Delta + \lambda_2(1-\gamma_2\Delta)} \\ = & \frac{\lambda_0\varphi\gamma_0\Delta}{\lambda_0\varphi\gamma_0\Delta + \lambda_2} \equiv \varphi^u, \end{aligned}$$

using the fact that $\lambda_1\gamma_1 = \lambda_2\gamma_2$. Observe that φ^u *does* depend upon Δ. Note that the fraction of quality-q_1 used cars of at any point in time, φ_1, will in general be different from φ^u, because cars of quality q_2 accumulate in the used-car market. Hence

$$\begin{aligned} V_u(\theta) &= -p_u + \varphi^u \left\{ \frac{1-e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta}\left[(1-\gamma_1\Delta)W_{u,1}(\theta) + \gamma_1\Delta(p_u + V_u(\theta))\right] \right\} \\ & \quad + (1-\varphi^u)\left\{ \frac{1-e^{-\rho\Delta}}{\rho} q_2\theta + e^{-\rho\Delta}\left[V_u(\theta) + (1-\gamma_2\Delta)p_u\right] \right\}, \\ W_{u,1}(\theta) &= \frac{1-e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta}\left\{(1-\gamma_1\Delta)W_{u,1}(\theta) + \gamma_1\Delta[p_u + V_u(\theta)]\right\}. \end{aligned}$$

To clarify: if the used car is q_1, then buyers enjoy it for one period; then, if it does not depreciate, they get the continuation value $W_{u,1}(\theta)$ determined by the assumption that the car is sold as soon as it depreciates. If the used car is q_2, it is sold immediately, but one must take into account the fact that the car may still die (hence the buyer may be unable to resell it).

Next, consider $\theta \in [\theta_{01}, \theta_0]$. These buyers buy a new car, and sell it when it depreciates to q_2. We must still keep track of the continuation values; however, now a new car is guaranteed to be of quality q_0.

$$\begin{aligned} V_{n,01}(\theta) &= -p_n + \frac{1-e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta}\left[(1-\gamma_0\Delta)W_{n,01,0}(\theta) + \gamma_0\Delta W_{n,01,1}(\theta)\right] \\ W_{n,01,1}(\theta) &= \frac{1-e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta}\left\{(1-\gamma_1\Delta)W_{n,01,1}(\theta) + \gamma_1\Delta[p_u + V_{n,01}(\theta)]\right\} \\ W_{n,01,0}(\theta) &= \frac{1-e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta}\left\{(1-\gamma_0\Delta)W_{n,01,0}(\theta) + \gamma_0\Delta W_{n,01,1}(\theta)\right\}. \end{aligned}$$

Finally, we consider buyers who buy new cars and keep q_0.

$$\begin{aligned} V_{n,0}(\theta) &= -p_n + \frac{1-e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta}\left\{(1-\gamma_0\Delta)W_{n,0}(\theta) + \gamma_0\Delta[p_u + V_{n,0}(\theta)\right\} \\ W_{n,0}(\theta) &= V_{n,0}(\theta) + p_n. \end{aligned}$$

To construct an equilibrium, consider an arbitrary $\varphi \in [0,1]$: $\varphi = 1$ cannot yield an equilibrium, because it would induce an efficient allocation, which, by Theorem 1 and Proposition 1, is impossible. For each such φ, it is possible to choose prices p_u and p_n such that

$$V_u(\theta_1) = 0, \quad V_{n,01}(\theta_{01}) = V_u(\theta_{01}).$$

We now consider three cases. (1) If $V_{n,0}(\theta_0) < V_{n,01}(\theta_0)$ for all φ (recall that φ determines the cutoff θ_0), then in particular this is true for $\varphi = 0$, where $\theta_0 = \bar{\theta}$; this implies that, for $\varphi = 0$, $V_{n,0}(\theta) < V_{n,01}(\theta)$ for all θ (the argument requires a decomposition analogous to the one in the proof of Lemma 2). Hence $\varphi = 0$ yields the right incentives to all consumers types when they do not own a car.

(2) If $V_{n,0}(\theta_0) > V_{n,01}(\theta_0)$ for all values of φ, then in particular this is the case at $\varphi = 1$, where $\theta_0 = \theta_{01}$; for this value of φ, it is then the case that $V_{n,0}(\theta_0) > V_{n,01}(\theta_0) = V_u(\theta_0)$. Hence there exists a price $p_n' > p_n$ such that $V_{n,0}(\theta_0) = V_u(\theta_0) > V_{n,01}(\theta_0)$; that is, at the prices p_n', p_u, all types $\theta \in [\theta_0 = \theta_{01}, \bar{\theta}]$ buy new cars and keep only q_0, and types $\theta \in [\theta_1, \theta_0]$ buy used cars.

(3) If, finally, there exist φ, φ' such that at the corresponding prices and cutoff types θ_0, θ_0', $V_{n,0}(\theta_0) < V_{n,01}(\theta_0)$ and $V_{n,0}(\theta_0') > V_{n,01}(\theta_0')$, then by continuity there exists φ'' such that equality obtains.

Thus, in all thee cases, for an appropriate choice of prices and $\varphi \in [0,1]$, consumers follow the policies described above when they do not own a car; to complete the argument, we now show that they also do so when they already own a car (i.e. they adopt the "right" keeping policies).

The argument for consumers who experiment with used cars is straightforward: if they currently own quality q_1 (resp. q_2) given that their best continuation policy is to buy another used car, they can only do worse (resp. better) in expectation by selling their current car. Thus, turn to type θ_{01}, assuming that $\varphi < 1$ (otherwise this case is irrelevant). It is clear that this type should not keep a car of quality q_2. If her current car instead is of quality q_1, her continuation value is

$$\begin{aligned} W_{n,01,1}(\theta_{01}) &= \frac{1-e^{-\rho\Delta}}{\rho} q_1\theta_{01} + e^{-\rho\Delta}\left\{(1-\gamma_1\Delta)W_{n,01,1}(\theta_{01}) + \gamma_1\Delta[p_u + V_{n,01}(\theta_{01})]\right\} = \\ &= \frac{1-e^{-\rho\Delta}}{\rho} q_1\theta_{01} + e^{-\rho\Delta}\left\{(1-\gamma_1\Delta)W_{n,01,1}(\theta_{01}) + \gamma_1\Delta[p_u + V_u(\theta_{01})]\right\} = \\ &= W_u(\theta_{01}); \end{aligned}$$

if she instead sells her car, then she can get at most $p_u + V_{n,01}(\theta_{01}) = p_u + V_u(\theta_{01}) \leq W_u(\theta_{01})$. The inequality follows because the l.h.s. is the value of receiving a used car, which may be of quality q_1 or q_2, and following the optimal keeping policy for used cars, whereas the r.h.s. is the value of receiving a car of quality q_1, then following the same optimal keeping policy. Hence, type θ_{01} should keep a car of quality q_1, and consequently she should also keep a car of quality q_0. This implies that all other types in $[\theta_{01}, \theta_0]$ also have the correct incentives.

Finally, consider type θ_0, assuming $\varphi > 0$ (otherwise this case is irrelevant). We must ensure that this type will be willing to sell quality q_1. If she does, she obtains

$$p_u + V_{n,0}(\theta). = p_u + V_{n,01}(\theta_0) \geq W_{n,01,1}(\theta_{01}),$$

where the inequality follows because $p_u + V_{n,01}(\theta_0)$ is the value of receiving a car of quality q_0, and keeping it until it depreciates to q_2, then buying a new car and continuing with the same keeping policy. Since $V_{n,0}(\theta_{01}) = V_{n,01}(\theta_0)$, $W_{n,01,1}(\theta_{01})$ can equivalently be viewed as the value of keeping a car of quality q_1 until it depreciates, then reverting to the designated policy for type θ_0. This shows that keeping q_1 is not a profitable deviation for type θ_0, and concludes the proof.

1.2 Equilibrium Under Selling with Observable Vintages

Notation is approximately as above. Now types in $[\theta_0, \bar{\theta}]$ buy new (i.e. vintage 0) cars and keep only q_0; their mass is v_0. Types in $[\theta_{01}, \theta_0]$ buy vintage 0 and keep q_0 and q_1; $v_{01,0}$ is the mass of such types who happen to own a quality-q_0 car, and $v_{01,1}$ is the mass of such types who own quality q_1. Types in $[\theta_1, \theta_{01}]$ buy vintage 1 and keep only q_1; $v_{1,1}$ and $v_{1,2}$ denote the masses of such types who own qualities q_1 and q_2 respectively. Finally, types in $[\theta_2, \theta_1]$ buy vintage 2 and keep quality q_2; their mass is v_2. We thus have, in steady state,

$$
\begin{aligned}
v_0 &= (1-\gamma_0)v_0 + (v_2 + v_{1,2})\gamma_2\Delta\varphi \\
v_{01,0} &= (1-\gamma_0\Delta)v_{01,0} + (v_2 + v_{1,2})\gamma_2\Delta(1-\varphi) \\
v_{01,1} &= (1-\gamma_1\Delta)v_{01,1} + v_{01,0}\gamma_0\Delta \\
v_{1,1} &= (1-\gamma_1\Delta)v_{1,1} + v_0\gamma_0\Delta \\
v_{1,2} &= v_{01,1}\gamma_1\Delta \\
v_2 &= (1-\gamma_2\Delta)v_2 + v_{1,1}\gamma_1\Delta + v_{1,2}(1-\gamma_2\Delta).
\end{aligned}
$$

To clarify, quality-q_1 cars of vintage 1 are cars previously owned by types in $[\theta_0, \bar{\theta}]$ that have just depreciated; quality-q_2 cars of vintage 1 instead are cars that were discarded by types in $[\theta_{01}, \theta_0]$. The latter cars are immediately resold, and hence become of vintage 2, provided they do not die: this explains the third term in the r.h.s. of the last equation. The remaining cars of vintage 2 are either surviving vintage-2 cars or vintage-1 cars that have just depreciated from q_1 to q_2.

We solve as above. In particular, $(v_0 + v_{01,0})\gamma_0\Delta = (v_2 + v_{1,2})\gamma_2\Delta$ and $(v_{01,1} + v_{1,1})\gamma_1\Delta = (v_2 + v_{1,2})\gamma_2\Delta$, and we obtain $v_0 + v_{01,0} = \lambda_0 Y$, $v_{01,1} + v_{1,1} = \lambda_1 Y$ and $v_{1,2} + v_2 = \lambda_2 Y$, with λ_i as above. Therefore

$$
\begin{aligned}
v_0 &= \lambda_0 Y\varphi \\
v_{01,0} &= \lambda_0 Y(1-\varphi) \\
v_{01,1} &= \lambda_1 Y(1-\varphi) \\
v_{1,1} &= \lambda_1 Y\varphi \\
v_{1,2} &= \lambda_1 Y(1-\varphi)\gamma_1 \\
v_2 &= \lambda_2 Y - \lambda_2 Y(1-\varphi)\gamma_2
\end{aligned}
$$

(note that $\lambda_i\gamma_i = \lambda_j\gamma_j$ for all $i, j = 0, \ldots 2$). The fraction of quality-q_1 cars of vintage 1 is $\varphi_1 = \frac{\varphi}{\varphi+(1-\varphi)\gamma_1\Delta}$, and the fraction of quality-q_0 cars in the hands of types $\theta \in [\theta_{01}, \theta_0]$ is $\varphi_0 = \frac{\lambda_0}{\lambda_0+\lambda_1}$.

Turn now to value functions and prices. For types $\theta \in [\theta_2, \theta_1]$,

$$V_2(\theta) = -p_2 + \frac{1-e^{-\rho\Delta}}{\rho} q_2\theta + e^{-\rho\Delta}\{(1-\gamma_2\Delta)[V_2(\theta)+p_2] + \gamma_2\Delta V_2(\theta)\}.$$

Next, consider $\theta \in [\theta_1, \theta_{01}]$. We must distinguish between buyers who currently own a quality-q_1 car, and those who currently own q_2 (and hence will immediately dispose of it). The key issue here is the composition of the supply of vintage-1 cars: $\lambda_0 Y \varphi \gamma_0 \Delta$ come from types $\theta \in [\theta_0, 1]$, and hence are of quality q_1; $\lambda_1 Y(1-\varphi)\gamma_1\Delta$ come from types $\theta \in [\theta_{01}, \theta_0]$, and hence are of quality q_2. Therefore, the fraction of quality-q_1 cars supplied is

$$\frac{\lambda_0 Y \varphi \gamma_0 \Delta}{\lambda_0 Y \varphi \gamma_0 \Delta + \lambda_1 Y (1-\varphi)\gamma_1 \Delta} = \frac{\varphi}{\varphi + 1 - \varphi} = \varphi,$$

where we use the fact that $\lambda_0\gamma_0 = \lambda_1\gamma_1$. Hence we can write

$$\begin{aligned}
V_1(\theta) &= -p_1 + \varphi\left\{\frac{1-e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta}[(1-\gamma_1\Delta)W_{1,1}(\theta) + \gamma_1\Delta(V_1(\theta)+p_2)]\right\} \\
&\quad +(1-\varphi)\left\{\frac{1-e^{-\rho\Delta}}{\rho} q_2\theta + e^{-\rho\Delta}[V_1(\theta) + (1-\gamma_2\Delta)p_2]\right\} \\
W_{1,1}(\theta) &= \frac{1-e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta}\left\{(1-\gamma_1\Delta)W_{1,1}(\theta) + \gamma_1\Delta[V_1(\theta)+p_2]\right\}.
\end{aligned}$$

To clarify: consider a buyer who currently has no car. If she buys a vintage-1 car, with probability φ she gets q_1; $W_{1,1}(\theta)$ represents her continuation payoff, assuming the car does not depreciate at the end of the period. With probability $1-\varphi$, she gets q_2, in which case she sells the car immediately, provided the car does not die at the end of the period; note that, in any case, the buyer will purchase a vintage-1 car in the next period if her current car is of quality q_2.

Now consider $\theta \in [\theta_{01}, \theta_0]$. Recall that these buyers sell their cars only when it depreciates to q_2.

$$\begin{aligned}
V_{01}(\theta) &= -p_0 + \frac{1-e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta}\left[(1-\gamma_0\Delta)W_{01,0}(\theta) + \gamma_0\Delta W_{01,1}(\theta)\right] \\
W_{01,1}(\theta) &= \frac{1-e^{-\rho\Delta}}{\rho} q_1\theta + e^{-\rho\Delta}\left\{(1-\gamma_1\Delta)W_{01,1}(\theta) + \gamma_1\Delta[p_1 + V_{01}(\theta)]\right\} \\
W_{01,0}(\theta) &= \frac{1-e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta}\left[(1-\gamma_0\Delta)W_{01,0}(\theta) + \gamma_0\Delta W_{01,1}(\theta)\right]
\end{aligned}$$

Note that the problem is exactly the same as the problem faced by consumers $\theta \in [\theta_{01}, \theta_0]$ in the no-vintages case: simply let $p_0 = p_n$ and $p_1 = p_u$.

Finally, $V_0(\theta)$ is exactly like $V_{n,0}$ in the no-vintage case:

$$\begin{aligned}
V_0(\theta) &= -p_0 + \frac{1-e^{-\rho\Delta}}{\rho} q_0\theta + e^{-\rho\Delta}\left\{(1-\gamma_0\Delta)W_0(\theta) + \gamma_0\Delta[p_1 + V_0(\theta)\right\} \\
W_0(\theta) &= V_0(\theta) + p_1.
\end{aligned}$$

To establish the existence of an equilibrium, we proceed as in the case of unobservable vintages. For every value of $\varphi \in [0,1]$, we can determine p_0, p_1, p_2 via the indifference conditions

$$V_2(\theta_2) = 0, \quad V_1(\theta_1) = V_2(\theta_1), \quad V_{01}(\theta_{01}) = V_1(\theta_{01}).$$

However, Theorem 1 and Proposition 1 imply that $\varphi = 1$ cannot correspond to an equilibrium, because it implies efficiency. Furthermore, the proof of Theorem 1 shows that, at $\varphi = 1$, type $\theta_{01} = \theta_0$ will strictly prefer to keep quality q_1 rather than resell her current car and buy another vintage-0 car. This easily implies that, for $\varphi = 1$, $V_0(\theta_0) < V_{01}(\theta_0)$. Hence, we only need to consider two cases: if $V_0(\theta_0) > V_{01}(\theta_0)$ for all φ, then $\varphi = 0$ yields the right incentives when consumers do not own a car; otherwise, $V_0(\theta_0) = V_{01}(\theta_0)$ for some $\varphi \in [0,1)$.

Incentives when consumers already own a car are verified as in the case of unobservable vintages, so the proof is omitted.

2 Omitted Proofs

2.1 Proof of Lemma 1

Note first that, by Eqs. (15), one can write

$$v^n = \begin{bmatrix} \gamma_{n,n}(\Delta) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} v^n + A_{n-1,n} v^{n-1}$$

for all $n = 1, \dots, N$, where $v^n = [v^n_n, \dots v^n_N]'$ and

$$A_{n-1,n} = \begin{bmatrix} \gamma_{n-1,n}\Delta & \gamma_{n,n}(\Delta) & 0 & \dots & \dots & 0 \\ \gamma_{n-1,n+1}\Delta & \gamma_{n,n+1}\Delta & \gamma_{n+1,n+1}(\Delta) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{n-1,N}\Delta & \gamma_{n,N}\Delta & \gamma_{n+1,N}\Delta & \dots & \dots & \gamma_{N,N}(\Delta) \end{bmatrix}.$$

Next, since $\gamma_{n,n}(\Delta) = 1 - G_{n,n+1}\Delta$,

$$\begin{aligned} 1' A_{n-1,n} v^{n-1} &= \begin{bmatrix} G_{n-1,n}\Delta - \gamma_{n-1,N+1}\Delta & 1 - \gamma_{n,N+1}\Delta & \dots & 1 - \gamma_{N,N+1}\Delta \end{bmatrix} v^{n-1} = \\ &= -\gamma_{n-1,n-1}(\Delta) v^{n-1}_{n-1} + \sum_{m=n-1}^{N} (1 - \gamma_{m,N+1}\Delta) v^{n-1}_m = \\ &= -\gamma_{n-1,n-1}(\Delta) v^{n-1}_{n-1} + 1' v^{n-1} - \sum_{m=n-1}^{N} \gamma_{m,N+1}\Delta v^{n-1}_m \end{aligned}$$

and hence

$$\begin{aligned} 1'v^n - \gamma_{n,n}(\Delta)v_n^n &= 1'v^{n-1} - \gamma_{n-1,n-1}(\Delta)v_{n-1}^{n-1} - \sum_{m=n-1}^{N} \gamma_{m,N+1}\Delta v_m^{n-1} = \\ &= 1'v^0 - \gamma_{0,0}(\Delta)v_0^0 - \sum_{\ell=0}^{n-1}\sum_{k=\ell}^{N} \gamma_{k,N+1}\Delta v_k^\ell. \end{aligned}$$

In particular, since vintage-N cars can only have quality q_N, $1'v^N = v_N^N$ and $G_{N,N+1} = \gamma_{N,N+1}$ and therefore also $\gamma_{N,N}(\Delta) = 1 - \gamma_{N,N+1}\Delta$. Hence

$$\gamma_{N,N+1}\Delta v^N = 1'v^N - \gamma_{N,N}(\Delta)v_N^N = 1'v^0 - \gamma_{0,0}(\Delta)v_0^0 - \sum_{\ell=0}^{N-1}\sum_{k=\ell}^{N} \gamma_{k,N+1}\Delta v_k^\ell$$

and therefore

$$\sum_{\ell=0}^{N}\sum_{k=\ell}^{N} \gamma_{k,N+1}\Delta v_k^\ell = 1'v^0 - \gamma_{0,0}(\Delta)v_0^0 = y.$$

This shows that, if the quantities v_m^n are defined via equation (15), they automatically satisfy equation (16). It is also easy to see that $v_n^n \geq \left(\prod_{\ell=1}^n \gamma_{\ell-1,\ell}\right)\chi_0 y$; furthermore, $v_m^n = 0$ if $y = 0$. Hence, as long as $\chi_0 > 0$, there exists y^* such that equation (17), too, is satisfied.

To prove the second part of the claim, note first that all quantities v_m^n are bounded, so $y \to 0$ as $\Delta \to 0$. This immediately implies that $v_m^0 \to 0$ for $m > 0$; proceeding by induction, assume that we have shown $v_m^{n-1} \to 0$ for $m > n-1$: then the last line of equation (15) implies that $v_m^n \to 0$ as well for $m > n$ (in particular, the terms in the summation corresponding to $\ell = n-1$ vanish because v_{n-1}^{n-1} is bounded). Furthermore, it is clear that $v_n^* = \sum_{\ell=0}^n v_n^\ell$ for all Δ; therefore, $|v_n^* - v_n^n| = \left|v_n^* - \sum_{\ell=0}^n v_n^\ell + \sum_{\ell=0}^n v_n^\ell - v_n^n\right| = \left|0 + \sum_{\ell=0}^{n-1} v_n^\ell\right| \to 0$ as $\Delta \to 0$.

2.2 Proof of Lemma 4

From equation (19), the claim is clearly true for $n = 0$. For $n > 0$, note first that the denominator of λ_m^n can be rewritten as follows:

$$\begin{aligned} \sum_{k=n}^{N}\left(\sum_{\ell=n-1}^{k-1} \gamma_{\ell,k}\Delta v_\ell^{n-1} + \gamma_{k,k}(\Delta)v_k^{n-1}\right) &= \sum_{\ell=n-1}^{N-1}\sum_{k=\ell}^{N} \gamma_{\ell,k}\Delta v_\ell^{n-1} + \sum_{\ell=n}^{N} \gamma_{\ell,\ell}(\Delta)v_\ell^{n-1} \\ &= \sum_{\ell=n-1}^{N-1} G_{\ell,\ell+1}\Delta v_\ell^{n-1} + \sum_{\ell=n}^{N} \gamma_{\ell,\ell}(\Delta)v_\ell^{n-1} \\ &= G_{n-1,n}\Delta v_{n-1}^{n-1} + \sum_{\ell=n}^{N} v_\ell^{n-1}. \end{aligned}$$

Accordingly, rewrite λ_n^n as follows:

$$\begin{aligned}\lambda_n^n &= \frac{\gamma_{n-1,n}\Delta v_{n-1}^{n-1} + \gamma_{n,n}(\Delta) v_n^{n-1}}{G_{n-1,n}\Delta v_{n-1}^{n-1} + \sum_{\ell=n}^N v_\ell^{n-1}} \\ &= \frac{\gamma_{n-1,n} v_{n-1}^{n-1} + \gamma_{n,n}(\Delta)\frac{v_n^{n-1}}{\Delta}}{G_{n-1,n} v_{n-1}^{n-1} + \sum_{\ell=n}^N \frac{v_\ell^{n-1}}{\Delta}}.\end{aligned}$$

Now Lemma 1 shows that $v_{n-1}^{n-1} \to v_{n-1}^* > 0$ as $\Delta \to 0$. Furthermore, we claim that $\sup_{\Delta>0} \frac{v_m^n}{\Delta} < \infty$ for all n and $m > n$. To see this, observe first that, from equations (16) and (17),

$$\frac{y}{\Delta} = \sum_{\ell=0}^N \sum_{k=\ell}^N \gamma_{k,N+1} v_k^\ell \leq \sum_{\ell=0}^N \sum_{k=\ell}^N v_k^\ell = Y < 1;$$

since $v_m^0 = \chi_m y$ for $m > 0$, this immediately implies that the claim is true for $n = 0$. Assuming that it is true for $n - 1 \geq 0$, for $m > n$, equation (15) implies that

$$\frac{v_m^n}{\Delta} = \sum_{\ell=n-1}^{m-1} \gamma_{\ell,m} v_\ell^{n-1} + \gamma_{m,m}(\Delta)\frac{v_m^{n-1}}{\Delta}$$

and the induction hypothesis implies that $\sup_{\Delta>0} \frac{v_m^{n-1}}{\Delta} < \infty$; since $\gamma_{m,m}(\Delta) \to 1$, the claim is true for n as well.

The proof of the Lemma can now be completed: we have

$$\begin{aligned}\lambda_n^n &\geq \frac{\gamma_{n-1,n} v_{n-1}^{n-1}}{G_{n-1,n} v_{n-1}^{n-1} + \sum_{\ell=n}^N \frac{v_\ell^{n-1}}{\Delta}} \\ &\geq \frac{\gamma_{n-1,n} v_{n-1}^{n-1}}{G_{n-1,n} v_{n-1}^{n-1} + \sum_{\ell=n}^N \sup_{\Delta>0} \frac{v_\ell^{n-1}}{\Delta}} \\ &\to \frac{\gamma_{n-1,n} v_{n-1}^*}{G_{n-1,n} v_{n-1}^* + \sum_{\ell=n}^N \sup_{\Delta>0} \frac{v_\ell^{n-1}}{\Delta}} > 0,\end{aligned}$$

and the claim follows.