Machina's Reflection Example and VEU Preferences: a Very Short Note

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1 Executive Summary

Machina [8] proposes examples of plausible preference patterns that cannot be accommodated by the Choquet expected utility model (Schmeidler [9]). In a recent paper, Baillon, L'Haridon and Placido [2] show that Machina's examples also pose a challenge for other popular models, including maxmin expected utility (Gilboa and Schmeidler [5]), variational preferences (Maccheroni, Marinacci and Rustichini [7]) and the "smooth ambiguity model" of Klibanoff, Marinacci and Mukerji [6].

This note focuses on the "reflection example," reproduced in Section 2 below; it argues that Machina's reflection example highlights important behavioral differences between notions of ambiguity aversion that are either implied by, or natural, in a variety of decision models.

I adopt the *vector expected utility* (henceforth VEU) model of Siniscalchi [10]. This is useful in the present context because VEU preferences admit a relatively "structured" representation,

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and yet can display a wide range of attitudes towards ambiguity. In particular, they can satisfy Schmeidler's classical "Uncertainty Aversion" axiom [9], as well as the less restrictive "Diversification" axiom of Chateauneuf and Tallon [3]; for VEU preferences, the latter notion of ambiguity aversion coincides with Ghirardato and Marinacci's "comparative definition" [4]. A VEU preference relation that satisfies Schmeidler's Uncertainty Aversion axiom is variational, and so Baillon et al. [2]'s result implies that no such preference can exhibit the pattern of interest in Machina's Reflection example. On the other hand, the main result in this brief note shows that this pattern *can* arise for VEU preferences that satisfy the Diversification axiom. Furthermore, the VEU representation employed here suggests an interesting, novel interpretation of the preferences of interest in Machina's example.

Finally, I note that the analysis in Baillon et al. [2] also implicitly recognizes a key role to ambiguity aversion. To provide some detail, these authors show that maxmin and variational preferences (which always satisfy Uncertainty aversion) cannot accommodate the pattern of interest in the Reflection example. Next, recall that smooth ambiguity preferences satisfy a natural notion of "smooth-ambiguity aversion" (loosely speaking, a preference for the reduction of second-order lotteries) if and only if the second-order utility function is concave. Baillon and coauthors show that this condition prevents the smooth-ambiguity model from accommodating the preferences of interest in Machina's Reflection example. It would be interesting to consider different notions of ambiguity aversion for the smooth-ambiguity model.

2 Machina's Example

Let the state space Ω consists of four points, $\omega_1 \dots \omega_4$. It is known that $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$ are equally likely (and not ambiguous). However, the relative likelihood of ω_1 vs. ω_2, and of ω_3 vs. ω_4, are not known. We identify the set of acts with $\mathbb{R}^4$.

Consider the monetary bets (acts) in Table 1.

Notice that f_1 and f_4 only differ by a "reflection," i.e. by exchanging prizes on states that are

	ω_1	ω_2	ω_3	ω_4
f_1	\$4,000	\$8,000	\$4,000	\$0
f_2	\$4,000	\$4,000	\$8,000	\$0
f_3	\$0	\$8,000	\$4,000	\$4,000
f_4	\$0	\$4,000	\$8,000	\$4,000

Table 1: Machina's reflection example. Reasonable preferences: $f_1 \prec f_2$ and $f_3 \succ f_4$

informationally symmetric. The same is true of f_2 and f_3. Hence, it is plausible to expect that $f_1 \sim f_4$ and $f_2 \sim f_3$. In particular, Machina [8] conjectures, and L'Haridon and Placido [1] verify, that a common pattern of "ambiguity-averse" preferences is $f_1 \prec f_2$ and $f_3 \succ f_4$.

Machina shows that Choquet EU prohibits this pattern. Baillon et al. [2] show that variational preferences do too, as do smooth-ambiguity preferences under the appropriate ambiguity-aversion assumption (concavity of the second-order utility).

3 VEU Anaysis

The VEU preference functional can be formally defined as follows in the present context: for every $f \in \mathbb{R}^4$, let[1]

$$V(f) = \mathrm{E}_p[u \circ f] + A(\mathrm{E}_p[\zeta_0 u \circ f], \ldots, \mathrm{E}_p[\zeta_n u \circ f]);$$

here, $p \in \Delta(\Omega)$ and E_p is the corresponding expectation operator, $u : \mathbb{R} \to \mathbb{R}$ is a Bernoulli utility function, $\zeta_0, \ldots, \zeta_n \in \mathbb{R}^4$ are random variables such that $\mathrm{E}_p[\zeta_i] = 0$ for all i, and $A : \mathbb{R}^n \to \mathbb{R}$ satisfies $A(0) = 0$ and $A(\phi) = A(-\phi)$. Furthermore, the map $a \mapsto \mathrm{E}_p[a] + A(\mathrm{E}_p[\zeta_0 a], \ldots, \mathrm{E}_p[\zeta_{n-1} a])$ is required to be monotonic on $\mathbb{R}^4$.

The random variables ζ_i are "adjustment factors" that represent *complementarities* among

[1]Notation: $a \circ b$ denotes the composition of two functions a and b with suitable domains and ranges. Also, if a, b are real functions on Ω, then ab is their statewise product: that is, the map $\omega \mapsto a(\omega)b(\omega)$.

	ω_1	ω_2	ω_3	ω_4
ζ_0	1	-1	0	0
ζ_1	0	0	1	-1

Table 2: The adjustment factors ζ_0 and ζ_1.

ambiguous events. The function A is an *adjustment function* that reflects attitudes towards ambiguity. For further details, consult [10].

In this context, Schmeidler's Ambiguity Aversion axiom is equivalent to A being *negative and concave*. On the other hand, Chateauneuf and Talllon's notion of Diversification is equivalent to Ghirardato and Marinacci's comparative notion, and correspond to A being *negative*. I shall now show that the preferences of interest in the Reflection example can be accommodated with a negative A—that is, within the VEU framework, these preferences are indeed consistent with a suitable, and well-understood, notion of ambiguity aversion.

First of all, in keeping with the informational symmetry assumptions discussed above, let p be uniform on Ω. Next, normalize the utility function u so that $u(\$0) = 0$, $u(\$8{,}000) = 4$, and $u(\$4{,}000) = 4\alpha$ for some $\alpha \in (0,1)$. With this normalization $\mathrm{E}_p[u \circ f_i] = 2\alpha + 1$ for all acts $i = 1, \ldots, 4$.

Next, define two adjustment factors $\zeta_0, \zeta_1 : \Omega \to \mathbb{R}$ according to Table 2. These definitons again reflect the informational assumptions in the example: ambiguity about ω_1 and ω_2 "cancels out," and so does ambiguity about ω_3 and ω_4. On the other hand, there is no interaction between states ω_1 and ω_3, between states ω_1 and ω_4, etc. In the language of Siniscalchi [10], the factors ζ_0 and ζ_1 correspond to two distinct and independent "sources of ambiguity."

Finally, let $v : \mathbb{R} \to \mathbb{R}$ be a function that satisfies the following properties:

- $v(0) = 0$ and $v(\phi) = v(-\phi)$ for all $\phi \in \mathbb{R}$;
- v is continuous and, for all $\phi > 0$, it is also differentiable at ϕ with $v'(\phi) \in (0,1]$;

- v is strictly concave on $\mathbb{R}_+$.

The first condition corresponds to the properties of adjustment functions in the VEU representation (consult the Appendix for details). The second condition further ensures that v is increasing (decreasing) on the non-negative (non-positive) reals, but not so fast as to violate monotonicity of the VEU functional to be defined below (see the appendix for explicit calculations). Observe that the first two conditions imply that $v(\phi) > 0$ for all $\phi \neq 0$. The third condition is crucial to deliver the prevalent pattern of preferences $f_1 \prec f_2$ and $f_3 \succ f_4$.[2] A simple example of a function satifying these requirements is

$$v(\phi) = \sqrt{1 + |\phi|} - 1.$$

It is now possible to define a suitable VEU functional $V : \mathbb{R}^4 \to \mathbb{R}$ by

$$V(f) = \mathrm{E}_p[u \circ f] - \frac{1}{2} v(\mathrm{E}_p[\zeta_0 u \circ f]) - \frac{1}{2} v(\mathrm{E}_p[\zeta_1 u \circ f]). \tag{1}$$

Notice that the map $a \mapsto -\frac{1}{2} v(\mathrm{E}_p[\zeta_0 a]) - \frac{1}{2} v(\mathrm{E}_p[\zeta_1 a])$ is negative for all $a \in \mathbb{R}^4$ (except at $a = 0$) but clearly *not* concave. Thus, the preferences defined here are ambiguity-averse according to Ghirardato and Marinacci's comparative definition (equivalently, they satisfy Diversification) but *not* according to Schmeidler's stronger Uncertainty Aversion axiom.

It remains to verify that this VEU functional is consistent with the preferences of interest in Machina's reflection example. As noted above, the baseline expectations of all four acts in Table 1 are all equal to $2\alpha + 1$; thus, the ranking of these acts is determined by the adjustments. These are computed in Table 3.

By assumption, $v(1-\alpha) = v(\alpha - 1)$; furthermore, strict concavity of v on the positive reals implies that

$$\frac{1}{2} v(1-\alpha) + \frac{1}{2} v(\alpha) > \frac{1}{2} v(0) + \frac{1}{2} v(1);$$

[2]Given the third condition (strict concavity), the second condition could be stated more parsimoniously; I do not do so here for simplicity.

Act	$\mathrm{E}_p[\zeta_0 f]$	$\mathrm{E}_p[\zeta_1 f]$	Adjustment
f_1	$\alpha-1$	α	$-\frac{1}{2}v(\alpha-1)-\frac{1}{2}v(\alpha)$
f_2	0	1	$-\frac{1}{2}v(0)-\frac{1}{2}v(1)$
f_3	1	0	$-\frac{1}{2}v(1)-\frac{1}{2}v(0)$
f_4	$-\alpha$	$1-\alpha$	$-\frac{1}{2}v(\alpha)-\frac{1}{2}v(1-\alpha)$

Table 3: Adjustments

thus, by Eq. (1), $V(f_1) < V(f_2)$ and $V(f_3) > V(f_4)$, consistently with the preferences of interest in Machina's reflection example (which are also found to be prevalent in the experiment run by L'Haridon and Placido [1]).

4 Comments

By inspecting Table 3, the preferences $f_1 \prec f_2$ and $f_3 \succ f_4$ may be interpreted as reflecting a desire to "minimize the sources of ambiguity one is subject to." Loosely speaking, the evaluation of the acts f_1 and f_4 is influenced both by ambiguity about ω_1 vs. ω_2, and by ambiguity about ω_3 vs. ω_4. Correspondingly, the coefficients $\mathrm{E}_p[\zeta_i f_1]$ and $\mathrm{E}_p[\zeta_i f_4]$ in Table 3 are non-zero for both $i=0$ and $i=1$. On the other hand, the acts f_2 and f_3 are each affected by a *single* source of ambiguity—respectively, ambiguity about ω_3 vs. ω_4 for f_1 and ambiguity about ω_1 vs. ω_2 for f_4. Again, Table 3 reflects this intuition. Since the function v is concave on $\mathbb{R}_+$, $-v$ is convex on $\mathbb{R}_+$, and this leads the VEU decision-maker portrayed by Eq. (1) to prefer a single source of ambiguity.

It is also worth observing that the VEU representation, together with the choice of adjustment factors in Table 3, is consistent with, but does not *force* this particular pattern of preferences to emerge. For instance, if the function v is instead assumed to be strictly *convex* on the non-negative reals, then the opposite pattern of preferences emerges: $V(f_1) > V(f_2)$ and

$V(f_3) < V(f_4)$. An example of a suitable function v is given by $v(\phi) = \frac{\phi^2}{1+|\phi|}$. Furthermore, if $v(\phi) = |\phi|$, then one readily obtains $f_1 \sim f_2$ and $f_3 \sim f_4$.

The adjustment factors chosen here, together with the symmtery properties of the VEU decision model, *do* force the indifferences $f_1 \sim f_4$ and $f_2 \sim f_3$, regardless of the choice of the function v. This is consistent with the informational symmetry highlighted by Machina.

Appendix

A Monotonicity for v as in the text

I show that, in the decision setting under consideration, and in particular with ζ_0, ζ_1 as defined in the text, the VEU functional in Eq. (1) is indeed monotonic.

For $j = 0, 1$, consider the functional $V_i(f) = \mathrm{E}_p[f] - v(\mathrm{E}_p[\zeta_i f])$. Thus, $V(f) = \frac{1}{2} V_i(f) + \frac{1}{2} V_i(f)$. It is then enough to show that V_i is monotonic. Fix one such i.

For all $f \in \mathbb{R}^4$ such that $\mathrm{E}_p[\zeta_i f] \neq 0, \partial V_i(f)/\partial f(\omega_j) = p(\{\omega_j\}) - v'(\mathrm{E}_p[\zeta_i f]) p(\{\omega_j\}) \zeta_i(\omega_j)$. This is positive provided

$$1 > v'(\mathrm{E}_p[\zeta_i f]) \zeta_i(\omega_j).$$

By assumption, $0 < v'(\phi) \leq 1$ for $\phi > 0$, and so $-1 \leq v'(\phi) < 0$ for $\phi < 0$. Furthermore, $|\zeta_i(\omega_j)| \leq 1$ for all $j = 1...4$. Hence, $v'(\mathrm{E}_p[\zeta_i f]) \zeta_i(\omega_j) \leq |v'(\mathrm{E}_p[\zeta_i f]) \zeta_i(\omega_j)| \leq |v'(\mathrm{E}_p[\zeta_i f])| |\zeta_i(\omega_j)| \leq 1$, and so $\partial V_i(f)/\partial f(\omega_j) > 0$. The claim now follows from Remark A.1 in [10], or directly by a simple continuity argument.

I also verify that the function $v(\phi) = \sqrt{1+|\phi|} - 1$ satisfies the required properties. It is immediate that $v(0) = 0$ and $v(-\phi) = v(\phi)$. The function is differentiable at all $\phi \neq 0$, and for $\phi > 0$, $v'(\phi) = \frac{1}{2} (1+\phi)^{-\frac{1}{2}} \leq \frac{1}{2}$, and clearly also $v'(\phi) > 0$. Strict concavity on $\mathbb{R}_+$ is immediate.

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