Risk Sharing in the Small and in the Large*

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Abstract

This paper analyzes risk sharing in economies with no aggregate uncertainty when agents have non-convex preferences. In particular, agents need not be globally risk-averse, or uncertainty-averse in the sense of Schmeidler (1989). We identify a behavioral condition under which betting is inefficient (i.e., every Pareto-efficient allocation provides full insurance, and conversely) if and only if agents' supporting probabilities (defined as in Rigotti, Shannon, and Strzalecki, 2008) have a non-empty intersection. Our condition is consistent with empirical and experimental evidence documenting violations of convexity in either outcomes or utilities. Our results show that the connection between speculative betting and inconsistent beliefs does not depend upon global notions of risk or ambiguity aversion.

1 Introduction

Consider an exchange economy with a single consumption good and no aggregate uncertainty. It is well understood that in such an economy risk-averse and (subjective) expected-

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utility-maximizing agents will choose to introduce individual uncertainty in the final allocation if and only if they have different beliefs. That is, betting can occur in equilibrium only if agents disagree on the probabilities of some events (Milgrom and Stokey, 1982). At the same time, it is also well understood that agents may not have unique probabilistic beliefs when some events relevant for the economy are more ambiguous than others (Ellsberg, 1961). The result on the connection between disagreement of beliefs and betting has been extended to take this into account. For instance, Billot, Chateauneuf, Gilboa, and Tallon (2000) show that risk-averse agents whose preferences satisfy the maxmin expected utility model of Gilboa and Schmeidler (1989) will bet if and only if they do not *share a belief*; i.e., if the sets of probabilities that they employ (in the maxmin representation) do not intersect. This result has been significantly extended by Rigotti et al. (2008, henceforth RSS). They showed that a similar result holds for any collection of agents whose preferences are suitably well-behaved and, notably, satisfy strict *convexity in consumption*: given any two contingent consumption plans f and g, any nondegenerate convex combination[1] $\alpha f + (1-\alpha) g$ is (strictly) preferred to either plan. In other words, agents have a (strict) preference for consumption smoothing across states.

Since these "risk-sharing[2]" results imply that, in equilibrium, agents will attain a state-independent (i.e., maximally smooth) consumption profile, one may expect convexity to play a key role. This paper shows that this is not the case. We identify a behavioral assumption that is sufficient to deliver the equivalence between risk sharing and consistency of the agents' "supporting probabilities" (defined below), but permits substantial departures from convexity in consumption (though it is implied by it).

Loosely speaking, our assumption only restricts the preferences of agents who are fully hedged against uncertainty. This allows for a broad range of attitudes towards risk and ambiguity, and enables us to address many empirical, experimental, and theoretical concerns about convexity. With expected-utility preferences and a single consumption good, convexity

[1] Note that convex combinations (here and in RSS) are *not* mixtures in the Anscombe and Aumann (1963) sense: they are the usual vector-space notion. They represent convex combinations of *consumption*, not *utilities*.

[2] Strictly speaking, the noted no-betting results for MEU and more general preferences concern agents' sharing of risk and ambiguity, or more broadly uncertainty. We use the terminology "risk sharing," rather than "uncertainty sharing," for consistency with the literature.

in consumption characterizes risk aversion. Beyond expected utility, it implies a combination of risk and ambiguity aversion (we will be more precise below). However, there is empirical evidence that investment decisions are sometimes risk-seeking, either because of intrinsic preferences (e.g., Kumar, 2009), or due to the nature of incentive contracts for fund managers (e.g., Chevalier and Ellison, 1997). In the context of insurance decisions, Wakker, Timmermans, and Machielse (2007) documents ambiguity-seeking behavior; for similar findings in asset markets, see Brenner and Izhakian (2012).

In experimental settings, the classic findings of Curley and Yates (1985) and Heath and Tversky (1991) raise questions about the pervasiveness of aversion to ambiguity (broadly defined). More recent papers cast doubts on the specific formalization of uncertainty aversion as convexity in *utilities*, due to Schmeidler (1989). To elaborate, most parametric representations of ambiguity-sensitive preferences associate with each contingent consumption bundle $f = (f_1, \ldots, f_S)$ a utility index $I(u(f))$, where $u(f) = \big(u(f_1), \ldots, u(f_S)\big)$ is the state-contingent utility vector associated with f and I is a function defined over utility vectors.[3] Schmeidler (1989) defines "uncertainty aversion" as quasiconcavity of I (hence, convexity in the induced preferences over state-contingent utility vectors). However, L'Haridon and Placido (2010) document patterns of behavior that, while intuitively consistent with aversion to ambiguity, cannot be represented by a utility index of the form $f \mapsto I(u(f))$, if I is quasiconcave and consistent with EU for unambiguous bundles f (Baillon, L'Haridon, and Placido, 2011).

Finally, from a theoretical perspective, the connection between convexity and aversion to risk or ambiguity is not clear-cut. For preferences that are probabilistically sophisticated (Machina and Schmeidler, 1992) but not EU, intuitive notions of risk aversion do *not* imply convexity in consumption (though they are implied by it): see e.g. Dekel (1989). For preferences that are not probabilistically sophisticated, Epstein (1999) and Ghirardato and Marinacci (2002, henceforth GM) question the identification of ambiguity aversion with convexity in utilities, as do Baillon et al. (2011), and provide alternative definitions.[4]

We illustrate how our results address these concerns by means of three examples in Sec-

[3] In particular, all parametric representations analyzed in of RSS (see their Section 2.4) have this form, with u strictly concave.

[4] We provide GM's definition in Appendix C.

tion 2. First, we exhibit an Edgeworth-box economy in which agents are "locally" risk- and ambiguity-seeking at their endowment point (which is uncertain). This is a stylized environment in which behavior is consistent with the experimental evidence described above on local departures from risk and ambiguity aversion. The example shows that our results apply, and equilibrium entails full risk-sharing. Second, we exhibit a probabilistically-sophisticated non-EU preference (based on Dekel, 1989) that is risk-averse but not convex in consumption, yet satisfies our key behavioral condition. Third, we exhibit a preference that is ambiguity-averse in the sense of GM (henceforth, "GM-ambiguity averse") and can accommodate the behavior documented in L'Haridon and Placido (2010) (and hence violates convexity). Once again, our condition is satisfied for this preference. These examples demonstrate that, as claimed above, our key behavioral assumption is consistent with a wide range of attitudes toward risk and ambigity. Furthermore, they show that risk sharing may still obtain despite significant departures from convexity in consumption.

We now describe our main results in greater detail. As noted above, under our condition, betting obtains if and only if agents' "supporting probabilities" are inconsistent. The notion of supporting probabilities coincides with RSS's definition of "subjective beliefs as supporting price vectors.[5]" In general, supporting probabilities may be different at different consumption bundles; RSS employ an axiom that ensures that they are costant across riskless consumptions. Our main risk-sharing result (Theorem 3) does not adopt this assumption;[6] however, to facilitate comparison with RSS, we also provide one that does (Theorem 4).

The condition we propose, *strict pseudoconcavity at certainty*, (SPC), admits geometric, economic, and decision-theoretic interpretations.[7] Suppose there are two states, and consider a constant consumption bundle that yields x units of the good in each state. Suppose further that the indifference curve through x is smooth at x.[8] Then, geometrically, SPC re-

[5]Our terminology avoids the use of the word "belief," which—because we adopt weaker behavioral assumptions than RSS—may be debated by some.

[6]We discuss how this difference affects the conditions for betting to be inefficient in Section 4.

[7]See Remark 1 and the ensuing discussion for details.

[8]Without smoothness, SPC is a condition on the elements of the Clarke (1983) differential of the representation at certainty. If preferences are convex, the Clarke subdifferential coincides with the convex-analysis differential, and SPC follows from strict convexity (Proposition 2).

quires that the entire indifference curve through x lie strictly above the tangent line. A natural economic interpretation is that, if the tangent at x is viewed as a budget line, then any point in the budget set is strictly worse than x. From a decision-theoretic perspective, SPC requires a (weak) form of "risk aversion at x:" the individual strictly prefers x to any other bundle f for which $P \cdot f = x$, where P is the probability corresponding to the slope of the indifference curve at x. Moreover, again if indifference curves are smooth at certainty, SPC admits a straightforward preference characterization that further clarifies its relationship with aversion to risk and uncertainty: see Proposition 2. For general preferences, a stronger behavioral condition is needed: see Appendix B. On the other hand, in Appendix C and in Online Appendix E.3 we provide some details on the relationship of SPC to other notions of aversion to ambiguity.

This paper is organized as follows. Section 2 provides examples of non-convex preferences that satisfy our assumptions. Section 3 introduces the formal setup. Section 4 contains the main risk-sharing results. Section 5 provides a sketch of the proof of Theorems 3 and 4, and contains additional results. Section 6 illustrates how condition SPC can be obtained for a specific parametric representation of preferences that we use in some of our examples and is not covered by RSS's results; this is generalized in Appendix C to all representation of the form $I(u(f))$, with u strictly concave. Appendix A contains further examples, and Appendix B provides a behavioral characterization of condition SPC. All proofs are in Appendix D. The Online Appendix contains detailed calculations for the examples, additional examples, and an analysis of the connection between SPC and notions of ambiguity aversion in the literature.

Related literature The relation with RSS and Billot et al. (2000) has already been discussed. A more detailed comparison of the differences between our analysis and the one in RSS can be found at the end of Section 5. We note that these papers allow for an arbitrary state space; for simplicity, we restrict attention to finite states.

Strzalecki and Werner (2011) extend and adapt the risk-sharing results in RSS to economies with aggregate uncertainty and convex preferences. An investigation of risk sharing in economies with aggregate uncertainty and non-convex preferences is left to future research.

Araujo, Chateauneuf, Gama-Torres, and Novinski (2014) study the existence of equilibrium in economies with aggregate uncertainty, where both uncertainty-averse and uncertainty-loving agents are present. They also study equilibrium risk sharing. Araujo, Bonnisseau, Chateauneuf, and Novinski (2015) study the existence of Pareto optima, and provide examples and results about risk sharing among risk-loving and risk-averse agents who are probabilistically sophisticated. Similarly, Assa (2015) studies risk sharing among agents with Choquet preferences whose capacities are non-convex distortions of a common probability. Our analysis is complementary; we do not require any agent to be globally uncertainty-averse, and do not require the presence of uncertainty-loving agents. Indeed condition SPC is inconsistent with global uncertainty appeal. Furthermore, we do not assume probabilistic sophistication.

Billot, Chateauneuf, Gilboa, and Tallon (2002) provide a version of Proposition 18 for Choquet-expected utility preferences (CEU; Schmeidler, 1989). They also prove a risk-sharing result for such preferences that does not assume convexity (or SPC) but requires large economies, with a continuum of agents of each "type."

Dominiak, Eichberger, and Lefort (2012) consider an economy with two CEU agents and riskless (full-insurance) endowments, extending the prior analysis of Kajii and Ui (2006) which assumed convexity. They provide a condition which is necessary and sufficient for the non-existence of Pareto-improving trades. Their analysis relies on the fact that there are only two agents in the economy, whose initial endowment is constant; on the other hand, it does not require either convexity or pointwise ambiguity aversion.

Marinacci and Pesce (2013) consider preferences that are both GM-ambiguity averse and invariant biseparable (Ghirardato, Maccheroni, and Marinacci, 2004). They study the impact of changes in GM-ambiguity aversion on efficient and equilibrium allocations. Though they do not focus on risk sharing, they independently derive a version of our Proposition 7. Chateauneuf, Dana, and Tallon (2000) obtain a similar result for the special case of CEU preferences. See however Example 4 in Appendix A on the implications of invariant biseparability.

There is a large literature on equilibrium analysis with non-convexities. For a survey that focuses on non-convex production sets, see Brown (1991). Our proof of Proposition 5 employs a result by Bonnisseau and Cornet (1988) that allows for non-convexities in consumption. An alternative approach to circumvent violations of convexity is to consider large economies (see e.g. Mas-Colell, Whinston, and Green, 1995, §17.I); we do not follow this approch, and instead consider a fixed, finite number of agents.

Finally, Ghirardato and Siniscalchi (2012) provides a behavioral foundation for the analysis in the present paper. Leveraging the results therein, Appendix B in this paper characterizes condition SPC in terms of the agent's preferences.

2 Risk sharing without convexity: examples

The following examples illustrate that our key behavioral assumption, condition SPC, is consistent with a wide range of attitudes toward risk and ambigity. They also show that risk sharing may still obtain despite significant departures from convexity in consumption.

The first example (Section 2.1) demonstrates that risk sharing can obtain in an economy in which agents are ambiguity-seeking (their indifference curves are locally concave) near the endowment point, which is non-constant. The second example (Section 2.2) describes a probabilistically sophisticated, risk-averse non-EU preference that does not satisfy convexity in consumption, but satisfies condition SPC. The third example (Section 2.3) describes a preference that is GM-ambiguity averse,[9] though not ambiguity-averse in the sense of Schmeidler (1989), and can thus account for the behavior in the reflection example of Machina (2009); again, condition SPC holds, whereas convexity in consumption fails.

In all three examples, as in the rest of the paper, we consider a finite state space S and preferences over contingent consumption bundles $f \in \mathbb{R}^S_+$.

2.1 Risk- and ambiguity-seeking behavior near the endowment point

We begin with a graphical illustration. Consider the single-good, two-state Edgeworth-box economy of Fig. 1. The endowment point ω is the midpoint between the allocations f and g, and lies below Agent 1's indifference curve going through these points. In particular, starting at ω, Agent 1 would strictly prefer to carry out the trade $g-\omega$ and move to g, even though this entails increasing the volatility of her consumption across the two states.

Despite this, preferences are such that the only points of tangency between 1's and 2's indifference curves are along the certainty line. Thus, at every efficient allocation, agents fully

[9] Indeed the preferences in all three examples of this section are GM-ambiguity averse.

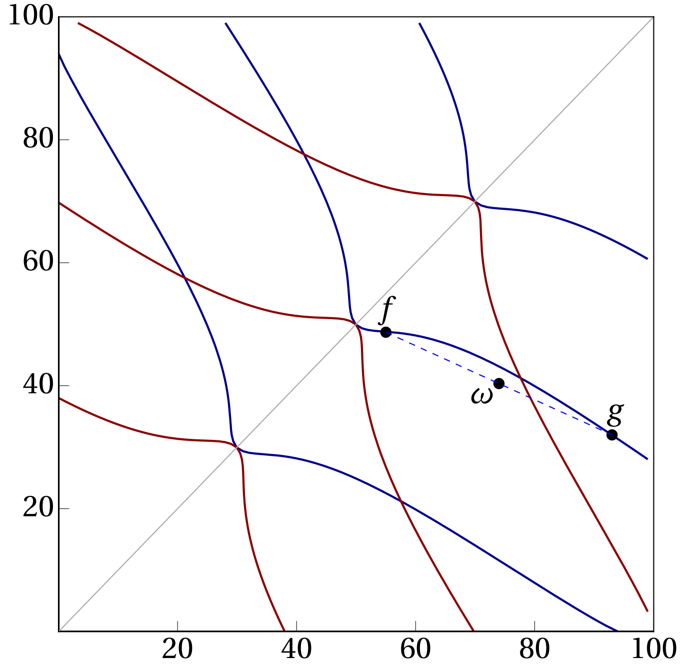

Figure 1: An economy with locally ambiguity-seeking agents

insure one another, i.e., they share risks. By the First Welfare Theorem, this holds a fortiori for all equilibrium allocations.

We now describe the preferences in Fig. 1 analytically. Both agents have VEU preferences (Siniscalchi, 2009), which are not necessarily convex:

$$V(f) = \sum_{s\in S} P_s u(f_s) + A\left(\sum_{s\in S} P_s \zeta_{0,s} u(f_s), \dots, \sum_{s\in S} P_s \zeta_{J-1,s} u(f_s)\right), \tag{1}$$

where u is a strictly increasing, differentiable, and strictly concave Bernoulli utility function, $P \in \Delta(S)$ is the *baseline prior*, $\zeta_0, \dots, \zeta_{J-1} \in \mathbb{R}^S$ are *adjustment factors* that satisfy $\sum_s P_s \zeta_{j,s} = 0$ for each j, and $A : \mathbb{R}^J \to \mathbb{R}$ (the *adjustment function*) satisfies $A(\phi) = A(-\phi)$ for all $\phi \in \mathbb{R}^J$.

We show in Proposition 9 that, if the adjustment function A is also smooth and non-positive, and an additional joint assumption on P, ζ, and A holds, the functional V thus defined satisfies all the assumptions of Theorem 4, our stronger risk-sharing result. Furthermore, the set of supporting probabilities consists of a single element, the baseline prior P. Thus, in an economy in which all agents have such VEU preferences, with possibly different parameters, risk-sharing obtains *if and only if they have a common baseline prior.*

Figure 1 is obtained using the following parameterization: for both agents, P is uniform, $J = 1$, $\zeta_0 \equiv \zeta = [-1, 1]$, and the adjustment function takes the form

$$A(\phi) = -\frac{1}{2}\theta \log\left(1 + \frac{\phi^2}{\theta}\right)$$

where $\theta \in (0, 4)$; note that $A \leq 0$. The two agents differ in the value of θ, and in their utility function u; in Fig. 1, $\theta_1 = 3$ and $u_1(x) = x^{0.9}$ for agent 1, and $\theta_2 = 3.5$ and $u_2(x) = x^{0.95}$ for agent 2. This parameterization satisfies all the assumptions of Proposition 9: see Appendix E.2.

The example in Section 2.3 below employs the same class of VEU preferences, and discusses its properties further.

2.2 Probabilistically sophisticated, non-expected utility preferences

This example is based upon the proof of Proposition 1 in Dekel (1989). Fix a probability distribution P over the state space S, and consider the preferences represented by

$$V(f) = g\left(\sum_s P_s u(f_s)\right) + g\left(\sum_s P_s f_s\right), \tag{2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly concave, and $g : \mathbb{R}_+ \cup \{u(r) : r \in \mathbb{R}_+\} \to \mathbb{R}$ is strictly increasing and differentiable.

These preferences are probabilistically sophisticated and risk-averse, in the (strong) sense that they exhibit aversion to mean-preserving spreads. To see this, let $\mathscr{D}$ be the set of all cumulative distribution functions (CDFs) on $[0, \infty)$, and define a functional $W : \mathscr{D} \to \mathbb{R}$ by letting

$$W(F) = g\left(\int u(x)dF(x)\right) + g\left(\int x\,dF(x)\right).$$

Then, for every bundle $f \in \mathbb{R}^S_+$, $V(f) = W(F_f)$, where F_f is the CDF induced by f and the probability P by letting $F_f(x) = P(\{s : f(s) \leq x\})$. Thus, a decision-maker with the preferences represented by Eq. (2) reduces uncertainty to risk. Online Appendix E.1 shows that the functional W satisfies monotonicity with respect to first-order stochastic dominance, a suitable form of continuity, and aversion to mean-preserving spreads.[10] Furthermore, suitable specifications of u and g can accommodate Allais-type behavior.

[10] Dekel (1989, Proposition 1) shows this for CDFs on a compact interval $[0, M]$. Since the set of consumption levels is unbounded above in the present paper, for completeness we extend the results in the Appendix.

The preferences in Equation (2) satisfy our condition SPC. We show in Online Appendix E.1 that, given any strictly concave u and strictly increasing g, the Clarke subdifferential of V at every constant bundle x is a multiple of the probability vector P. It then suffices to show that, for every non-constant bundle f, if $x = P \cdot f$, then $V(f) < V(x)$ (see Remark 1). But this follows from Jensen's inequality, because u is strictly concave and g is strictly increasing: $\sum_s u(f_s)P_s < u(x)$, and so $V(f) = g(\sum_s P_s u(f_s)) + g(P \cdot f) < g(u(x)) + g(x) = V(1_S x)$. Hence, Theorem 3 implies that agents with preferences as in Equation (2) (but possibly different functions u and g) will engage in mutually beneficial bets if and only if their beliefs P differ.

However, some parametric specifications within this class do not satisfy convexity in consumption. For instance, the indifference curves in Figure 2 are drawn for a two-point state space, with $u(x)$ a positive affine transformation[11] of $-\frac{1}{1+x}$, P uniform, and $g(r) = e^{2(r-3)}$.

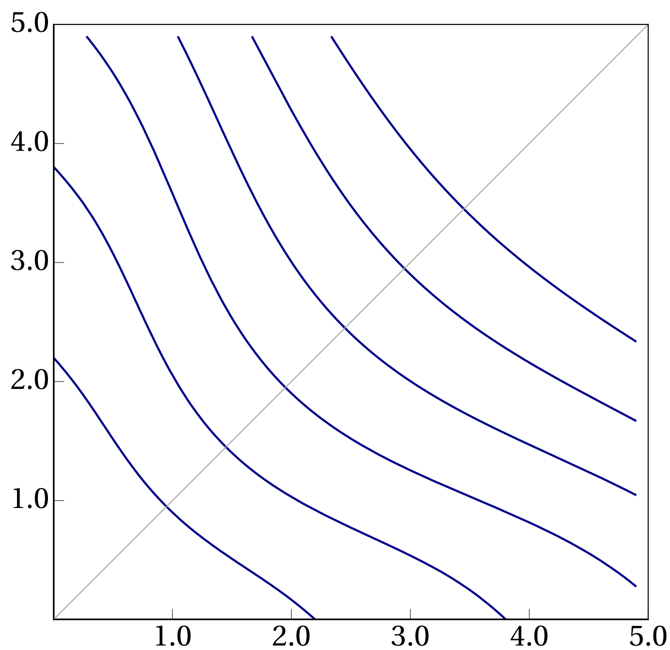

Figure 2: Risk-averse, non-convex preferences (see §2.2)

[11]Similarly to the construction in Dekel (1989), we fix two bundles f_1, f_2 such that $P \cdot f_1 > P \cdot f_2$ and f_2 is strictly preferred to f_1 by an EU decision maker with beliefs P and utility $v(x) = -\frac{1}{1+x}$. To obtain u, we normalize v so that $P \cdot u \circ f_1 = P \cdot f_2$ and $P \cdot u \circ f_2 = P \cdot f_1$.

2.3 Machina's reflection paradox

Consider the family of VEU preferences in Eq. (1). As illustrated in Section 2.1, these preferences are not necessarily convex in either consumption or utility; however, under suitable assumptions on the parameters P, ζ, A, they satisfy the assumptions of our risk-sharing results. We now demonstrate that these preferences can accommodate the modal preferences in the "reflection example" of Machina (2009) (see also L'Haridon and Placido, 2010; Baillon et al., 2011). Let $S = \{s_1, s_2, s_3, s_4\}$ and assume that the events $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous and equally likely, but no further information is provided as to the relative likelihood of s_1 vs. s_2 and s_3 vs. s_4. Furthermore, the draw of s_1 vs. s_2 and s_3 vs. s_4 are perceived as being independent. Consider the bets in Table 1.

	s_1	s_2	s_3	s_4
f^1	\$4,000	\$8,000	\$4,000	\$0
f^2	\$4,000	\$4,000	\$8,000	\$0
f^3	\$0	\$8,000	\$4,000	\$4,000
f^4	\$0	\$4,000	\$8,000	\$4,000

Table 1: Machina's reflection example. Reasonable preferences: $f^1 \prec f^2$ and $f^3 \succ f^4$

Machina (2009) argues on the basis of symmetry considerations that the preference ranking $f^1 \prec f^2$ and $f^3 \succ f^4$ is plausible and intuitively consistent with aversion to ambiguity; L'Haridon and Placido (2010) verify that these rankings do occur in an experimental setting. However, Baillon et al. (2011) show that this behavior cannot be accommodated by preference models that satisfy uncertainty aversion a la Schmeidler (1989), and (a) are consistent with EU in the absence of ambiguity, and (b) respect the symmetry of betting preferences in this example (e.g., s_1 and s_2 are deemed equally likely and ambiguous, etc.). We now provide a parameterization of the VEU preferences in Eq. (1) that does generate these rankings, and is consistent with (a) and (b). The parameterization we provide is also consistent with GM's "comparative" notion of aversion to ambiguity.[12]

[12] A similar example is provided in Siniscalchi (2009), but the VEU preferences described therein are not smooth and violate SPC, our main preference assumption.

Assume a uniform baseline prior P and two adjustment factors $\zeta_0, \zeta_1 \in \mathbb{R}^S$:

$$\zeta_0 = [1, -1, 0, 0] \qquad \text{and} \qquad \zeta_1 = [0, 0, 1, -1].$$

The adjustment function is a two-factor analog of the one considered in Section 2.1:

$$A(\phi) = A(\phi_0, \phi_1) = -\frac{1}{2}\theta \sum_{j=0,1} \log\left(1 + \frac{\phi_j^2}{\theta}\right)$$

where $\theta \in (0,4)$. Finally, let $u(0) = 0$, $u(8{,}000) = 4$, and $u(4{,}000) = 4\alpha$, for some $\alpha \in (\frac{1}{2}, 1)$. Appendix E.2 shows that this specification of the parameters P, A, ζ_0, ζ_1 yields a strictly monotonic preference. Furthermore, while this parameterization does not satisfy the Uncertainty Aversion axiom of Schmeidler (1989), it is ambiguity-averse in the sense of GM: see Siniscalchi (2009), Proposition 2.[13] Finally, Appendix E.2 shows that the rankings $f^1 \prec f^2$ and $f^3 \succ f^4$ obtain iff $0 < \theta < \frac{\alpha(1-\alpha)}{2}$.

3 Setup

We consider an Arrow-Debreu economy with finitely many states S, no aggregate uncertainty, a single good that can be consumed in non-negative quantity, and N consumers.

3.1 Decision-theoretic assumptions

We begin by describing consumers' preferences. To simplify notation, in this section we do not use consumer indices. We complete the description of the economy in section 3.2.

Behavior is described by a preference relation $\succcurlyeq$ over bundles (contingent consumption plans) $f \in \mathbb{R}^S_+$. We assume that $\succcurlyeq$ is represented by a function $V : \mathbb{R}^S_+ \to \mathbb{R}$: that is, for every pair $f, g \in \mathbb{R}^S_+$, $f \succcurlyeq g$ if and only if $V(f) \geq V(g)$.

Given $n \geq 1$, an open subset B of $\mathbb{R}^n$, and a function $F : B \to \mathbb{R}$, the **Clarke subdifferential** of F at $b \in B$ (Clarke, 1983) is

$$\partial F(b) = \text{cl conv}\left\{\lim_{k\to\infty} d^k : \exists (b^k) \to b \text{ such that } d^k = \nabla F(b^k)\, \forall k\right\}. \tag{3}$$

[13] For VEU preferences, $A \leq 0$ characterizes preferences that are ambiguity-averse in the sense of GM; on the other hand, uncertainty aversion (convexity) requires that A be non-positive and concave.

That is, $\partial F(b)$ is the set of all limits of gradients of the function F, taken along sequences (b^k) for which $\nabla F(b^k)$ exists for all k.[14] The Clarke subdifferential is non-empty at any $b \in B$ where F is locally Lipschitz. If F is monotonic,[15] then its Clarke subdifferential consists of non-negative vectors (cf. e.g. Rockafellar, 1980, Theorem 6, Corollary 3). Appendix D.1 provides additional results on the characterization of Clarke subdifferentials. Finally, a function $F : B \to R$, with $B \subset \mathbb{R}^n$ open, is **nice** at $b \in B$ if $(0,\ldots,0) \notin \partial F(b)$. Loosely speaking, niceness is a form of "infinitesimal non-satiation;" in the differentiable case, it means that F has a non-zero gradient at b.

We summarize our basic decision-theoretic assumptions in the following:

Assumption 1 The relation $\succcurlyeq$ admits a representation V satisfying the following properties:

1. V is strongly monotonic: that is, $f \geq g$ and $f \neq g$ imply $V(f) > V(g)$;

2. V is locally Lipschitz at every $f \in \mathbb{R}^S_{++}$.

Ghirardato and Siniscalchi (2012) (henceforth GS) argue that most parametric models of ambiguity-sensitive preferences admit a representation where V satisfies property 2; for instance, this is the case if V is monotonic and concave, or if it is constant-additive.[16] GS also provide axioms that ensure the existence of a locally Lipschitz representation; see Appendix B for details. Since strong monotonicity can clearly be expressed in terms of preferences, it follows that all of the requirements in Assumption 1 admit a behavioral characterization.

RSS identify a set of measures that plays a key role in the analysis of risk sharing. Denote by $\Delta(S)$ the unit simplex in $\mathbb{R}^S$. For every $f \in \mathbb{R}^S_+$, let

$$\pi(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ V(g) \geq V(f) \Longrightarrow P \cdot g \geq P \cdot f\}. \quad (4)$$

That is, $\pi(f)$ is the set of (normalized) prices such that any bundle that is weakly preferred to f is not less expensive than f. This is the usual notion of "quasi-optimality" in equilib-

[14] This is well-posed because, by Rademacher's theorem, if F is Lipschitz in an open neighborhood of b, it is differentiable almost everywhere on that neighborhood; see Clarke (1983), p. 63.

[15] That is, $g \geq g'$ implies $F(g) \geq F(g')$.

[16] V is constant-additive if, for every act f and constant x, $V(f + 1_S x) = V(f) + x$.

rium theory. Alternatively, we can interpret each $P \in \pi(f)$ as representing a risk-neutral SEU preference whose better-than set at f contains the better-than set of $\succcurlyeq$ at f.

RSS refer to $\pi(f)$ as the agent's "subjective beliefs." Since we make weaker assumptions on preferences than RSS, we adopt the more neutral terminology *supporting probabilities*.

3.2 The economy

An economy is a tuple $(N, (\succcurlyeq_i, \omega_i)_{i\in N})$, where N is the collection of agents, and for every i, agent i is characterized by preferences $\succcurlyeq_i$ over $\mathbb{R}^S_+$ and has an endowment $\omega_i \in \mathbb{R}^S_+$. As in RSS, we assume that *there is no aggregate uncertainty*: formally, $\sum_i \omega_i = 1_S \bar{x}$ for some $\bar{x} > 0$.

An *allocation* is a tuple $(f_1, \ldots, f_N)$ such that $f_i \in \mathbb{R}^S_+$ for each $i \in N$; as usual f_i is the contingent-consumption bundle assigned to agent i. The allocation $(f_1, \ldots, f_N)$ is *feasible* if $\sum_i f_i = \sum_i \omega_i$; it is a *full-insurance allocation* if, for every consumer i, $f_i = 1_S x_i$ for some $x_i \in \mathbb{R}_+$; it is *Pareto-efficient* if it is feasible, and there is no other feasible allocation $(g_1, \ldots, g_N)$ such that $g \succcurlyeq f$ for all i, and $g_j \succ_j f_j$ for some j.

For each $i \in N$, we denote by V_i and, respectively, $\pi_i(\cdot)$ the representation of i's preferences and her sets of supporting probabilities.

4 Risk Sharing

To begin, it is useful to restate the main result of RSS for convex preferences.[17] For completeness, recall that a preference $\succcurlyeq$ is *strictly convex* if, for all bundles $f, g \in \mathbb{R}^S_+$ with $f \neq g$, $f \succcurlyeq g$ implies $\alpha f + (1-\alpha)g \succ g$ for all $\alpha \in (0,1)$, or, equivalently, if the representation V of $\succcurlyeq$ is *strictly quasiconcave*: that is, $f \neq g$ and $V(f) \geq V(g)$ imply $V(\lambda f + (1-\lambda)g) > V(g)$ for all $\lambda \in (0,1)$.

Theorem 1 (cf. RSS, Proposition 9) *Suppose that, for each $i \in N$, Assumption 1 holds, and that furthermore $\succcurlyeq_i$ is strictly convex and $\pi_i(1_S x) = \pi_i(1_S)$ for every $x > 0$. Then the following*

[17] Strictly speaking, the assumptions in Theorem 1 are slightly stronger than those in RSS's Proposition 9. Specifically, we maintain the assumption that each V_i is locally Lipschitz; RSS only assume continuity. We retain all our assumptions to streamline the exposition. Also note that all the parametric representations analyzed in RSS are concave, hence locally Lipschitz.

are equivalent:

(i) There exists an interior, full-insurance Pareto-efficient allocation;

(ii) Every Pareto-efficient allocation is a full-insurance allocation;

(iii) Every feasible, full-insurance allocation is Pareto-efficient;

(iv) $\bigcap_i \pi_i(1_S) \neq \emptyset$.

The implications (ii) ⇒ (iii) ⇒ (i) hold for all strongly monotonic and continuous preferences.[18] However, the implication (i) ⇒ (iv) is proved by invoking the Second Welfare Theorem, which requires convexity. RSS's argument for (iv) ⇒ (ii) also invokes strict convexity.

We now introduce our main assumption on preferences.

Definition 1 The function V is **strictly pseudoconcave at** $f \in \mathbb{R}^S_{++}$ if

$$\forall g \in \mathbb{R}^S_+ \setminus \{f\}, \qquad V(g) \geq V(f) \quad \Longrightarrow \quad \forall Q \in \partial V(f), \quad Q \cdot (g - f) > 0. \tag{5}$$

The functional V satisfies **strict pseudoconcavity at certainty (SPC)** if it is strictly pseudoconcave at $1_S x$ for all $x > 0$.

The intuition for this condition is sharpest in case V is continuously differentiable at a point f, in which case the Clarke subdifferential equals the gradient of V at f. Then, V is strictly pseudoconcave at f if, whenever a bundle g is weakly preferred to f, moving from f in the direction of g by a small (infinitesimal) amount is strictly beneficial. Appendix B provides a behavioral characterization of strict pseudoconcavity that formalizes this intuition leveraging Theorem 7 in GS.[19]

In the Introduction we described alternative intepretations of condition SPC. These build upon the following characterization:

Remark 1 If V is monotonic and locally Lipschitz at $f \in \mathbb{R}^S_{++}$, then it is strictly pseudoconcave at f if and only if

$$\forall g \in \mathbb{R}^S_+ \setminus \{f\}, \forall Q \in \partial V(f): \qquad Q \cdot g = Q \cdot f \quad \Longrightarrow \quad V(g) < V(f). \tag{6}$$

[18] For (ii) ⇒ (iii), the key step is in Remark 4, which follows from standard results.

[19] The notion of (non-strict) pseudoconcavity was introduced by Mangasarian (1965) for differentiable functions; for a definition of (strict) pseudoconvexity for non-smooth functions and related results, see e.g. Penot and Quang (1997).

Furthermore, if V is monotonic, locally Lipschitz and strictly pseudoconcave at f, then V is nice at f.

Thus, first, the hyperplane associated with the vector Q and going through f is tangent to the indifference curve of V going through f, and strictly supports it. Second, interpreting Q as a price vector, any point on the associated budget line (hence, by monotonicity, any point in the budget set) is strictly worse than f. Third, in the special case of a constant bundle $f = 1_S x$, for every $Q \in \partial V(1_S x)$, if one interprets the normalized vector $P = Q/(Q \cdot 1_S)$ as a "local probability," then any bundle g with P-expected value x is worse than x. This can be interpreted as a form of "risk aversion at $1_S x$.[20]"

As these interpretations suggest, strict pseudoconcavity at f is *not* just a local condition. It connects the local behavior of V at f, represented by the Clarke subdifferential $\partial V(f)$, with its global behavior: notice that g can be any bundle, not necessarily close to f. However, the condition is imposed on a specific bundle f; in particular, SPC requires that it hold for all constant bundles. This is in contrast with (strict) quasiconcavity, which imposes a restriction on every pair of bundles.

To further illustrate, the following result shows that condition SPC holds in particular when preferences are strictly convex, as is assumed in RSS's risk-sharing result. It also provides a simple preference characterization that applies if V is smooth at a point; this result further clarifies the connection with quasiconcavity.

Proposition 2 *Suppose that V satisfies Assumption 1.*

1. *If V is strictly quasiconcave, then it is strictly pseudoconcave at every $f \in \mathbb{R}^S_{++}$ where it is nice.*[21] *In particular, SPC holds if and only if V is nice at $1_S x$ for every $x > 0$.*

2. *If V is strictly pseudoconcave at $f \in \mathbb{R}^S_{++}$, then for every $g \in \mathbb{R}^S_+ \setminus \{f\}$,*

$$V(g) \geq V(f) \implies \exists \bar{\lambda} \in (0,1): \quad \forall \lambda \in (0,\bar{\lambda}), \quad V(\lambda g + (1-\lambda) f) > V(f). \tag{7}$$

[20]Under condition SPC, for every $x > 0$, the set $\pi(1_S x)$ coincides with the collection of normalized vectors $P = Q/(Q \cdot 1_S)$, $Q \in \partial V(1_S x)$: see Proposition 8 in Section 5. Thus, the interpretation just given applies to the elements of $\pi(1_S x)$ as well.

[21]For the converse implication, Penot and Quang (1997) show that a locally Lipschitz function on a Banach space that satisfies strict pseudoconcavity *everywhere* is strictly quasiconcave.

3. *If V is continuously differentiable on a neighborhood of $f \in \mathbb{R}^S_{++}$ and $\nabla V(f) \neq 0$, then it is strictly pseudoconcave at f if and only if Eq.* (7) *holds for all $g \in \mathbb{R}^S_+ \setminus \{f\}$.*

Thus, under niceness, assuming *global* strict quasiconcavity ensures that condition SPC holds. However, condition SPC restricts the behavior of the functional V and of its differential *only at certainty*. This allows for violations of (strict or weak) quasiconcavity elsewhere on its domain. As the example in Section 2.3 illustrates, such violations are consistent with interesting patterns of behavior. We provide additional examples in Appendix A.

Eq. (7) states that, if g is preferred to f, then a version of strict quasiconcavity holds around f: mixtures of g and f are strictly preferred to f when the weight on g is small. Strict pseudoconcavity at f always implies this condition, and under smoothness, it is equivalent to it.[22] However, many popular models of ambiguity-sensitive preferences are *not* smooth, especially at certainty. The notion of strict pseudoconcavity is stronger than Eq. (7) (see Example 5 in Online Appendix E.4) and allows us to handle non-smooth preferences as well. For further discussion, see Appendix B, where we provide a behavioral characterization of strict pseudoconcavity for general preferences.

We can now state our main result.

Theorem 3 *Suppose that, for each $i \in N$, Assumption 1 holds, and furthermore V_i satisfies SPC. Then the following are equivalent:*

(ii) Every Pareto-efficient allocation is a full-insurance allocation;

(iii) Every feasible, full-insurance allocation is Pareto-efficient;

(iv) For every feasible, full insurance allocation $(1_S x_1, \ldots, 1_S x_N)$,

$$\bigcap_i \pi_i(1_S x_i) \neq \emptyset.$$

Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.

Items (ii)–(iv) in the statement above correspond to items (ii)–(iv) in Theorem 1. This numbering is intentional: (i) no longer implies the other conditions. See Example 6 in Appendix E.5 for further discussion.

[22] We thank an anonymous referee for suggesting this condition.

In addition to replacing strict convexity with condition SPC, Theorem 3 differs from RSS's risk-sharing result in two related aspects. On one hand, RSS assume that the sets of supporting probabilities $\pi_i(1_S x_i)$ are constant at certainty; there is no corresponding assumption in Theorem 3 (cf. Example 7 in Appendix E.5). On the other hand, the condition in item (iv) of Theorem 1 (RSS's result) involves agents' preferences alone, whereas condition (iv) in Theorem 3 involves both preferences and endowments—agents' sets of supporting probabilities must have a non-empty intersection at all *feasible*, full-insurance allocations. Absent additional assumptions on preferences, a non-empty intersection at *one* such allocation is not enough to ensure that betting is inefficient: see Example 6 in the Online Appendix.

It turns out that, if we additionally adopt RSS's assumption that supporting probabilities at certainty are constant, then we can similarly state condition (iv) purely in terms of preferences. Consider the following definition:

Definition 2 V_i satisfies **condition TIC** (Translation Invariance at Certainty) if $\pi_i(1_S x) = \pi_i(1_S)$ for all $x > 0$.

RSS introduce an axiom that implies condition TIC for convex preferences. In Appendix B, we show that their axiom implies TIC for non-convex preferences as well.

Theorem 4 *Suppose that, for each $i \in N$, Assumption 1 holds, and furthermore V_i satisfies SPC and TIC. Then the following are equivalent:*

(i) There exists an interior, full-insurance Pareto-efficient allocation;

(ii) Every Pareto-efficient allocation is a full-insurance allocation;

(iii) Every feasible, full-insurance allocation is Pareto-efficient;

(iv) $\bigcap_i \pi_i(1_S) \neq \emptyset$.

Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.

Thus, under assumptions SPC and TIC, we obtain a close counterpart to RSS's risk-sharing result (Theorem 1). In particular, note that, unlike in Theorem 3, under TIC condition (i) *does* imply the other conditions, as in RSS's result. The preferences in the example in Section 2.3 satisfy both TIC and SPC, though they are not convex.

5 Analysis of the main results, and the role of SPC

This section provides a discussion of the key steps in the proof of Theorem 3. Omitted proofs and remaining details can be found in Appendix D.4. The analysis highlights the role of condition SPC, and also provides results on risk sharing that may be of independent interest. For this reason, each individual result specifies the assumptions on preferences that are needed. At the end of this section, we compare our proof strategy with the one in RSS.

Recall that the key implications in Theorem 3 are (iii) ⇒ (iv)—if full-insurance allocations are efficient, then agents share some supporting probability—and (iv) ⇒ (ii)—if agents share some supporting probability, then only full-insurance allocations are efficient (i.e., betting is inefficient).

We begin with a result that is reminiscent of the implication (iii) ⇒ (iv). It generalizes the standard result that smooth indifference curves must be tangent at any interior Pareto-efficient allocation. With smooth, convex preferences, the common slope at the point of tangency determines a supporting price vector; as we discuss momentarily, a "local price vector" is also identified in the non-convex, non-smooth case, though the sense in which it "supports" the allocation is more delicate. With this caveat, the following result can also be viewed as a local version of the Second Welfare Theorem.[23]

Proposition 5 *For each $i \in N$, assume that V_i is locally Lipschitz and monotonic. Let $(f_i)_{i\in N}$ be an interior allocation such that each functional V_i is nice at f_i. If $(f_i)_{i\in N}$ is Pareto-efficient, then there exists a vector $p \in \mathbb{R}^S_+ \setminus \{0\}$ and, for each $i \in N$, scalars $\lambda_i > 0$ and vectors $Q_i \in \partial V_i(f_i)$ such that $p = \lambda_i Q_i$ for every i.*

The proof builds upon Bonnisseau and Cornet (1988), who show that, under the stated assumptions, there is a vector p such that $-p$ lies in the intersection of the Clarke normal cones of the upper contour set of V_i at the bundle f_i (see Appendix D.2 for a precise statement and definitions of the required terms). By analogy with the convex case, this suggests interpreting p as a "local price vector.[24]"

[23]This result does not require the assumption of no aggregate uncertainty.

[24]If preferences are convex, the Clarke normal cone coincides with the normal cone in the sense of convex analysis.

For our purposes, it is convenient to restate the above result slightly. Recall that the elements of the sets $\pi_i(\cdot)$ that appear in Theorem 3 are probabilities—that is, they are non-negative vectors normalized to lie in the unit simplex. On the other hand, the elements of the Clarke subdifferential of the functional V_i are arbitrary non-negative vectors—they are not normalized. Following GS, the **normalized Clarke subdifferential** of V_i at $f \in \mathbb{R}^S_{++}$ is

$$C_i(f) = \left\{ \frac{Q}{Q(S)} : Q \in \partial V_i(f),\, Q \neq 0_S \right\}. \tag{8}$$

GS provide a behavioral characterization of the normalized Clarke subdifferential (see Appendix B). This notion allows us to restate Proposition 5 as follows:

Corollary 6 *Let $(f_i)_{i\in N}$ be an interior allocation such that each functional V_i is nice at f_i. If $(f_i)_{i\in N}$ is Pareto-efficient, then $\bigcap_{i\in N} C_i(f_i) \neq \emptyset$.*

The difference between this result and the implication (iii) ⇒ (iv) in Theorem 3 is that Corollary 6 involves the normalized Clarke subdifferentials $C_i(\cdot)$ instead of the sets $\pi_i(\cdot)$. Specifically, Proposition 16 below shows that $\pi_i(f) \subseteq C_i(f)$ for every bundle f; thus, the conclusion in Corollary 6 is weaker than implication (iii) ⇒ (iv) in Theorem 3.

In light of this Corollary, one may conjecture that, if the intersection of normalized Clarke subdifferentials is non-empty at every full-insurance allocation, then betting is inefficient. (Such a conclusion would be analogous to the implication (iv) ⇒ (ii).)

Example 1 in Appendix A shows that this is not the case. Intuitively, if the normalized Clarke subdifferentials have non-empty intersection at an allocation, then *locally* there are no mutually beneficial trades. However, the notion of Pareto efficiency involves more than just local comparisons: there may be Pareto-superior allocations sufficiently far from the given one.[25] Thus, to establish a converse to Corollary 6, we need to ensure that the probabilities in the sets $C_i(\cdot)$ provide *some* global information about preferences as well—at least when constant bundles are considered.

The discussion in Section 4 suggests that condition SPC—a classical notion from optimization theory—can help establish the required connection between local and global behavior of preferences. The following result confirms that this is, indeed, the case.

[25] This is indeed the case in Example 1: consider the allocation $\left(1_S x^h, 1_S(\bar{x} - x^h)\right)$.

Proposition 7 *Assume that, for every $i \in N$, V_i is locally Lipschitz and strongly monotonic, and satisfies Condition SPC. Assume further that, for every feasible, full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$, it is the case that $\bigcap_i C_i(1_S x_i) \neq \emptyset$. Then every Pareto-efficient allocation provides full insurance. Moreover, such an allocation is a competitive equilibrium allocation (with transfers).*

In words, under Condition SPC, if, at every feasible, full-insurance allocation, the agents' normalized differentials intersect, then *betting is Pareto-inefficient*. Indeed, as noted above, by standard arguments, if every Pareto-efficient allocation provides full insurance, then it is also the case that every feasible, full-insurance allocation is Pareto-efficient. Therefore, under the assumptions of Proposition 7, the set of Pareto-efficient allocations coincides with the set of feasible, full-insurance allocations.

Corollary 6 and Proposition 7 characterize the inefficiency of betting in terms of the normalized Clarke subdifferentials of the representing functions V_i. Theorem 3 instead is formulated in terms of the sets $\pi_i(\cdot)$ of supporting probabilities, as is RSS's original result. Thus, the final step in our argument is to relate the sets $\pi_i(\cdot)$ and $C_i(\cdot)$.

Once again, strict pseudoconcavity plays a central role:

Proposition 8 *Fix $i \in N$ and $f \in \mathbb{R}^S_{++}$. Assume that V_i is monotonic and locally Lipschitz. Then V_i is strictly pseudoconcave at f if and only if it is nice at f and $C_i(f) = \pi_i(f)$.*

In particular, under the assumption that each V_i is nice at constant bundles, condition SPC is *equivalent* to the requirement that normalized Clarke subdifferential and the set of supporting probabilities coincide at every constant bundle. This provides the required link between Clarke subdifferentials and supporting probabilities.

We now leverage Propositions 5, 7, and 8 to prove Theorems 3 and 4.

Begin with Theorem 3. As just noted, for every i and $x_i > 0$, since SPC holds, Proposition 8 implies that V_i is nice at $1_S x_i$ and $\pi_i(1_S x_i) = C_i(1_S x_i)$. The implication (ii) ⇒ (iii) is standard. Now assume (iii) and fix a feasible, full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$. Then this allocation is Pareto-efficient. By Proposition 5, $\bigcap_i C_i(1_S x_i) \neq \emptyset$; since $\pi_i(1_S x_i) = C_i(x_i)$ for all i, (iv) holds. Finally, assume (iv): then, since $C_i(1_S x_i) = \pi_i(1_S x_i)$, by Proposition 7, every Pareto-efficient allocation provides full-insurance, i.e., (ii) holds.

Turn now to Theorem 4. Under SPC and TIC, $\pi_i(1_S x) = \pi_i(1_S) = C_i(1_S x) = C_i(1_S)$. Therefore, the condition in (iv) of Theorem 4 is equivalent to the condition in (iv) of Theorem 3. Hence, the equivalence of (ii), (iii) and (iv) follows from Theorem 3. For (i) $\Rightarrow$ (iv), if $(1_S x_1, \ldots, 1_S x_N)$ is an interior, full-insurance Pareto-efficient allocation, since each V_i is nice at $1_S x_i$ by SPC, Proposition 5 implies that $\bigcap_i C_i(1_S x_i) \neq \emptyset$, so (iv) holds by the equalities established above. Finally, (iii) $\Rightarrow$ (i) is immediate.

We emphasize that, if condition SPC is not satisfied, then sharing a supporting probability—i.e., $\bigcap_i \pi_i(1_S x_i) \neq \emptyset$ for all $(1_S x_1, \ldots, 1_S x_N)$—is *neither necessary nor sufficient* for betting to be inefficient. Example 2 in Appendix A shows that it is not necessary. To see that it is not sufficient, consider an Edgeworth-box economy where both agents have risk-neutral expected-utility preferences with the *same* subjective probability P. In this economy, every feasible allocation is Pareto-efficient, including ones that do not provide full insurance; yet, both agents' sets of supporting probabilities (at any bundle) consist of the sole probability P.[26]

Finally, we compare our analysis with RSS's proof of their risk-sharing result (Proposition 9 in their paper). RSS first establish equivalent characterizations of the sets $\pi(\cdot)$ of supporting probabilities. Given these preliminary results, their Proposition 9 "can be understood as [a] straightforward consequence of the basic welfare theorems" (RSS, p. 1178). In particular, the Second Welfare Theorem is employed in the proof that (i) $\Rightarrow$ (iv). Convexity is used in both steps, as well as in the proof that (iv) $\Rightarrow$ (ii).

We also use our characterization of supporting probabilities as normalized differentials to prove Theorems 3 and 4. However, the structure and details of our arguments are different, due to the fact that we do not assume convexity (or, in Theorem 3, translation invariance at certainty). Proposition 8 provides the alternative characterization of supporting probabilities we need; this is of course different from the characterization obtained in RSS. Then, we use our Proposition 5 in lieu of the second welfare theorem, and Proposition 7 to prove that (iv) $\Rightarrow$ (ii).

[26] We owe this example to a referee of an earlier version of this paper.

6 An illustration: smooth VEU preferences

A convenient class of preferences that satisfies the assumptions of Theorem 4, but is not necessarily covered by RSS's result, is the family of VEU preferences (Siniscalchi, 2009) that are continuously differentiable (hence, regular) and GM-ambiguity-averse, but not necessarily convex. In the present setting, a *VEU representation* is a function $V : \mathbb{R}^S_+ \to \mathbb{R}$ such that[27]

$$\forall f \in \mathbb{R}^S_+, \qquad V(f) = \sum_{s\in S} P_s u(f_s) + A\left(\sum_{s\in S} P_s \zeta_{0,s} u(f_s), \ldots, \sum_{s\in S} P_s \zeta_{J-1,s} u(f_s)\right), \tag{9}$$

where $P \in \Delta(S)$ (the *baseline prior*), $u : \mathbb{R}_+ \to \mathbb{R}$ is a Bernoulli utility function, $0 \le J \le |S|$, each $\zeta_j \in \mathbb{R}^S$ (an *adjustment factor*) satisfies $P(\zeta_j) = 0$, and $A : \mathbb{R}^J \to \mathbb{R}$ (the *adjustment function*) satisfies $A(\phi) = A(-\phi)$ for all $\phi \in \mathbb{R}^J$. The preferences in the examples of Sections 2.1 and 2.3 are a special case in which $J = 1$.

Proposition 9 *Assume that V is a VEU representation such that u is strictly increasing, differentiable and strictly concave, A is continuously differentiable with $A(\phi) \le 0$ for all $\phi \in \mathbb{R}^J$, $P(\{s\}) > 0$ for all s and, for all $a \in u(X)^S$ and $s \in S$, $1 + \sum_{0\le j<J} \frac{\partial A}{\partial \phi_j}(P(\zeta_0 a), \ldots, P(\zeta_{J-1} a))\zeta_j(s) > 0$.[28] Then V satisfies Assumption 1; furthermore, SPC holds. Finally, $\pi(1_S) = \{P\}$.*

Thus, under the assumptions of Proposition 9, our risk-sharing results apply. In particular, since VEU preferences also satisfy constant-additivity, TIC holds, so we can invoke Theorem 4. One specific implication is that, in an economy where all agents have VEU preferences satisfying these conditions, risk-sharing obtains *if and only if agents have the same baseline prior P* (but possibly different adjustment factors and functions).

Appendix C considers a broad parametric class of preferences wherein the representation V can be decomposed into a strictly concave Bernoulli utility $u : \mathbb{R}_+ \to \mathbb{R}$ on consumption and a suitable functional I defined over "utility profiles" $u \circ f$. The Expected-Utility, maxmin EU, Choquet, and smooth-ambiguity representations also admit a decomposition of this kind; Eq. (9) shows that so does the VEU representation. Proposition 9 is obtained as a special case within this class.

[27] If $a, b : u(X)^S \to \mathbb{R}$, "$ab$" denotes the function that assigns the value $a(s)b(s)$ to each state s.

[28] This last condition ensures that I is strictly monotonic.

A Examples

A.1 Risk sharing and shared probabilities

We provide two examples that emphasize the different roles of the conditions on shared probabilities discussed in Section 5. Example 1 shows that the condition $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is not sufficient for risk sharing. On the other hand, Example 2 shows that the condition $\bigcap_i \pi_i(1_S x_i) \neq \emptyset$ is not necessary. In both examples, condition SPC is not satisfied by at least one agent.

Example 1 Consider the following two-agent economy. Let $S = \{s_1, s_2\}$; agent 1's preferences are represented by

$$V_1(h) = \max\left(\left[\frac{1}{2}\sqrt{h_1} + \frac{1}{2}\sqrt{h_2}\right]^2, \epsilon + \min_{p \in [0.3, 0.7]} [p h_1 + (1-p) h_2]\right)$$

for some $\epsilon > 0$. Agent 2 has risk-neutral expected-utility preferences, with probability $P_2 = (\frac{1}{3}, \frac{2}{3})$. Figure 3 represents this economy in an Edgeworth box. The solid indifference curves refer to agent 1's preferences and the dashed ones represent agent 2's preferences; as usual, for agent 2, utility increases in the south-western direction. The indifference curves of agent

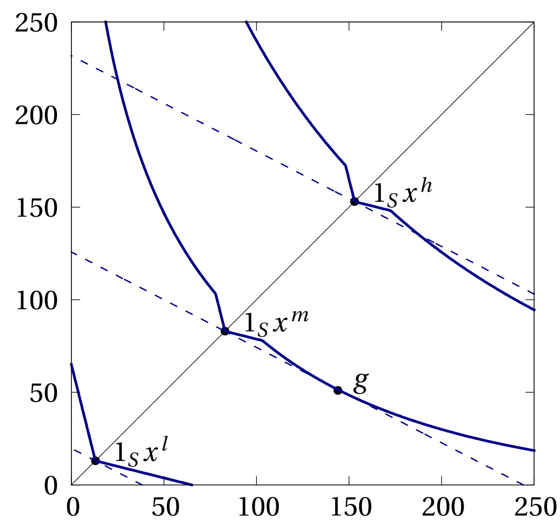

Figure 3: $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is not sufficient for risk sharing.

1 have a small inward "dent" at certainty; in a neighborhood of the 45° line, this preference coincides with the risk-neutral MEU preference with priors $D_1 = \{P \in \Delta(S) : 0.3 \leq P(s_1) \leq 0.7\}$.

Notice that the allocation $\left(1_S x^h, 1_S(\bar{x}-x^h)\right)$ provides full insurance. The normalized Clarke subdifferential of V_1 at $1_S x^h$ (indeed, everywhere on the 45° line) is D_1; hence, it contains that of V_2, which coincides with the probability P_2. Moreover, $\pi_2(1_S(\bar{x}-x^h)) = \{P_2\}$ (see RSS), but $P_2 \notin \pi_1(1_S x^h)$: this follows because, as is apparent from Figure 3, P_2 does not support 1's indifference curve at x^h. Thus, $\pi_1(1_S x^h) \cap \pi_2(1_S(\bar{x}-x^h)) = \emptyset$.

However, $\left(1_S x^h, 1_S(\bar{x}-x^h)\right)$ is not Pareto-efficient. Furthermore, the allocation $(g, 1_S \bar{x} - g)$ is Pareto-efficient, but does not provide full insurance. □

Example 2 Let $S = \{s_1, s_2\}$. Assume that agent 2 has EU preferences, with a prior P_2 that assigns probability 0.4 to state s_1 (on the horizontal axis) and power utility $u(x) = x^{0.2}$. Consumer 1 has preferences represented by

$$V_1(h) = \max\left(\frac{1}{2}h_1 + \frac{1}{2}h_2, \delta + \min_{p\in[0,1]}[p h_1 + (1-p)h_2]\right).$$

Thus, agent 1's preferences are risk-neutral EU, with a uniform prior, except within δ of the certainty line. See Figure 4.

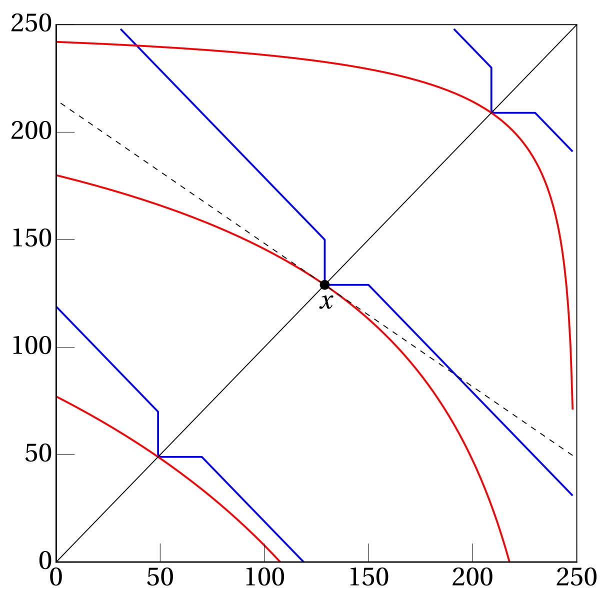

Figure 4: $\bigcap_i \pi_i(1_S x_i) \neq \emptyset$ is not necessary for risk sharing.

For agent 2, the supporting probability set is equal to $\pi_2(1_S x_2) = \{P_2\}$ at every $x_2 > 0$; this follows from RSS's characterization of supporting probabilities for EU preferences. For agent

1, $\pi_1(1_S x) \neq \emptyset$ for all $x > 0$: in particular, the uniform probability belongs to every such set (note that any bundle that is at least as good as x lies strictly above the line going through the point $1_S x$ with slope -1.)

However, $P_2 \notin \pi_1(1_S x_1)$ for x_1 sufficiently large. For instance, consider the point on the certainty line labelled x in Figure 4. Observe that the tangent to 2's indifference curve at x (the dashed black line) crosses 1's indifference curve going through x. This occurs because P_2 is not uniform, and the dent in 1's preferences at certainty (which depends upon the parameter δ) is sufficiently small.

Thus, $\pi_1(1_S x) \cap \pi_2(1_S(\bar{x} - x)) = \emptyset$. However, the value of δ in Figure 4 is chosen so that, given the curvature of 2's utility function, the agents' indifference curves are not tangent anywhere except at certainty. That is, betting is inefficient in this economy: a feasible allocation is Pareto-efficient if and only if it provides full insurance.[29] □

B Behavioral characterization of conditions SPC and TIC

Throughout this section of the Appendix, and the next, we identify a probability distribution $P \in \Delta(S)$ with the linear function it induces on $\mathbb{R}^S$. Thus, we write $P(f)$ instead of $P \cdot f$.

Fix a preference $\succcurlyeq$ represented by a functional V that satisfies Assumption 1.

We first recall an axiom from RSS, and show that it implies condition TIC.

Axiom 1 (Translation Invariance at Certainty) *For all $x, x' \in \mathbb{R}_+$ and $g \in \mathbb{R}^S$: if there is $\lambda > 0$ such that $1_S x + \lambda g \in \mathbb{R}^S_+$ and $1_S x + \lambda g \succcurlyeq x$, then there is $\lambda' > 0$ such that $1_S x' + \lambda' g \in \mathbb{R}^S_+$ and $1_S x' + \lambda' g \succcurlyeq x$.*

Proposition 10 *Assume that $\succcurlyeq$ is represented by V. If Axiom 1 holds, then V satisfies condition TIC.*

Proof: Fix $x, x' \in \mathbb{R}_{++}$ and $P \in \pi(1_S x)$. Consider $g \in \mathbb{R}^S_+$ such that $V(g) \geq V(1_S x')$, i.e., $g \succcurlyeq 1_S x'$. Equivalently, $1_S x' + 1 \cdot (g - 1_S x') \succcurlyeq 1_S x'$. Therefore, by Axiom 1, there exists $\lambda > 0$ such that

[29] In Appendix D.4, we define the set $\pi_i^s(f)$ of *strict* supporting beliefs at $f \in \mathbb{R}^S_{++}$. It is always the case that $\pi_i^s(f) \subseteq \pi_i(f)$. Thus, a fortiori, this example shows that the non-empty intersection of these sets at certainty is not necessary for risk sharing.

$1_S x+\lambda(g-1_S x')\in\mathbb{R}^S_+$ and $1_S x+\lambda(g-1_S x)\succcurlyeq 1_S x$. Since $P\in\pi(1_S x)$, $P(1_S x+\lambda(g-1_S x'))\geq P(1_S x)=x$, i.e., $x+\lambda[P(g)-x']\geq x$, i.e., $P(g)\geq x'$. Since g was arbitrary, $P\in\pi(1_S x')$.

Therefore, $\pi(1_S x)\subseteq\pi(1_S x')$. Repeating the argument switching x and x' yields the required conclusion. ■

Observe that Axiom 1 is only sufficient for condition TIC.[30]

Next, we characterize SPC using a key notion from GS. That paper considers a preference defined over acts mapping states to a convex subset X of a vector space, endowed with a mixture operation, that admits a Bernoulli separable representation (I,u). The utility u is affine with respect to the assumed mixture operation on X. We take $X=\mathbb{R}_+$ and convex combination as the mixture operation. As a result, the utility function is affine on $\mathbb{R}_+$, so it can be taken to be the identity; consequently, in the notation of this paper, $I=V$. This implies that $I=V$ subsumes both risk and ambiguity attitudes, whereas in GS risk attitudes are captured by u. Moreover, since u is taken to be the identity, convergence of acts as defined in GS is convergence in the usual Euclidean topology.

Definition 3 For any pair of acts $f,g\in\mathbb{R}^S_+$ and prize $x\in\mathbb{R}_+$, say that f is a *(weakly) better deviation than g near x*, written $f\succcurlyeq^*_x g$, if, for every $(\lambda^n)_{n\geq 0}\subset[0,1]$ and $(h^n)_{n\geq 0}$ such that $\lambda^n\downarrow 0$ and $h^n\to 1_S x$,

$$\lambda^n f+(1-\lambda^n)h^n\succcurlyeq\lambda^n g+(1-\lambda^n)h^n\quad\text{eventually.}$$

The basic intuition is that f is a better deviation than g at x if, starting from an initial riskless consumption bundle $1_S x$, the DM prefers to move by a vanishingly small amount in the direction of the bundle f rather than in the direction of the bundle g. Furthermore, this remains true if the initial bundle is not exactly $1_S x$, but is close to it. We then have:

Proposition 11 *For every $x>0$, V is strictly pseudoconcave at $1_S x$ if and only if it is nice at $1_S x$ and*

$$\forall g\in\mathbb{R}^S_+,\qquad g\succcurlyeq 1_S x\qquad\Longrightarrow\qquad\exists\delta>0:\quad g\succcurlyeq^*_x 1_S(x+\delta)\tag{10}$$

[30] A full characterization can be obtained under condition SPC, leveraging the results in GS.

To gain intuition, the "local quasiconcavity" property of Eq. (7) in Proposition 2 requires that, if $g \succcurlyeq 1_S x$, then, for all sequences $(\lambda^n) \downarrow 0$,

$$\lambda^n g + (1-\lambda^n) 1_S x \succ 1_S x \quad \text{eventually}$$

which can be written

$$\lambda^n g + (1-\lambda^n) 1_S x \succ \lambda^n 1_S x + (1-\lambda^n) 1_S x \quad \text{eventually.}$$

Thus, moving from x toward g by a small amount is strictly better than moving toward x—that is, not moving away from x at all. Since we do not assume differentiability (cf. Proposition 2 part 3), we need to strenghten this requirement by asking that moving toward g be better than moving toward x starting from bundles close to x as well. (This is also the main insight behind the definition of the Clarke directional derivative.) We thus ask that, for all sequences $(h^n) \to 1_S x$,

$$\lambda^n g + (1-\lambda^n) h^n \succ \lambda^n 1_S x + (1-\lambda^n) h^n \quad \text{eventually.}$$

Finally, the above strict preference is not enough to ensure that, in the limit, g is a strictly better deviation than x from x. To address this, we require that g be better than $x + \delta$ for some (possibly small) $\delta > 0$:

$$\lambda^n g + (1-\lambda^n) h^n \succcurlyeq \lambda^n 1_S (x+\delta) + (1-\lambda^n) h^n \quad \text{eventually.}$$

This is precisely the statement that $g \succcurlyeq_x 1_S(x+\delta)$.

Proof: We first show that the preference $\succcurlyeq^*_x$ is "translation-invariant" in the sense that, for every $f, g \in \mathbb{R}^S_+$ and $\Delta > 0$, $f \succcurlyeq^*_x g$ iff $f + 1_S \Delta \succcurlyeq^*_x g + 1_S \Delta$. Supplementary Appendix S.E of GS shows that $\succcurlyeq^*_x$ satisfies Independence: that is, for all $f', g', h' \in \mathbb{R}^S_+$ and $\lambda \in (0,1)$, $f' \succcurlyeq^*_x g'$ iff $\lambda f' + (1-\lambda) h' \succcurlyeq^*_x \lambda g' + (1-\lambda) h'$. Let $f' = 2f$ and $g' = 2g$; then $f', g' \in \mathbb{R}^S_+$. Take $h' = 2 1_S \Delta$. Then $f + 1_S \Delta \succcurlyeq^*_x g + 1_S \Delta$ is equivalent to $\frac{1}{2} f' + \frac{1}{2} h' \succcurlyeq^*_x \frac{1}{2} g' + \frac{1}{2} h'$; by independence, this holds iff $f' \succcurlyeq^*_x g'$. Now apply independence again with $h'' = 0_S$ to conclude that $f' \succcurlyeq^*_x g'$ iff $\frac{1}{2} f' + \frac{1}{2} h'' \succcurlyeq^*_x \frac{1}{2} g' + \frac{1}{2} h''$. But the latter preference statement is equivalent to $f \succcurlyeq^*_x g$. This proves the claim.

Now fix $\Delta > 0$ throughout. Assume first that V is nice at $1_S x$ and Eq. (10) holds, and consider $g \in \mathbb{R}^S_+$ such that $g \succcurlyeq x$. Then there is $\delta > 0$ such that $g \succcurlyeq^*_x 1_S(x+\delta)$. By the above claim, $g' \equiv g + 1_S \Delta \succcurlyeq^*_x 1_S(x + \delta + \Delta)$. Furthermore, in the language of GS, g' is an interior act:

that is, there are $y, y' \geq 0$ such that $y > g'(s) > y'$ for all s. Since $x, \delta, \Delta > 0$, so is $1_S(x+\delta+\Delta)$. Let (f, f') be a spread of $g', 1_S(x+\delta+\Delta)$: that is, $f(s) > g'(s)$ and $x+\delta+\Delta > f'(s)$ for all s. Since $\succcurlyeq^*_x$ is monotonic, $f \succcurlyeq^*_x f'$. Then, by Theorem 7 in GS, $P(g)+\Delta = P(g') \geq P(1_S[x+\delta+\Delta]) = x+\delta+\Delta$ for every $P \in C(1_S x)$. Equivalently, $P(g) \geq x+\delta$ for all $P \in C(1_S x)$. Since V is nice at $1_S x$, this implies that, for every $Q \in \partial V(1_S x)$, $Q(g)/Q(S) \geq x+\delta$, or $Q(g-1_S x) \geq Q(S)\delta$. Since $\partial V(1_S x)$ is compact, $\min_{Q' \in \partial V(1_S x)} Q'(S)$ is attained by some $Q^* \in \partial V(1_S x)$; since V is monotonic and nice at $1_S x$, $Q^*(S) > 0$. Therefore, $Q(g-1_S x) \geq Q(S)\delta \geq Q^*(S)\delta > 0$ for all $Q \in \partial V(1_S x)$, which shows that V is strictly pseudoconcave at $1_S x$.

Conversely, assume that V is strictly pseudoconcave at $1_S x$ and consider $g \in \mathbb{R}^S_+$ such that $g \succ 1_S x$. Then V is nice at $1_S x$, because it is monotonic (see Remark 1). Furthermore $Q(g - 1_S x) > 0$ for all $Q \in \partial V(1_S x)$. Equivalently, $Q([g+1_S\Delta]-1_S[x+\Delta]) > 0$ for all $Q \in \partial V(1_S x)$. Since $\partial V(1_S x)$ is compact, there are $Q^+, Q^- \in \partial V(1_S x)$ such that $Q^-([g+1_S\Delta]-1_S[x+\Delta]) = \min_{Q' \in \partial V(1_S x} Q'([g+1_S\Delta]-1_S[x+\Delta])$ and $Q^+(S) = \max_{Q' \in \partial V(1_S x)} Q'(S)$. By strict pseudoconcavity at $1_S x$, $Q^-([g+1_S\Delta]-1_S[x+\Delta]) > 0$; and by niceness at $1_S x$, $Q^+(S) > 0$. Let

$$\eta = \frac{Q^-[g+1_S\Delta]-1_S[x+\Delta]}{Q^+(S)} > 0.$$

Observe that, for every $Q \in \partial V(1_S x)$,

$$Q([g+1_S\Delta]-1_S[x+\Delta]) \geq Q^-([g+1_S\Delta]-1_S[x+\Delta]) = \eta Q^+(S) \geq \eta Q(S).$$

Now let $\epsilon > 0$ be such that $\epsilon < \Delta$ and $\epsilon < \frac{1}{2}\eta$. Then

$$Q([g+1_S(\Delta-\epsilon)]-1_S[x+\Delta+\epsilon]) = Q([g+1_S\Delta]-1_S[x+\Delta]) - Q(S)(2\epsilon) \geq (\eta-2\epsilon)Q(S)$$

for every $Q \in \partial V(1_S x)$. Furthermore, by the choice of ϵ, the act $g+1_S(\Delta-\epsilon)$ is interior, and $\delta \equiv \eta - 2\epsilon > 0$. Further rewrite this as

$$Q([g+1_S(\Delta-\epsilon)]-1_S[x+\Delta+\epsilon+\delta]) \geq 0$$

for all $Q \in \partial V(1_S x)$. Hence,

$$P(g+1_S(\Delta-\epsilon)) \geq P(1_S[x+\Delta+\epsilon+\delta])$$

for all $P \in C(1_S x)$. The act $1_S[x+\Delta+\epsilon+\delta]$ is of course also interior. Then, by Theorem 7 in GS, for all spreads (f, f') of $(g+1_S[\Delta-\epsilon], 1_S[x+\Delta+\epsilon+\delta])$, $f \succcurlyeq^*_x f'$. One particular such spread is $(g+1_S\Delta, 1_S[x+\Delta+\delta])$. Thus, $g+1_S\Delta \succcurlyeq^*_x 1_S[x+\Delta+\delta]$. By the above claim, this holds iff $g \succcurlyeq^*_x 1_S[x+\delta]$. Since g was arbitrary, Eq. (10) holds. ■

C Preference representations with strictly concave utility

Most representations of ambiguity-sensitive preferences used in applications decompose the functional V into a Bernoulli utility function $u:\mathbb{R}_+\to\mathbb{R}$, and an aggregator $I:\mathbb{R}^S\to\mathbb{R}$: that is, for every $f\in\mathbb{R}^S_+$,

$$V(f)=I(u\circ f),$$

where $u\circ f=\big(u(f(s))\big)_{s\in S}\in\mathbb{R}^S$ is the utility vector associated with the bundle f. For instance, MEU preferences admit such a representation, with $I:\mathbb{R}^S\to\mathbb{R}$ given by

$$I(a)=\min_{P\in D}\int a\,dP \qquad \forall a\in\mathbb{R}^S$$

where $D\subseteq\Delta(S)$. Analogously, for smooth ambiguity-averse preferences (Klibanoff, Marinacci, and Mukerji, 2005),

$$I(a)=\phi^{-1}\left(\int_{\Delta(S)}\phi\left(\int_S a\,dP\right)d\mu\right) \qquad \forall a\in u(X)^S,$$

where μ is a (second-order) probability on $\Delta(S)$ and ϕ is a concave (second-order) utility defined on the range of u. As noted in Section 6, the VEU representation also admits such a decomposition.

This Appendix provides conditions that ensure that both Assumption 1 and SPC hold for all preferences that admit a decomposition (I,u). Begin with basic assumptions about the functional representation.

Assumption 2

1. $u:\mathbb{R}_+\to\mathbb{R}$ is strictly increasing, strictly concave, and differentiable;

2. $I:\mathbb{R}^S\to\mathbb{R}$ is normalized, strongly monotonic, locally Lipschitz and nice at each $1_S u(x)$, $x>0$.

The following key assumption is the counterpart of Condition SPC. Unlike the latter, it refers to the function I, which is defined over utility vectors, rather than acts.

Definition 4 Fix an act $f \in \mathbb{R}^S_{++}$. The function I is **∂-quasiconcave at** $u \circ f$ if

$$\forall g \in \mathbb{R}^S_+, \qquad I(u \circ g) \geq I(u \circ f) \quad \Longrightarrow \quad \forall Q \in \partial I(u \circ f), \quad Q \cdot (u \circ g - u \circ f) \geq 0. \tag{11}$$

The function I satisfies **differential quasiconcavity at certainty (DQC)** if it is ∂-quasiconcave at $1_S u(x)$ for all $x > 0$.

To compare with SPC, consider an act g such that $V(g) = I(u \circ g) \geq I(u \circ f) = V(f)$. Strict pseudoconcavity at f requires that $Q \cdot (g - f) > 0$ for all $Q \in \partial V(f)$; on the other hand, ∂-quasiconcavity at $u \circ f$ requires that $Q \cdot (u \circ g - u \circ f) \geq 0$ for all $Q \in \partial I(u \circ f)$. As noted above, these assumptions pertain to different objects: in general, $\partial V(f) \neq \partial I(u \circ f)$. Yet the interpretation is similar: if g is at least as good as f, then moving from the utility vector $u \circ f$ towards the vector $u \circ g$ should increase the value of I, at least weakly. Note also that SPC requires a strict inequality, whereas DQC allows for a weak one. We now show that, when u is strictly concave, this is sufficient to imply that V satisfies SPC. This is the main result of this section.

Proposition 12 *If (I, u) satisfy Assumption 2, then V satisfies Assumption 1 and is nice at every $1_S x$, $x > 0$. If in addition I satisfies DQC, then SPC holds.*

(The term "∂-quasiconcave" is due to Penot and Quang, 1997.)

Condition DQC holds in two useful special cases. We need two additional definitions. The first is due to GM, to which we refer the reader for interpretation.

$$\text{Core } I = \{P \in \Delta(S) : \forall f \in \mathbb{R}^S_+,\ I(u \circ f) \leq P \cdot (u \circ f)\}. \tag{12}$$

GM deem *ambiguity-averse* a preference represented by a functional I such that Core $I \neq \emptyset$.

The second is due to Clarke (1983). A locally Lipschitz function $J : \mathbb{R}^S \to \mathbb{R}$ is **regular** at $b \in \mathbb{R}^S$ if its directional derivative

$$J'(b; a) = \lim_{t \downarrow 0} \frac{J(b + ta) - J(b)}{t} \tag{13}$$

is well-defined for all $a \in \mathbb{R}^S$, and coincides with $\max_{Q\in\partial J(b)} Q \cdot a$: see Clarke (1983, Def. 2.3.4). If J is continuously differentiable at b, then it is regular there (Clarke, 1983, Corollary to Proposition 2.2.1, and Proposition 2.3.6 (a)).

Corollary 13 *Suppose that (I, u) satisfy Assumption 2. Then I satisfies DQC, and hence SPC holds, if one of the conditions below is satisfied:*

1. *I is quasiconcave, or*
2. Core $I \neq \emptyset$ *and I is regular at every $1_S u(x)$, $x > 0$.*

The functional I in the MEU representation is normalized, strongly monotonic, globally Lipschitz, and nice everywhere on its domain; furthermore, it is concave. The functional I in the smooth ambiguity-averse representation is also normalized, and if ϕ is continuously differentiable, satisfies $\phi'(r) > 0$ for all $r \in u(\mathbb{R}_{++})$, and is strictly concave, then I is also strongly monotonic, locally Lipschitz, nice at certainty, and quasiconcave.[31] Thus, condition 1 of Corollary 13 holds, and SPC is satisfied. Analogous conclusions hold for other parametric uncertainty-averse models, such as variational preferences (Maccheroni, Marinacci, and Rustichini, 2006) and confidence-function preferences (Chateauneuf and Faro, 2009). Of course, these prefence models also satisfy the assumptions in RSS. Proposition 9 instead provides conditions under which Corollary 13 part 2 applies for the VEU representation.

While Corollary 13 provides easy-to-check sufficient conditions for SPC, these conditions are not necessary. Thus, Theorems 3 and 4 cover a broader set of preferences than the ones that admit a decomposable representation satisfying the assumptions in Corollary 13. Example 3 in the Online Appendix E illustrates this.

Moreover, the conditions in Corollary 13 are restrictive in conjunction with certain structural properties of the functional I. For instance, this is the case for Choquet-expected utility preferences (Schmeidler, 1989). Example 4 in the Online Appendix E provides the details.

[31] A direct calculation shows that, under the stated assumptions on ϕ, I is continuously differentiable, and thus locally Lipschitz; furthermore, its gradient at any $1_S\gamma$, $\gamma \in u(\mathbb{R}_{++})$, is $\int_{\Delta(S)} P\, d\mu \neq 0_S$, so I is nice at certainty. Finally, since $a \mapsto \int_{\Delta(S)} \phi(\int a\, dP) d\mu$ is concave and ϕ^{-1} is strictly increasing, I is quasiconcave.

D Proofs

As in the previous Appendix, for $P \in \Delta(S)$ and $f \in \mathbb{R}^S$, we write $P(f)$ instead of $P \cdot f$. For expositional reasons, Proposition 16 is proved "out of order," in Appendix D.3.

D.1 Preliminaries: Clarke derivatives and differentials

We introduce additional notation and definitions related to differentials and their properties.

Fix a function $F : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on an open set $B \subset \mathbb{R}^n$. The **Clarke upper derivative** of F at $b \in B$ in the direction $a \in \mathbb{R}^n$ is

$$F^{\circ}(b;a) = \limsup_{t \downarrow 0, c \to b} \frac{F(c+ta) - F(c)}{t}; \tag{14}$$

Clarke (1983) shows that the set $\partial F(b) \equiv \{Q \in \mathbb{R}^n : Q \cdot a \leq F^{\circ}(b;a)\}$ is such that $F^{\circ}(b;a) = \max_{Q \in \partial F(b)} Q(a)$. Furthermore, it admits the characterization given in Eq. (3) in Section 3.

The **Clarke lower derivative** (cf. Ghirardato et al., 2004, pp. 150 and 157) is instead

$$F^{\ell}(b;a) = \liminf_{t \downarrow 0, c \to b} \frac{F(c+ta) - F(c)}{t}; \tag{15}$$

It is readily verified that $F^{\ell}(b;a) = -F^{\circ}(b;-a)$ and, therefore, $F^{\ell}(b;a) = \min_{Q \in \partial F(b)} Q(a)$ for all $b \in B$ and all $a \in \mathbb{R}^n$.

D.2 Preliminaries: Clarke tangent and normal cones; supporting probabilities

The following geometric notions will be useful. For every bundle $f \in \mathbb{R}^S_+$, let

$$U(f) = \{g \in \mathbb{R}^S_+ : g \succcurlyeq f\},$$

the upper countour set of the preference $\succcurlyeq$ at f. For every set $C \subset \mathbb{R}^S_+$ and bundle $f \in \mathbb{R}^S_+$, let

$$d_C(f) = \inf\{\|f - g\| : g \in C\}$$

The *Clarke tangent cone* to C at some $f \in C$ is

$$T_C(f) = \{v \in \mathbb{R}^S : (d_C)^0(f;v) = 0\},$$

i.e. the set of directions v for which the Clarke derivative of the distance function (which is Lipschitz and convex) is zero. The following characterization (Clarke, 1983, Theorem 2.4.5) is useful:

$$T_C(f) = \{v \in \mathbb{R}^S : \forall (f^k, t^k) \subset C \times \mathbb{R}_{++} \text{ s.t. } f^k \to f,\ t^k \downarrow 0,\ \exists (v^k) \subset \mathbb{R}^S \text{ s.t. } v^k \to v,\ f^k + t^k v^k \in C\ \forall k\}.$$

Finally, define the *Clarke normal cone* to C at f by polarity:

$$N_C(f) = \{Q \in ba(S) = \mathbb{R}^S : Q(v) \leq 0\ \forall v \in T_C(f)\}.$$

Specializing to our environment, we have

$$T(f) \equiv T_{U(f)}(f) = \big\{v \in \mathbb{R}^S : \forall (f^k, t^k) \subset \mathbb{R}^S_+ \times \mathbb{R}_{++} \text{ s.t. } f^k \succcurlyeq f\ \forall k,\ f^k \to f,\ t^k \downarrow 0,$$
$$\exists (v^k) \subset \mathbb{R}^S \text{ s.t. } v^k \to v,\ f^k + t^k v^k \succcurlyeq f\ \forall k\big\}.$$

and it is convenient to define

$$N(f) \equiv N_{U(f)}(f) = \{Q \in \mathbb{R}^S : Q(v) \leq 0\ \forall v \in T(f)\}.$$

Loosely speaking, $T(f)$ is the set of directions v with the property that any sequence of bundles preferred to f and converging to it can be perturbed in the direction v without leaving the upper contour set of f. More informally, moving from bundles near f in the direction v by a small amount leads to an act that is at least as good as f. Then, if Q is in the normal cone, $-Q$ is a price vector that assigns non-negative value to such changes.

The following two results pertain to the Clarke normal cone. Note that the first does not require any specific assumption on the functional V.

Remark 2 For every bundle $f \in \mathbb{R}^S_{++}$, $-\pi(f) \subseteq N(f)$.

Proof: Fix $P \in \pi(f)$. Consider $v \in T(f)$, the constant sequence $f^k \equiv f$, and an arbitrary sequence $(t^k) \downarrow 0$. Since $v \in T(f)$, there exists a sequence $(v^k) \to v$ such that, for every k, $f^k + t^k v^k \succcurlyeq f$, i.e., $V(f + t^k v^k) \geq V(f)$. Since $P \in \pi(f)$, $P(f + t^k v^k) \geq P(f)$, and therefore $P(v^k) \geq 0$ for every k. By continuity, $P(v) \geq 0$. Therefore, $-P \in N(f)$. ■

Remark 3 For every bundle $f \in \mathbb{R}^S_{++}$, if V is locally Lipschitz and nice at f, then $N(f) \subseteq \bigcup_{\lambda \geq 0} \lambda\big(-\partial V(f)\big)$. Thus, for any such f, if $R \in N(f) \setminus \{0_S\}$, there is $\lambda > 0$ and $Q \in \partial V(f)$ such that $R = -\lambda Q$.

Proof: Let $W = -V$, and note that $U(f) = \{g \in \mathbb{R}^S_+ : W(g) \leq W(f)\}$. By Proposition 2.3.1 in Clarke (1983), $\partial W(f) = -\partial V(f)$. Thus, if V is nice at f, so is W; moreover, by Corollary 1 to Theorem 2.4.7 in Clarke (1983), $N(f) \subset \bigcup_{\lambda \geq 0} \lambda \partial W(f) = \bigcup_{\lambda \geq 0} \lambda\big(-V(f)\big)$, as claimed.

Hence, if $R \in N(f)$, there is $Q \in \partial V(f)$ and $\lambda > 0$ such that $R = -\lambda Q$. ■

The following result restates the definition of $\pi(f)$ for $f \in \mathbb{R}^S_{++}$.

Lemma 14 *Assume that V is strongly monotonic and continuous. For every $f \in \mathbb{R}^S_{++}$, $\pi(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ P(f) \geq P(g) \Longrightarrow V(f) \geq V(g)\}$.*

Proof: Denote the set on the rhs of the Remark by $\hat{\pi}(f)$. Suppose that $P \in \pi(f)$. We show that, for every $g \in \mathbb{R}^S_+$, $V(g) > V(f)$ implies $P(g) > P(f)$, so $P \in \hat{\pi}(f)$. Fix g and suppose $V(g) > V(f)$. Since $P \in \pi(f)$, $P(g) \geq P(f)$. By contradiction, suppose $P(g) = P(f)$. Then, there must be a state s such that $g(s) \geq f(s)$, thus $g(s) > 0$, and $P(\{s\}) > 0$. By continuity of V, there is $\epsilon > 0$ such that $g(s) - \epsilon > 0$ and the bundle g' defined by $g'(s) = g(s) - \epsilon$ and $g'(s') = g(s')$ for $s' \neq s$ satisfies $V(g') > V(f)$. But $P(g') = P(g) - P(\{s\})\epsilon < P(g) = P(f)$, which contradicts the assumption that $P \in \pi(f)$. Thus $P(g) > P(f)$.

Conversely, suppose that $P \in \hat{\pi}(f)$. We show that, for every $g \in \mathbb{R}^S_+$, $P(f) > P(g)$ implies $V(f) > V(g)$, so $P \in \pi(f)$. Fix g and suppose that $P(f) > P(g)$. Since $P \in \hat{\pi}(f)$, $V(f) \geq V(g)$. By contradiction, suppose $V(f) = V(g)$. Then there is $\epsilon > 0$ such that $P(f) > P(g + 1_S\epsilon)$; however, by strong monotonicity $V(g + 1_S\epsilon) > V(g) = V(f)$, which contradicts the assumption that $P \in \hat{\pi}(f)$. Hence, $V(f) > V(g)$. ■

Corollary 15 *For every $f \in \mathbb{R}^S_{++}$ and $P \in \pi(f)$, $P(\{s\}) > 0$ for all $s \in S$.*

Proof: Fix $f \in \mathbb{R}^S_{++}$, $s \in S$ and $P \in \pi(f)$. Define g by $g(s) = f(s) + 1$ and $g(s') = f(s')$ for all $s' \neq s$. If $P(\{s\}) = 0$, then $P(f) = P(g)$, and therefore, by Lemma 14, $V(f) \geq V(g)$: this contradicts the assumption that V is strongly monotonic. ■

D.3 Preliminaries on strict quasiconcavity and strict pseudoconcavity

Proof of Remark 1: suppose that V is strictly pseudoconcave at f and consider arbitrary $g \in \mathbb{R}^S_{++}, Q \in \partial V(f)$ such that $g \neq f$ and $Q(g) = Q(f)$. If $V(g) \geq V(f)$, strict pseudoconcavity at f implies $Q(g-f) > 0$, contradiction: thus, $V(g) < V(f)$, so Eq. (6) holds at f.

Conversely, assume that Eq. (6) holds at f, and consider $g \in \mathbb{R}^S_{++} \setminus \{f\}, Q \in \partial V(f)$ such that $V(g) \geq V(f)$. Suppose that $Q(g-f) \leq 0$. There are two cases. If $Q(1_S) = 0$, then by monotonicity $Q = 0_S$. Therefore, $Q(g) = Q(f)$ and so Eq. (6) implies $V(g) < V(f)$, contradiction. If instead $Q(1_S) \neq 0$, so $Q(1_S) > 0$ by monotonicity, let $g' = g + 1_S Q(f-g)/Q(1_S) \geq g$; this is well-defined because $Q(1_S) > 0$. Then $Q(g') = Q(g) + Q(f-g)/Q(1_S) \cdot Q(1_S) = Q(f)$, so by Eq. (6) $V(g') < V(f)$. But $g' \geq g$ and monotonicity imply $V(g) < V(f)$, contradiction. Thus, $Q(g-f) > 0$.

Finally, suppose that V is strictly pseudoconcave at f. Let $g = f + 1_S \epsilon$ for $\epsilon > 0$. By monotonicity, $V(g) \geq V(f)$. By strict pseudoconcavity, if $0_S \in \partial V(f)$, then $0_S \cdot (g-f) > 0$, contradiction. Thus, V is nice at f. ■

The following result clarifies the relationship between the sets $C(f)$ and $\pi(f)$, for $f \in \mathbb{R}^S_+$, in particular under the assumptions of strict quasiconcavity and strict pseudoconcavity. Proposition 8 in Section 5 follows from it as an immediate corollary. In addition, we introduce the set of "strict" supporting probabilities; this plays a role in the proof of Proposition 2, as well as in Proposition 18 of Appendix D.4 below. For every bundle f, let

$$\pi^s(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+ \setminus \{f\},\ V(g) \geq V(f) \Longrightarrow P(g) > P(f)\}. \tag{16}$$

Thus, for every $g \in \mathbb{R}^S_+$ such that $V(g) \geq V(f)$, probabilities $P \in \pi(f)$ satisfy $P \cdot g \geq P \cdot f$; however, probabilities $P \in \pi^s(f)$ satisfy the stronger condition $P \cdot g > P \cdot f$. It follows that $\pi^s(f) \subseteq \pi(f)$, but the inclusion may be strict for more general preferences (consider the case of risk-neutral EU preferences).

We also introduce terminology to refer to the condition in Eq. (7) in Proposition 2. Say that V is **strictly quasiconcave at** $f \in \mathbb{R}^S_{++}$ if, for every $g \in \mathbb{R}^S_+ \setminus \{f\}$ such that $V(g) \geq V(f)$, there is

$\bar{\lambda} \in (0,1)$ such that $V(\lambda g + (1-\lambda)f) > V(f)$ for all $\lambda \in (0,\bar{\lambda})$—that is, if. Eq. (7) holds for g. This condition is of course implied by strict quasiconcavity.

Proposition 16 *Assume that V is locally Lipschitz and monotonic.*[32] *Fix $f \in \mathbb{R}^S_{++}$.*

1. *for every $f \in \mathbb{R}^S_{++}$, $\pi^s(f) \subseteq \pi(f)$; if V is strictly quasiconcave at f, then $\pi^s(f) = \pi(f)$.*
2. *for every $f \in \mathbb{R}^S_{++}$, if V is nice at f, then $\pi(f) \subseteq C(f)$;*
3. *if V is strictly pseudoconcave at f, then V is nice at f, and $C(f) \subseteq \pi^s(f)$;*
4. *conversely, if V is nice at f, and $C(f) \subseteq \pi^s(f)$, then V is strictly pseudoconcave at f.*

Thus, V is strictly pseudoconcave at f if and only if it is nice at f and $C(f) = \pi^s(f) = \pi(f)$.

Proof of Proposition 16: (1) fix $f \in \mathbb{R}^S_{++}$. The statement that $\pi^s(f) \subseteq \pi(f)$ is immediate from the definitions. Now suppose that V is strictly quasiconcave at f. Fix $P \in \pi(f)$ and $g \in \mathbb{R}^S_{++} \setminus \{f\}$ such that $V(g) \geq V(f)$. Since $P \in \pi(f)$, $P(g) \geq P(f)$. By contradiction, suppose that $P(g) = P(f)$. We consider two cases.

First, if $V(g) > V(f)$, then by continuity of V there is $\alpha \in (0,1)$ such that $V(\alpha g) > V(f)$. However, since $f \in \mathbb{R}^S_{++}$, $P(g) = P(f) > 0$ and so $P(\alpha g) = \alpha P(g) < P(g) = P(f)$. This contradicts the fact that $P \in \pi(f)$.

Second, if $V(g) = V(f)$, then by strict quasiconcavity of V at f, there is $\lambda \in (0,1)$ such that $g' = \lambda g + (1-\lambda)f$ satisfies $V(g') > V(f)$. Moreover, $P(g') = \lambda P(g) + (1-\lambda)P(f) = P(f)$ because $P(g) = P(f)$. But then, applying the argument in the preceding paragraph to g' yields a contradiction.

(2): fix $f \in \mathbb{R}^S_{++}$ and consider $P \in \pi(f)$. By Remark 2, $-P \in N(f)$. By Remark 3, if V is nice at f, then there are $\lambda > 0$ and $Q \in \partial V(f)$ such that $-P = \lambda(-Q)$, i.e., $P = \lambda Q$. Furthermore, $1 = P(S) = \lambda Q(S)$, so $\lambda = Q(S)^{-1}$ and $P = \frac{Q}{Q(S)} \in C(f)$, as required.

(3): that V is nice at f was shown in Remark 1. Let $P \in C(f)$ and consider $g \in \mathbb{R}^S_{+} \setminus \{f\}$ such that $V(g) \geq V(f)$. By strict pseudoconcavity, this implies that, for every $Q \in \partial V(f)$, $Q(g-f) > 0$, i.e., $Q(g) > Q(f)$. In particular, since $P = Q/Q(S)$ for some $Q \in \partial V(f)$ with $Q(S) > 0$ (by monotonicity, $Q \neq 0$ implies $Q > 0$), $P(g) > P(f)$. Thus, $P \in \pi^s(f)$.

[32] Monotonicity is only used in the proof of parts 3 and 4, and hence in the last statement.

(4): consider $g \in \mathbb{R}^S_+$ such that $V(g) \geq V(f)$ and $Q \in \partial V(f)$. Since V is nice at f and V is monotonic, $Q(S) > 0$ and so $Q/Q(S) \in C(f)$. By assumption, also $Q/Q(S) \in \pi^s(f)$, and therefore $V(g) \geq V(f)$ implies $Q(g)/Q(S) > Q(f)/Q(S)$, i.e., $Q(g) > Q(f)$, or $Q(g-f) > 0$. Since $Q \in \partial V(f)$ was arbitrary, V is strictly pseudoconcave at f.

Finally, assume that V is strictly pseudoconcave at f. By part 3, V is nice at f, and $C(f) \subseteq \pi^s(f)$; by the first part of 1, $\pi^s(f) \subseteq \pi(f)$; and by part 2, since V is nice at f, $\pi(f) \subseteq C(f)$; therefore, $\pi^s(f) = \pi(f) = C(f)$. Conversely, assume that V is nice at f and $\pi^s(f) = \pi(f) = C(f)$: then by part 4, V is strictly pseudoconcave at f. ■

Proof of Proposition 2:

(1): We first show that V is ∂-quasiconcave (Definition 4).

Fix $f \in \mathbb{R}^S_{++}$ and $g \in \mathbb{R}^S_+$ such that $V(g) \geq V(f)$. Also fix $\epsilon > 0$ and let $g_\epsilon = g + 1_S \epsilon$. By strong monotonicity, $V(g_\epsilon) > V(g) \geq V(f)$. Consider sequences $(c^k) \subset \mathbb{R}^S_+$ and $(t^k) \subset \mathbb{R}_{++}$ such that $c^k \to f$ and $t^k \downarrow 0$. Note that

$$t^k[g_\epsilon - f] + c^k = t^k[g_\epsilon - f + c^k] + (1-t^k)c^k$$

and, since $c^k \to f \in \mathbb{R}^S_{++}$, eventually $g_\epsilon - f + c^k \in \mathbb{R}^S_{++}$; furthermore, by continuity $V(g_\epsilon - f + c^k) \to V(g_\epsilon)$ and $V(c^k) \to V(f)$. Therefore, for k sufficiently large, $V(g_\epsilon - f + c^k) > V(c^k)$. Then, by (strict) quasiconcavity, for all such k,

$$V(t^k[g_\epsilon - f] + c^k) = V(t^k[g_\epsilon - f + c^k] + (1-t^k)c^k) \geq V(c^k).$$

It follows that

$$V^\ell(f; g_\epsilon - f) = \liminf_{c \to f, t \downarrow 0} \frac{V(t[g_\epsilon - f] + c) - V(c)}{t} \geq 0.$$

Finally, since this holds for all $\epsilon > 0$, by continuity of $V^\ell(f; \cdot)$, $V^\ell(f; g - f) \geq 0$ as well. Since $V^\ell(f; g-f) = \min_{Q \in \partial V(f)} Q(g-f)$, it follows that $Q(g) \geq Q(f)$ for all $Q \in \partial V(f)$.

We now show that this implies that $C(f) \subseteq \pi(f)$. Since, for all $g \in \mathbb{R}^S_+$, $V(g) \geq V(f)$ implies $Q(g) \geq Q(f)$ for all $Q \in \partial V(f)$, this is true in particular for $Q \in \partial V(f)$ such that $Q(S) > 0$. Therefore, if $P \in C(f)$, then $P(g) \geq P(f)$. Hence, $P \in \pi(f)$, as claimed.

To complete the proof, Proposition 16 part 1 implies that, if V is strictly quasiconcave, then $\pi^s(f)=\pi(f)$ for all $f\in\mathbb{R}^S_{++}$. Conclude that $C(f)\subseteq\pi^s(f)$ for all such f. By Proposition 16 part 4, if in addition V is nice at f, then it is strictly pseudoconcave at f.

(2): Suppose that V satisfies strict pseudoconcavity at f. Since V is locally Lipschitz, there is $\hat{\lambda}\in(0,1)$ such that, for all $\lambda\in(0,\hat{\lambda})$, $\lambda g+(1-\lambda)f$ lies in a neighborhood of f where V satisfies a Lipschitz condition of rank L. Then, by (Clarke, 1983, Proposition 2.1.2), for every such λ and every $Q_\lambda\in\partial V(\lambda g+(1-\lambda)f)$, $\|Q_\lambda\|\leq L$.

We claim that there is $\bar{\lambda}\in(0,\hat{\lambda})$ such that, for all $\lambda\in(0,\bar{\lambda})$ and all $Q_\lambda\in\partial V(\lambda g+(1-\lambda)f)$, $Q_\lambda(g-f)>0$. Suppose not: then, for every $k\geq 1$, there are $\lambda_k\in(0,\frac{1}{k})$ and $Q_k\in\partial V(\lambda_k g+(1-\lambda_k)f)$ such that $Q_k(g-f)\leq 0$. Since (Q_k) lies in a compact set, it has a convergent subsequence, $(Q_{k(\ell)})$ with limit Q. Furthermore, $\lambda_{k(\ell)}g+(1-\lambda_{k(\ell)})f\to f$. By Clarke (1983, Proposition 2.1.5) $Q\in\partial V(f)$. Furthermore, by continuity of expectations, $Q(g)\leq Q(f)$. This contradicts the assumption that V satisfies strict pseudoconcavity at f.

By the Mean Value Theorem (Clarke, 1983, Theorem 2.3.7), for every $\lambda\in(0,\bar{\lambda})$ there is $\lambda'\in(0,\lambda)$ and $Q\in\partial V(\lambda' g+(1-\lambda')f)$ such that $V(\lambda g+(1-\lambda)f)-V(f)=Q(\lambda g+(1-\lambda)f)-Q(f)=\lambda[Q(g)-Q(f)]$. By the claim just proved, the last term is strictly positive. Therefore, $V(\lambda g+(1-\lambda)f)>V(f)$.

(3): Suppose that V is continuously differentiable on a neighborhood U of f, with $\nabla V(f)\neq 0$. It is enough to show that Eq. (7) implies that V is strictly pseudoconcave at f. The argument is similar to that for part 1, but since we can use directional derivatives, the details are straightforward.[33] We first show that V is ∂-quasiconcave at f. Fix $g\in\mathbb{R}^S_+\setminus\{f\}$ with $V(g)\geq V(f)$. The derivative of V at f in the direction $g-f$ is

$$V'(f;g-f)=\lim_{t\to 0}\frac{V(t(g-f)+f)-V(f)}{t}=\lim_{t\to 0}\frac{V(tg+(1-t)f)-V(f)}{t}\geq 0,$$

where the inequality follows because, by Eq. (7), for t small $V(tg+(1-t)f)>V(f)$. Since V is continuously differentiable at f, $\partial V(f)=\{\nabla V(f)\}$ and $0\leq V'(f;g-f)=\nabla V(f)\cdot(g-f)$. Thus, V is ∂-quasiconcave at f.

[33] Note also that, in part 1, global quasiconcavity—rather than just Eq. (7)—is required precisely because the definition of the Clarke directional derivative requires considering points in a neighborhood of f that are not necessarily on the segment joining f and g.

To conclude the proof, as in the argument for part 1, since $\nabla V(f) \neq 0$, $\nabla V(f)/(\nabla V(f) \cdot 1_S) \in \pi(f)$; and since Eq. (7) implies that V is strictly quasiconcave at f, $\pi(f) = \pi^s(f)$. But then $V(g) \geq V(f)$ implies $\nabla V(f) \cdot (g - f) > 0$, so V is strictly pseudoconcave at f. ■

D.4 Results in Section 5

The key step in the proof of Proposition 5 is contained in the following result.

Lemma 17 *Assume that each V_i is monotonic. If $(f_i)_{i\in N}$ is a Pareto-efficient allocation, then there exists a price vector $p \in \mathbb{R}^S_+ \setminus \{0\}$ such that $-p \in N_i(f_i)$ for all $i \in N$.*

Proof: Apply Prop. 2.1 (a) and (e) and Theorem 2.1 in Bonnisseau and Cornet (1988) to get $-p \in \bigcap_{i\in N} N_i(f_i)$. We only need to show that p is non-negative. By monotonicity, $\mathbb{R}^S_+ \subset T_i(f_i)$: to see this, note that, if $v \in \mathbb{R}^S_+$, then for any sequence (f^k, t^k) such that $f^k \succcurlyeq_i f_i$, $f^k \to f_i$, and $t \downarrow 0$, the constant sequence $v^k = v$ satisfies $f^k + t^k v^k \geq f^k \succcurlyeq_i f_i$ for all k.

Now consider $v \in \mathbb{R}^S_+$ s.t. $v_s = 0$ iff $p_s \geq 0$, and $v_s = 1$ otherwise. If $p_s < 0$ for some s, then $p \cdot v < 0$, i.e. $-p \cdot v > 0$, which contradicts the fact that $v \in T_i(f_i)$ and $-p \in N_i(f_i)$ for all i. Thus, $p \geq 0$. ■

Proof of Proposition 5 and Corollary 6: Lemma 17 yields $p \in \mathbb{R}^S_+ \setminus \{0_S\}$ such that $-p \in N_i(f_i)$ for all i; by Remark 3, $-p \in \bigcup_{\lambda>0} \lambda\big(-\partial V_i(f)\big)$ for all $i \in N$. Thus, $p = \lambda_i Q_i$ for every i, where $\lambda_i > 0$ and $Q_i \in \partial V_i(f)$; then $Q_i(S) = \frac{\sum_s p_s}{\lambda_i}$, and therefore $\frac{Q_i}{Q_i(S)} = \frac{\lambda_i^{-1} p}{\lambda_i^{-1} \sum_s p_s} = \frac{p}{\sum_s p_s} \equiv P$; hence, $P \in \bigcap_i C_i(f)$. ■

The next Remark follows from standard arguments; we include the proof for completeness.

Remark 4 Assume that each V_i is continuous and strongly monotonic. If a feasible allocation $(f_1, \dots, f_N)$ is not Pareto-efficient, then it is Pareto-dominated by a Pareto-efficient allocation.

Proof: By assumption, there exists a feasible allocation $(g_1,\ldots,g_N)$ that Pareto-dominates $(f_1,\ldots,f_N)$. Assume wlog that $g_1 \succ_1 f_1$. Consider the following problem: maximize $V_1(h_1)$ subject to $(h_1,\ldots,h_N)$ being feasible and $h_i \succcurlyeq_i g_i$ for all $i = 2,\ldots,N$. Notice that the allocation $(g_1,\ldots,g_N)$ satisfies these constraints. By standard arguments (e.g. Mas-Colell et al., 1995, §16.F), since preferences are continuous and strongly monotonic, a solution $(h_1^*,\ldots,h_N^*)$ to this problem exists and is Pareto-efficient. Furthermore, for every $i > 1$, $h_i^* \succcurlyeq_i g_i \succcurlyeq_i f_i$, and $h_1^* \succcurlyeq_i g_1 \succ_1 f_1$; that is, $(h_1^*,\ldots,h_N^*)$ is a Pareto-efficient allocation that Pareto-dominates $(f_1,\ldots,f_N)$. ■

We now prove Proposition 7. To do so, we state and prove a slightly more general result. By Proposition 16, if V is locally Lipschitz and monotonic, and in addition SPC holds, then $C(1_S x) = \pi^s(1_S x)$ for all $x \geq 0$. It follows that we can replace the conditon $\bigcap_i C_i(1_s x_i) \neq \emptyset$ in Proposition 7 with $\bigcap_i \pi_i^s(1_S x_i)$. It turns out that this is the key condition for betting to be inefficient. (Furthermore, local Lipschitzianity can be relaxed to continuity.) We then have:

Proposition 18 *Assume that, for every $i \in N$, V_i is continuous and strongly monotonic. Assume further that, for every feasible, full-insurance allocation $(1_S x_1,\ldots,1_S x_N)$, it is the case that $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$. Then every Pareto-efficient allocation provides full insurance. Moreover, such an allocation is a competitive equilibrium allocation (with transfers).*

Proof: Assume that $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$ for every feasible, full-insurance allocation $(1_S x_1,\ldots,1_S x_N)$, with $x_i \geq 0$ for all $i \in N$.

We first show that every Pareto-efficient allocation must provide full insurance. To do so, consider a feasible allocation $(f_1,\ldots,f_N)$. We show that, if this allocation does not provide full insurance, there is a full-insurance allocation that Pareto-dominates it.

For every $i \in N$, let c_i be the certainty equivalent of f_i: that is, $V_i(1_S c_i) = V_i(f_i)$. There are two cases to consider.

Case 1: $\sum_i c_i \geq \bar{x} > 0$. Define a new allocation $(1_S x_1,\ldots,1_S x_N)$ as follows: for every $i \in N$, let $x_i = \frac{\bar{x}}{\sum_j c_j} c_i$. Then $\sum_i x_i = \frac{\bar{x}}{\sum_j c_j} \sum_i c_i = \bar{x}$, i.e., $(1_S x_1,\ldots,1_S x_N)$ is feasible. Since $(f_1,\ldots,f_N)$ is not a full-insurance allocation, there is at least one agent i for whom f_i is non-constant; wlog let that be agent 1. By strong monotonicity, $V_1(1_S c_1) = V_1(f_1) > V_1(0_S)$; since V_1 is strongly monotonic, $c_1 > 0$, and therefore, $x_1 = \frac{\bar{x}}{\sum_j c_j} c_1 > 0$. By assumption, there is $P \in \bigcap_i \pi_i^s(1_S x_i)$; it is immediate from the definitions that $\pi_i^s(1_S x_i) \subseteq \pi_i(1_S x_i)$, so $P \in \pi_i(1_S x_i)$. By Corollary 15, since in particular $P \in \pi_1(1_S x_1)$ and $x_1 > 0$, P is strictly positive.

For every $i \in N$, by construction $V_i(f_i) = V_i(1_S c_i) \geq V_i(1_S x_i)$. Since $P \in \pi_i^s(1_S x_i) \subseteq \pi_i(1_S x_i)$, for all i, $P(f_i) \geq x_i$. For $i = 1$, f_1 is non-constant and hence distinct from $1_S x_1$, $P(f_1) > x_1$.

Conclude that $\sum_i P(f_i) > \sum_i x_i = \bar{x}$. However, $\sum_i P(f_i) = P(\sum_i f_i) = P(1_S \bar{x}) = \bar{x}$, because $(f_1, \ldots, f_N)$ is feasible: contradiction. Thus, this case cannot occur.

Case 2: $\sum_i c_i < \bar{x}$. Let $\epsilon = \frac{\bar{x} - \sum_i c_i}{N}$: then, the full-insurance allocation $\big(1_S(c_1+\epsilon), \ldots, 1_S(c_N+\epsilon)\big)$ is feasible and Pareto-dominates $(f_1, \ldots, f_N)$ by strong monotonicity, as claimed.

Conversely, consider a feasible, full-insurance allocation $(1_S y_1, \ldots, 1_S y_N)$, and suppose that it is not Pareto-efficient. Then, by Remark 4, it is Pareto-dominated by a Pareto-efficient allocation; by the result just proved, under the maintained assumptions, this allocation must be a full-insurance allocation, say $(1_S x_1, \ldots, 1_S x_N)$. Since preferences are strongly monotonic, this implies that $x_i \geq y_i$ for all i, and the inequality is strict for at least one i. But then $\sum_i x_i > \sum_i y_i = \bar{x}$, i.e., $(1_S x_1, \ldots, 1_S x_N)$ is not feasible: contradiction. Thus, every full-insurance allocation is Pareto-efficient.

Finally, let $(1_S x_1, \ldots, 1_S x_N)$ be a full-insurance, hence Pareto-efficient allocation. Fix $P \in \bigcap_i \pi_i^s(1_S x_i)$. Since $\sum_i x_i = \bar{x} > 0$, there must be some $i \in N$ for whom $x_i > 0$; since $P \in \pi_i^s(1_S x_i)$, by Corollary 15 and the fact that $\pi_i^s(1_S x_i) \subseteq \pi_i(1_S x_i)$, P is strictly positive.

Now suppose that, for some $i \in N$ and $g \in \mathbb{R}_+^S$, $g \succ_i 1_S x_i$. Since $P \in \pi_i^s(1_S x_i)$, $P(g) > x_i$. Equivalently, $P(g) \leq x_i$ implies $1_S x_i \succcurlyeq_i g$. We can then let $t = P(1_S x_i) - P(\omega_i) = x_i - P(\omega_i)$: we get $\sum_i t = \sum_i x_i - \sum_i P(\omega_i) = \bar{x} - P(\sum_i \omega_i) = \bar{x} - P(1_S \bar{x}) = 0$. Hence $t_1, \ldots, t_N$ define feasible transfers. Since preferences are strongly monotonic (hence local non-satiated), consumers will exhaust their budget $P(\omega_i) + t_i = x_i$, and the argument just given shows that they will demand $1_S x_1, \ldots, 1_S x_N$. ■

We provide this result not so much for the sake of generality, but because it helps highlight the precise role of Condition SPC in our analysis.

If SPC does not hold, then the condition in Proposition 18, while sufficient, is *not* necessary for betting to be Pareto-inefficient. Example 2 in Appendix A demonstrates this (see footnote

29). On the other hand, as noted in Section 5, the condition that $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is necessary for the full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$ to be Pareto-efficient (Proposition 5), but it is not sufficient (Example 1). This points to a gap between Propositions 5 and 18. SPC closes this gap, because it ensures that the strict supporting probabilities $\pi_i^s(1_S x_i)$ and the normalized Clarke subdifferential $C_i(1_S x_i)$ coincide for every $x_i > 0$: see Proposition 16.

Finally, we point out one subtlety in the proof of Proposition 18. Consider the following intuition, which is based on "risk aversion" (see the discussion following Remark 1) and is also related to the approach taken by RSS. Fix a non-constant allocation $(f_1, \ldots, f_N)$. Suppose that there exists a *common* probability P such that every agent i strictly prefers the expectation $P \cdot f_i$ to her bundle f_i. Then the full-insurance allocation $(P \cdot f_1, \ldots, P \cdot f_N)$ is feasible, and Pareto-dominates $(f_1, \ldots, f_N)$. The problem is that this approach leads to a circularity. On one hand, for any probability P, one can of course define the constant, feasible allocation $(P \cdot f_1, \ldots, P \cdot f_N)$. However, it is not the case, in general, that $P \in \bigcap_i C_i(1_S(P \cdot f_i))$ [which, by SPC, would imply that indeed, $(P \cdot f_1, \ldots, P \cdot f_N)$ Pareto-dominates $(f_1, \ldots, f_N)$]. On the other hand, for every full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$, by assumption there exists $P \in \bigcap_i C_i(1_S x_i)$. However, it is not the case in general that $x_i = P \cdot f_i$ for every i. Our second key insight is that one can also construct a dominating full-insurance allocation by rescaling the *certainty equivalents* of the bundles $f_1, \ldots, f_N$. This avoids any circularity.

D.5 Results in Appendix C and Proof of Proposition 9

Throughout this section, we assume that I is normalized, strongly monotonic, and locally Lipschitz, and that u is strictly increasing, strictly concave and twice differentiable.

The normalized Clarke subdifferential of I at $h \in \mathbb{R}^S_+$ is

$$C^u(h) = \left\{ \frac{Q^u}{Q^u(S)} : Q^u \in \partial I(u \circ h),\, Q^u \neq 0_S \right\}. \tag{17}$$

Remark 5 For every $i \in N$, the Clarke subdifferential at $f \in \mathbb{R}^S_{++}$ of $V = I \circ u$ is

$$\partial V(f) = \left\{ Q \in \mathbb{R}^S : \forall h \in \mathbb{R}^S,\, Q(h) = \sum_s Q^u(s) u'(f(s)) h(s) \text{ for some } Q^u \in \partial I(u \circ f) \right\}.$$

Proof: Let $\mathbb{U} = u(\mathbb{R}_+)$. The map $F : \mathbb{R}^S_+ \to \mathbb{U}^S$ defined by $F(f) = (u(f_1), \ldots, u(f_S))$ is strictly differentiable (pp. 30-31 Clarke, 1983) and, furthermore, it maps every neighborhood of f

to a neighborhood of $F(f)$.[34] Hence, since $V = I \circ F$, by Theorem 2.3.10 in Clarke $\partial V(f) = \partial I(u \circ f) \circ D_s F(f)$; that is, more explicitly, every $Q \in \partial V(f)$ is defined by

$$\forall h \in \mathbb{R}^S, \qquad Q(h) = \sum_s Q^u(s) u'(f_s) h_s$$

for some $Q^u \in \partial I(u \circ f)$. ■

We define a set of probabilities that is related to $\pi(\cdot)$, but employs the decomposition of V in terms of I and u.

$$\pi^c(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ I(u \circ g) \geq I(u \circ f) \Longrightarrow P(u \circ g) \geq P(u \circ f)\}. \tag{18}$$

Recall that one can interpret $P \in \pi(f)$ as a risk-neutral SEU preference whose better-than set at f contains the better-than set of $\succcurlyeq$ at f. Similarly, $P \in \pi^c(f)$ identifies an SEU preference, with risk attitudes described by u, whose better-than set at f contains that of $\succcurlyeq$ at f.

The following result is a consequence of the concavity of u.

Remark 6 For all $x \in \mathbb{R}_{++}$, $\pi^c(1_S x) \subseteq \pi(1_S x)$.

Proof: Fix $P \in \pi^c(1_S x)$ and suppose that $g \in \mathbb{R}^S_+$ satisfies $V(g) \geq V(1_S x)$. Then $I(u \circ g) \geq I(u(x)) = u(x)$, and since $P \in \pi^c(1_S x)$, $P(u \circ g) \geq P(1_S u(x)) = u(x)$. Since u is (strictly) concave, $u(P(g)) \geq P(u \circ g)$, so $u(P(g)) \geq u(x)$. Since u is strictly increasing, $P(g) \geq x$. Thus, $P \in \pi(1_S x)$. ■

This is our "portmanteau" theorem.

Proposition 19 *For every $x > 0$:*

1. *I is nice at $1_S u(x)$ if and only if V is nice at $1_S x$.*

2. *$C^u(1_S x) = C(1_S x)$*

3. *$\pi^c(1_S x) \subseteq \pi^s(1_S x)$*

[34]To see this, fix a strictly positive bundle f and consider the set $\{g \in \mathbb{R}^S_+ : f_s - \epsilon < g_s < f_s + \epsilon \forall s \in S\}$, which is open. The image of this set via F is $\{v \in \mathbb{U}^S : u(f_s - \epsilon) < v_s < u(f_s + \epsilon) \forall s \in S\}$, because u is continuous and strictly increasing. This set is also open.

4. *if* I *is* ∂ *-quasiconcave at* $1_S u(x)$*, then* $C(1_S x) \subseteq \pi^c(1_S x)$

Furthermore, Core $I = \bigcap_{x>0} \pi^c(1_S x)$.

Corollary 20 *Assume that* I *is nice and* ∂ *-quasiconcave at* $1_S u(x)$ *for every* $x > 0$*. Then, for every such* $x > 0$*,* V *is nice at* $1_S x$*, and* $\pi(1_S x) = \pi^c(1_S x) = \pi^s(1_S x) = C(1_S x)$*. Furthermore,* V *satisfies SPC.*

Proof: (1): by Remark 5,

$$\partial V(1_S x) = \left\{Q \in \mathbb{R}^S : \forall h \in \mathbb{R}^S,\ Q(h) = u'(x)Q^u(h) \text{ for some } Q^u \in \partial I(u \circ f)\right\}.$$

Since $u'(x) > 0$ by assumption, $0_S \in \partial V(1_S x)$ iff $0_S \in \partial I(1_S u(x))$.

(2): Again from Remark 5,

$$C(1_S x) = \left\{\frac{Q}{Q(S)} : Q \in \partial V(1_S x),\ Q \neq 0_S\right\} = \left\{\frac{u'(x)Q^u}{u'(x)Q^u(S)} : Q^u \in \partial I(1_S u(x)),\ Q^u \neq 0_S\right\} = C^u(1_S x),$$

where by part 1, $Q^u = 0_S$ iff $0_S \in \partial I(1_S u(x))$.

(3): by Remark 6, $\pi^c(1_S x) \subseteq \pi(1_S x)$. Also recall that, since I is strongly monotonic and u is strictly increasing, $V = I \circ u$ is strongly monotonic. Finally, since I and u are both continuous, so is V.

Fix $P \in \pi^c(1_S x)$. Consider $g \in \mathbb{R}^S_+$ such that $g \neq 1_S x$ and $V(g) \geq V(1_S x)$.

Suppose first that g is constant, i.e., $g = 1_S y$ for some $y \geq 0$. Since $V(1_S y) = V(g) \geq V(1_S x)$, $y \geq x > 0$ by strong monotonicity of $V = I \circ u$. Since $1_S y = g \neq 1_S x$, $y > x$. Therefore, $P(g) = y > x$.

Now suppose that g is non-constant. As noted above, $P \in \pi^c(1_S x)$ implies $P \in \pi(1_S x)$. Since $V = I \circ u$ is strongly monotonic and continuous, by Corollary 15 $P \gg 0$. Then, since g is non-constant and u is strictly concave, $u(P(g)) > P(u \circ g)$. To see this, suppose that $s, s' \in S$ are such that $g(s) \neq g(s')$. Then, since $P(s) > 0$ and $P(s') > 0$,

$$\sum_{t \in \{s,s'\}} \frac{P(t)}{P(\{s,s'\})} u(g(t)) < u\left(\sum_{t \in \{s,s'\}} \frac{P(t)}{P(\{s,s'\})} g(t)\right),$$

and therefore

$$\begin{aligned}
P(u\circ g)=&\sum_{t\in S}P(t)u(g(t))=\\
&=[1-P(\{s,s'\}]\sum_{t\in S\setminus\{s,s'\}}\frac{P(t)}{1-P(\{s,s'\})}u(g(t))+P(\{s,s'\})\sum_{t\in\{s,s'\}}\frac{P(t)}{P(\{s,s'\})}u(g(t))<\\
&<[1-P(\{s,s'\}]u\left(\sum_{t\in S\setminus\{s,s'\}}\frac{P(t)}{1-P(\{s,s'\})}g(t)\right)+P(\{s,s'\})u\left(\sum_{t\in\{s,s'\}}\frac{P(t)}{P(\{s,s'\})}g(t)\right)\leq\\
&\leq u\left(\sum_{t\in S}P(t)g(t)\right)=u(P(g)).
\end{aligned}$$

To conclude the argument, $P\in\pi^{c}(1_S x)$ and $V(g)\geq V(1_S x)$ imply $P(u\circ g)\geq u(x)$; but since $u(P(g))>P(u\circ g)$ and u is strictly increasing, $P(g)>x$.

Since g was chosen arbitrarily, $P\in\pi^{s}(1_S x)$.

(4): fix $P\in C(1_S x)$. By part 2, $P\in C^{u}(1_S x)$, so there exists $Q^{u}\in\partial I(1_S u(x))$ such that $P=Q^{u}/Q^{u}(S)$. Consider $f\in\mathbb{R}^{S}_{+}$ such that $I(u\circ f)\geq u(x)$. Since I satisfies ∂-quasiconcavity at $1_S u(x)$, in particular $Q^{u}(u\circ f-1_S u(x))\geq 0$. This implies that $P(u\circ f)\geq u(x)$. Since f was arbitrary, $P\in\pi^{c}(1_S x)$.

For the last statement, fix $P\in\text{Core }I$ and $x>0$. Consider $f\in\mathbb{R}^{S}_{+}$ such that $V(f)\geq V(1_S x)$, i.e., since $V=I\circ u$ and I is normalized, $I(u\circ f)\geq u(x)$. Since $P\in\text{Core }I$, $P(u\circ f)\geq I(u\circ f)$. Therefore, $P(u\circ f)\geq u(x)$. Since f was arbitrary, $P\in\pi^{c}(1_S x)$.

Conversely, suppose that $P\in\bigcap_{x>0}\pi^{c}(1_S x)$. Fix $f\in\mathbb{R}^{S}_{+}$. If $f=0_S$, then $P(u\circ f)=u(0)=I(u\circ f)$, where the second equality follows because I is normalized. If instead $f\neq 0_S$, let c be the certainty equivalent of f: that is, $I(u\circ f)=u(c)$. By strong monotonicity, $c>0$. Therefore, $P\in\pi^{c}(1_S c)$. Then, $I(u\circ f)=u(c)$ implies that $P(u\circ f)\geq u(c)$. But then $P(u\circ f)\geq u(c)=I(u\circ f)$, so $P\in\text{Core }I$.

Turn now to the Corollary. By assumption, I is nice and ∂-quasiconcave at every $1_S u(x)$, $x>0$. By part 1, V is nice at $1_S x$, $x>0$. Moreover, for every such $x>0$, by parts 3 and 4, $C(1_S x)\subseteq\pi^{c}(1_S x)\subseteq\pi^{s}(1_S x)$; by Proposition 16 part 4, V is strictly pseudoconcave at $1_S x$. Therefore, V satisfies SPC. Furthermore, by Proposition 16 parts 1 and 2, $\pi^{s}(1_S x)\subseteq\pi(1_S x)\subseteq C(1_S x)$. Therefore, since $C(1_S x)\subseteq\pi^{c}(1_S x)$, all these sets are equal. ■

Proof of Proposition 12: since u is strictly increasing and I is strongly monotonic, V is strongly monotonic. Since u is concave and strictly increasing, it is locally Lipschitz (cf. Clarke, 1983, Prop. 2.2.6). Since I is locally Lipschitz, $V = I \circ u$ is also locally Lipschitz (cf. Clarke, 1983, p. 42). Therefore, Assumption 1 holds. Moreover, since I is nice at every $1_S u(x)$, $x > 0$, Proposition 19 part 1 implies that V is nice at every $1_S x$, for $x > 0$. Finally, by Corollary 20, since I satisfies DQC, it is ∂-quasiconcave at every $1_S u(x)$, $x > 0$; thus, V is strictly pseudoconcave at every $1_S x$, $x > 0$, i.e., SPC holds. ■

Proof of Corollary 13. Note: Part (1) follows from a result in Penot and Quang (1997); however, since their assumptions are formulated somewhat differently from ours, invoking their result requires some work. We provide a direct proof.

It is convenient to let $\mathbb{U} = u(\mathbb{R}_+) = \{r : \exists x \geq 0, r = u(x)\}$. Fix $\gamma \in \text{int}(\mathbb{U})$ and $a \in \mathbb{U}^S$ such that $I(a) \geq \gamma$. For both conditions, we use the properties of the Clarke lower derivative in Eq. (15); in particular, it is enough to show that $I^\ell(1_S\gamma; a - 1_S\gamma) \geq 0$.

(1): fix $\epsilon > 0$ such that $a + 1_S\epsilon \in \mathbb{U}^S$ (this must exist, because $\mathbb{U} = u(\mathbb{R}_+)$ does not contain its supremum). By strong monotonicity, $I(a + 1_S\epsilon) > \gamma$. Consider sequences $(c^k) \subset \mathbb{U}^S$ and $(t^k) \subset \mathbb{R}_{++}$ such that $c^k \to 1_S\gamma$ and $t^k \downarrow 0$. Note that

$$t^k[(a+1_S\epsilon)-1_S\gamma]+c^k = t^k[(a+1_S\epsilon)-1_S\gamma+c^k]+(1-t^k)c^k$$

and, since $c^k \to 1_S\gamma$, eventually $(a+1_S\epsilon)-1_S\gamma+c^k \in \mathbb{U}^S$; furthermore, by continuity $I(a+1_S\epsilon - 1_S\gamma + c^k) \to I(a+1_S\epsilon)$ and $I(c^k) \to I(1_S\gamma) = \gamma$. Therefore, for k sufficiently large, $I(a+1_S\epsilon - 1_S\gamma + c^k) > I(c^k)$. Then, by quasiconcavity, for all such k,

$$I(t^k[(a+1_S\epsilon)-1_S\gamma]+c^k) = I(t^k[(a+1_S\epsilon)-1_S\gamma+c^k]+(1-t^k)c^k) \geq I(c^k).$$

It follows that

$$I^\ell(1_S\gamma;(a+1_S\epsilon)-1_S\gamma) = \liminf_{c\to 1_S\gamma, t\downarrow 0} \frac{I(t[(a+1_S\epsilon)-1_S\gamma]+c)-I(c)}{t} \geq 0.$$

Finally, by continuity of $I^\ell(1_S\gamma;\cdot)$, $I^\ell(1_S\gamma; a - 1_S\gamma) \geq 0$ as well.

(2): if I is regular, $I^{\ell}(1_S\gamma; a - 1_S x) = -I^{\circ}(1_S\gamma; 1_S\gamma - a) = -I'(1_S\gamma; 1_S\gamma - a)$; furthermore, if $I(a) \geq I(1_S\gamma) = \gamma$, by normalization, for any $P \in \text{Core } I \neq \emptyset$,

$$-I^{\ell}(1_S\gamma; a - 1_S\gamma) = I'(1_S\gamma; 1_S\gamma - a) = \lim_{t\downarrow 0} \frac{I(1_S\gamma + t[1_S\gamma - a]) - I(1_S\gamma)}{t} =$$
$$= \lim_{t\downarrow 0} \frac{I(1_S\gamma + t[1_S\gamma - a]) - \gamma}{t} \leq \lim_{t\downarrow 0} \frac{P(1_S\gamma + t[1_S\gamma - a]) - \gamma}{t} =$$
$$= \lim_{t\downarrow 0} \frac{\gamma + t\gamma - tP(a) - \gamma}{t} = \gamma - P(a) \leq I(a) - P(a) \leq 0,$$

as required. ■

Proof of Proposition 9: observe first that, for all $\phi \in \mathbb{R}^J$,

$$\nabla I(a) \equiv \left(\frac{\partial I(a)}{\partial a(s)}\right)_{s\in S} = \left(P(\{s\})\left[1 + \sum_{0\leq j<J} \frac{\partial A(P(\zeta_0 a), \ldots, P(\zeta_{J-1} a))}{\partial \phi_j} \zeta_j(s)\right]\right)_{s\in S}. \tag{19}$$

Thus, the last condition in the Proposition is simply the requirement that all partial derivatives be strictly positive almost everywhere on $u(\mathbb{R}_+)^S$. Thus, I is strongly monotonic.

Next, we show that $\nabla A(0_J) = 0_J$. Fix $0 \leq j < J$. Since A is continuously differentiable at 0_J, satisfies $A(0_J) = 0$ and is symmetric about 0_J,

$$\nabla A(0_J)\cdot 1_j = \lim_{t\downarrow 0} \frac{A(0_J + t1_j) - A(0_j)}{t} = \lim_{t\downarrow 0} \frac{A(t1_j)}{t} = \lim_{t\downarrow 0} \frac{A(t(-1_j))}{t} = \lim_{t\downarrow 0} \frac{A(0_J + t(-1_j)) - A(0_J)}{t} = \nabla A(0_J)\cdot(-1_j),$$

which clearly requires that $\nabla A \cdot 1_j = \frac{\partial A(0_J)}{\partial \phi_j} = 0$, as claimed. Since $P(\zeta_j 1_S x) = xP(\zeta_j) = 0$, it follows that $\nabla I(1_S x) = P$ for all $x > 0$. Hence I is nice at certainty, and $C^u(1_S x) = \{P\}$; by Proposition 19 part 2, also $C(1_S x) = \{P\}$.

Since $A \leq 0$, it is immediate that $P \in \text{Core } I$; furthermore, I is smooth, hence regular. Therefore, I and u satisfy Assumption 2; furthermore, it satisfies condition 2 in Corollary 13, so SPC holds.

Finally, by Corollary 13 part 2, I satisfies DQC, so by Corollary 20, $\pi(1_S x) = \pi^c(1_S x) = C(1_S x) = \{P\}$, and so, by Proposition 19, Core $I = \{P\}$ as well. ■

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