Some Notes on Relativistic Forces

Will M. Farr

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The relativistic version of Newton's third law is

$$ma^\mu = f^\mu. \tag{1}$$

Recall that the acceleration is given by

$$a^\mu = \frac{dv^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau}; \tag{2}$$

we use τ instead of t so that the coordinates, velocity, and acceleration all have the same transformation properties under Lorentz transformations. The quantity f^μ is the four-force, or simply force. We can also write Equation (1) as

$$\frac{dp^\mu}{d\tau} = f^\mu. \tag{3}$$

However, the usual concept of force as measured in any frame is defined by the rate of change of the momentum with *time*, not proper time along some trajectory. Hartle calls the spatial parts of this quantity the three-force:

$$\frac{d\mathbf{p}}{dt} \equiv \mathbf{F}. \tag{4}$$

Because of the relationship between t and τ, we have

$$f^i = \gamma F^i. \tag{5}$$

Now recalling that the $\vec{a} \cdot \vec{v} = 0$, implying that $\vec{f} \cdot \vec{v} = 0$, we must have

$$\vec{f} = \left(\gamma \mathbf{F} \cdot \mathbf{u}, \gamma \mathbf{F}\right), \tag{6}$$

where $\mathbf{u}$ is the three velocity (i.e. $\vec{v} = (\gamma, \gamma \mathbf{u})$).

When writing down the "usual" force laws, we mean the three-force $\mathbf{F} = d\mathbf{p}/dt$, for example:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right). \tag{7}$$

(This equation can be written relativistically as $f^\mu = qF^{\mu\nu} v_\nu$ using the field strength tensor—this I did not mention in class. You can check that it all works out given the definitions above.) The relevant question to ask about a force law if you are not exactly sure which force is meant is: am I thinking of the change of momentum with time in some frame? If yes, then you want the three-force, $\mathbf{F}$, from Equation (4). The name of the three-force can be a source of confusion because, unlike the three-velocity $\mathbf{u} = d\mathbf{x}/dt$, it does involve relativistic concepts: the momentum in Equation (4) is the spatial part of the relativistic momentum.