SOLVING THE COLLISIONLESS BOLTZMANN EQUATION IN GENERAL RELATIVITY

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Abstract. We have developed a new numerical method for determining the dynamical evolution of a collisionless system in full general relativity. The method exploits Liouville's theorem to determine the evolution of the distribution function of matter in phase space directly. The distribution function is governed by the collisionless Boltzmann equation coupled to Einstein's equations for the gravitational field. The method accurately tracks the increasingly complicated, fine-grained structure developed by the distribution function because of phase mixing. It can be used to study Newtonian as well as fully relativistic systems. We restrict our analysis to spherically symmetric systems in this paper, but the gravitational field can be arbitrarily strong and the matter velocities arbitrarily close to the speed of light. Applications include violent relaxation, the stability of relativistic star clusters, and the collapse of unstable relativistic star clusters to black holes. We report evidence that some relativistic star clusters with arbitrarily large central redshifts are stable to spherical perturbations.

INTRODUCTION

Recently, Shapiro and Teukolsky (1985a, b, c, 1986, hereafter ST1, 2, 3, 4) have demonstrated how to calculate numerically (at least in spherical symmetry) the dynamical evolution of self-gravitating, collisionless systems in general relativity. Their method yields a solution to the collisionless Boltzmann (Vlasov) equation coupled to Einstein's equations for the gravitational field.

Collisionless relativistic systems may very well exist in nature. Quasars, active galactic nuclei (AGNs), and other intense extragalactic radio sources are believed to be powered by supermassive black holes (see e.g. Begelman, Blandford and Rees 1984). But the formation of these supermassive black holes remains a mystery. This is a topical issue, since there is now solid observational evidence (see Dressler and Richstone 1988, and Kormendy 1988) that the nuclei of some nearby galaxies do indeed contain massive ($M \gtrsim 10^6 M_\odot$) black holes. It has long been recognized (cf. Zel'dovich and Podurets 1965) that such massive black holes can form as a consequence of the dynamical instability of a relativistic star cluster. In ST3 (see also Kochanek, Shapiro, and Teukolsky 1987, and Quinlan and Shapiro 1988 and this volume), recent Newtonian Fokker-Planck calculations of the gravothermal catastrophe together with the relativistic calculations of ST1,2 were combined to make such a scenario more than plausible.

The computational method of ST combines the techniques of numerical relativity with those of N-body particle simulations. It is basically a relativistic generalization of the particle-mesh methods used to study Newtonian collisionless star clusters (cf. Sellwood 1987 for an excellent review). Particle methods have the characteristic feature that all numerical calculations are done in *real space*, even though they are used to solve the collisionless Boltzmann equation, which expresses the evolution of a system in *phase space*. It is in fact impossible with particle methods to determine the distribution function of the system in phase space. Nevertheless it is usually assumed that the particles "naturally" provide an adequate statistical coverage of phase space. However, it is not clear to what extent a particle simulation (especially with small N) actually reproduces the solution of the collisionless Boltzmann equation, which in some sense should correspond to the $N \rightarrow \infty$ limit. For any finite value of N, artificial random statistical fluctuations will be present in all computed quantities. Recently there has been some indication that these fluctuations can sometimes lead to spurious results. For example, Nishida (1986) found that the bar instability of a thin stellar disk can be suppressed by the presence of a small bulge component, even though previous particle simulations had led to the opposite conclusion.

Another approach to study collisionless systems is to use what we will refer to as a "phase space method " (Sellwood (1987) uses the term "collisionless Boltzmann code" instead). A phase space method does not rely on any statistical representation of the system by particles. Instead, *a phase space method explicitly constructs the (smooth) distribution function of matter in phase space*. The source terms of the field equations are obtained by numerical quadratures over velocity space. This has the great advantage of eliminating random statistical fluctuations in the data while at the same time providing us with the full distribution function of the system. Unfortunately, very few phase space methods have so far been successfully developed, even in the much simpler framework of Newtonian gravity, and all of them are still in their infancy. The reason is the extreme complexity of working in phase space instead of real space. The large number of dimensions in phase space (already three in spherical symmetry where real space has only one) would discourage many attempts. In addition, distribution functions often have rather irregular structures that can be hard to accurately represent numerically (e.g. on a grid in phase space). Such irregular structures can be due to discontinuities in the initial data or to phase mixing.

Given these difficulties it is not surprising that no one has attempted to develop a phase space method in the framework of general relativity, where further important complications are introduced by the need to integrate forward in time Einstein's equations for the gravitational field. The new method we have developed allows us to obtain for the first time the complete evolution of the distribution function for relativistic collisionless systems. These first results are restricted to spherical symmetry, but the matter velocity can approach the speed of light and the gravitational field can become arbitrarily strong. In particular, several cases include catastrophic collapse of the system to a black hole.

Here, we will restrict ourselves to a very brief description of the numerical method, followed by two applications. The first application presents the time evolution of the distribution function for an unstable relativistic star cluster undergoing catastrophic collapse to a black hole. The second application demonstrates the existence of stable relativistic star clusters with arbitrarily large central redshifts. A detailed description of our method, together with a more extensive list of applications, will be published elsewhere (Rasio, Shapiro, and Teukolsky 1988a, b).

MATHEMATICAL FORMULATION AND METHOD

We adopt the notations of Misner, Thorne and Wheeler (1973; hereafter MTW) and set $c = G = 1$ throughout. The metric is written in the ADM form, with isotropic radial coordinate,

$$\begin{aligned} ds^2 = &-(\alpha^2 - A^2\beta^2)dt^2 + 2A^2\beta dr dt \\ &+ A^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \end{aligned} \tag{1}$$

Here α and β are the lapse and shift functions of ADM; see also Smarr and York (1978a, b). As coordinates in phase space, we use the radial velocity u_r, the "angular momentum at infinity" j, defined by

$$j \equiv \sqrt{u_\theta^2 + \frac{u_\phi^2}{\sin^2\theta}}, \tag{2}$$

and the angle

$$\psi \equiv \tan^{-1}\left(\frac{u_\theta \sin\theta}{u_\phi}\right), \tag{3}$$

measuring the orientation of the transverse velocity. In spherical symmetry, the distribution function f cannot depend on ψ, and j is a conserved quantity, so that the collisionless Boltzmann equation can be simply written

$$\frac{\partial f}{\partial t} + \left(\frac{dr}{dt}\right)\frac{\partial f}{\partial r} + \left(\frac{du_r}{dt}\right)\frac{\partial f}{\partial u_r} = 0, \tag{4}$$

since $\partial f/\partial\psi = 0$ and $dj/dt = 0$. The coefficients dr/dt and du_r/dt are given explicitly, in terms of the metric coefficients, by

$$\frac{dr}{dt} = \frac{u_r}{A^2 u^0} - \beta, \tag{5}$$

$$\frac{du_r}{dt} = -u^0\alpha\,\alpha_{,r} + u_r\,\beta_{,r} + \frac{u_r^2}{u^0}\frac{A_{,r}}{A^3} + \frac{j^2}{u^0}\left(\frac{1}{r^3A^2} + \frac{A_{,r}}{r^2A^3}\right), \tag{6}$$

where

$$u^0 = \frac{1}{\alpha}\sqrt{1 + \frac{u_r^2}{A^2} + \frac{j^2}{A^2r^2}}. \tag{7}$$

Equation (6) is simply the geodesic equation corresponding to the metric (1), while equation (7) is obtained from the normalization condition $u_\mu u^\mu = -1$.

The four basic steps for propagating the distribution function f from time t_1 to $t_2 > t_1$ are: (1) Compute the matter source terms of the Einstein field equations (e.g. the mass-energy density ρ), by integrating f at t_1 over velocity space (u_r, j), (2) Integrate the field equations, thereby determining the new metric coefficients A, α, and β at time t_1, (3) Extrapolate to determine (guess) the value of these quantities over the interval (t_1, t_2), and finally (4) Compute f at time t_2 using equations (4)–(7). The key idea in our method is to use Liouville's theorem for step (4), which we write as

$$f(r_2, u_2, t_2) = f(r_1, u_1, t_1), \tag{8}$$

where (r_1, u_1) is the position in phase space at time t_1 of a test particle that *will reach* the position (r_2, u_2) at time t_2. Since dr/dt and du_r/dt can be evaluated for all $t \leq t_2$, one can actually *construct* the trajectory of such a test particle. Since f is also known at all times $t \leq t_2$ one can therefore determine $f(r_2, u_2, t_2)$ for all (r_2, u_2). In the Newtonian regime, this idea was first implemented in a numerical scheme by Fujiwara (1981, 1983). In this scheme, the distribution function is constructed on a *grid of points* in phase space. Equation (8) is used, with $t_1 = t_2 - \Delta t$, where Δt is the timestep. If (r_2, u_2) is a grid point at time t_2, then (r_1, u_1) is in general not a grid point at time t_1, and it is necessary to *interpolate* on the grid at t_1 to determine $f(r_1, u_1, t_1)$.

We first tried to develop a relativistic generalization of Fujiwara's method, but found that in many cases (even Newtonian) it develops numerical instabilities and leads to grossly inaccurate results (Inagaki *et al.*, 1984, and White, 1986, have reported similar problems). The inaccuracy is entirely due to the use of interpolation on a multidimensional grid. One very simple way of not having to use any interpolation, is to extend the value of t_1 in equation (8) to $t = 0$. Indeed, at $t = 0$ the distribution function f is known *to arbitrary accuracy*, since it must be given as an initial condition. Moreover, there is no need in this case to introduce *any grid at all* in phase space, since intermediate values of f are never used and therefore need not be stored. One can directly compute all quadratures over velocity space at any time by using a *self-adaptive* quadrature routine. When the routine asks for the value of f at some point in phase space, this point is simply tracked along a dynamical path all the way back to $t = 0$, where f can be accurately evaluated from the initial data. Moreover, such a self-adaptive quadrature routine is *ideally suited* to problems involving discontinuous distribution functions or large degrees of phase mixing: more points can be added in phase space to maintain high accuracy whenever and wherever required by the structure of the distribution function. The only disadvantage of this scheme is its obviously large computational cost. Indeed, the computation time *per iteration* increases with time, since longer and longer trajectories have to be constructed to evaluate f at a given point. However, at least to some extent, this merely reflects the real increase of complexity occuring in the physical system.

BLACK HOLE FORMATION IN PHASE SPACE

A well-known example of a relativistic collisionless system is an equilibrium cluster of compact stars characterized by a *truncated isothermal* distribution function,

$$F(E) = \begin{cases} K \exp(-E/T), & E \leq E_{max}; \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where K is a normalization constant, E is the conserved "energy at infinity" of a star, T is the "temperature at infinity" of the cluster, and E_{max} is the value of E at the surface of the cluster. Following historical tradition (Zel'dovich and Podurets 1965, Fackerell 1966, Ipser 1969, ST2), we examine the one-parameter sequence of models obtained by imposing the constraint $E_{max} = m_0 - 0.5T$, where m_0 is the rest mass of a star. Figure 1 shows the time evolution of the central redshift for several models along the sequence. The transition between stability and instability is clearly located at central redshift $Z_c = 0.42$, in complete agreement with the semi-analytic calculations of Ipser (1969), and the particle simulations of ST2.

Figure 1. Time evolution of the central redshift Z_c for several clusters initially taken along the truncated isothermal sequence. Here the unit of time $t_{dyn} \equiv (3\pi/32)^{1/2}\alpha_c^{-1}\rho_c^{-1/2}$ is the central (minimum) free-fall proper time scale. The three models at the bottom are stable: our calculations reveal no change of structure on a dynamical timescale. The five models having initial redshifts above 0.42 are unstable: they collapse to black holes in a few dynamical times.

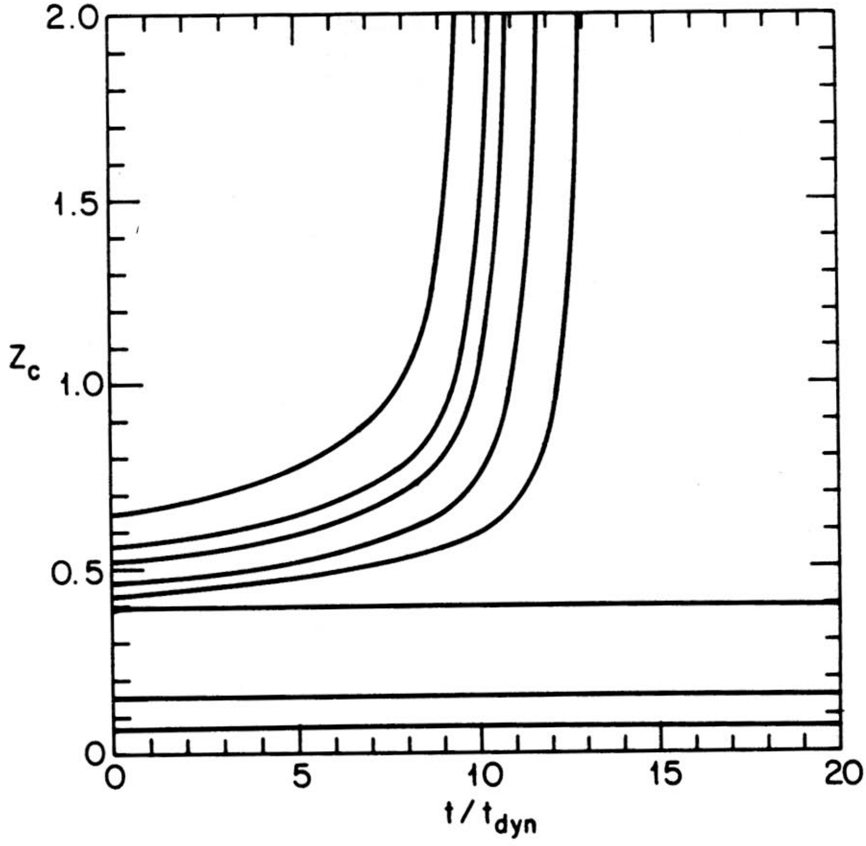
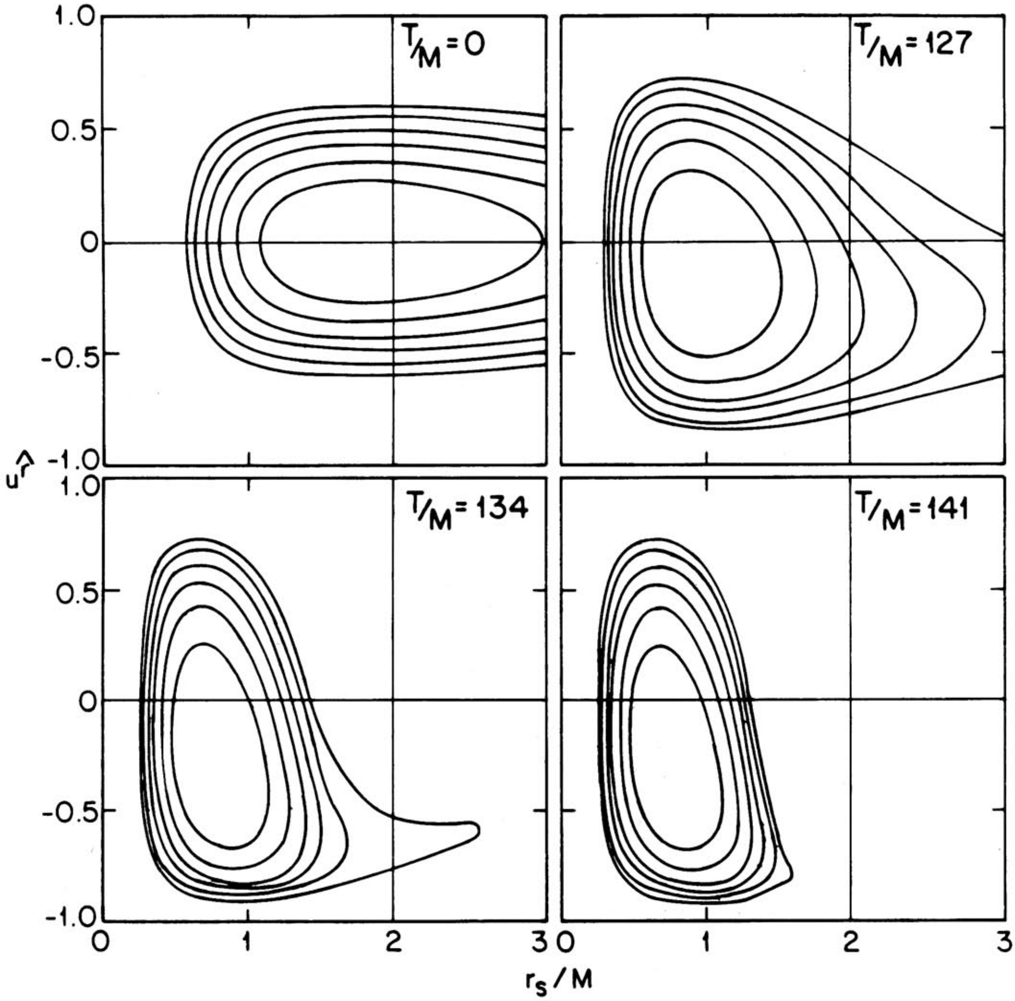
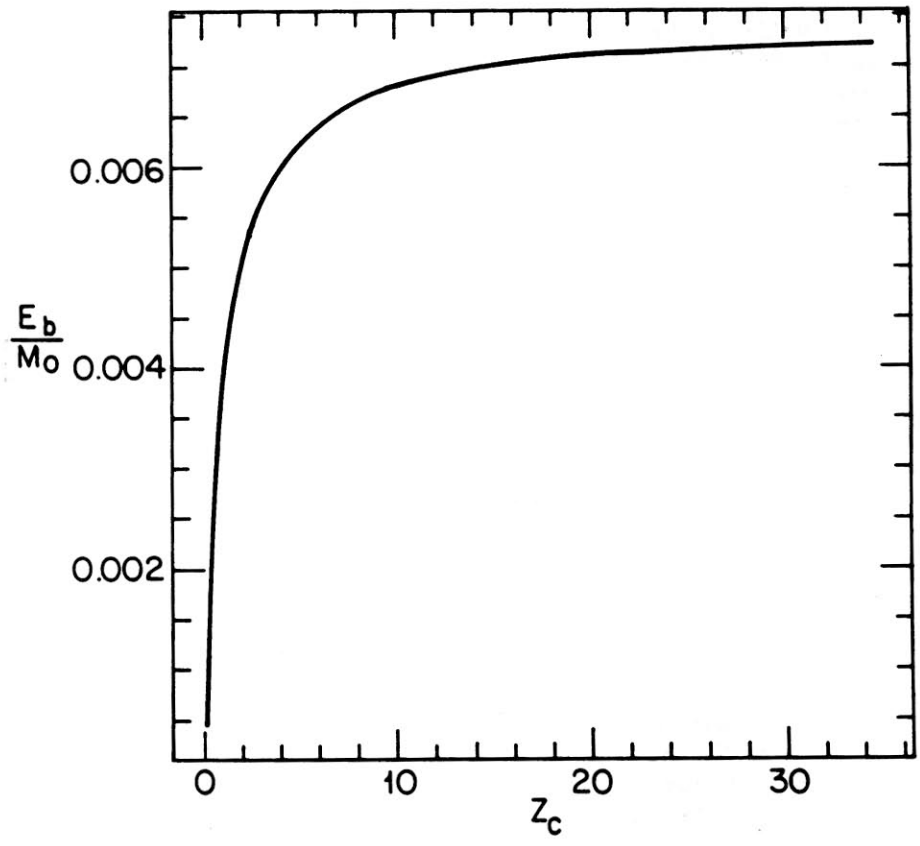


Figure 2. Time evolution of the distribution function for a cluster undergoing catastrophic collapse to a black hole. Initially, the cluster has a truncated isothermal distribution with central redshift $Z_c = 0.52$. The Schwarzschild radius r_s (in units of the total mass-energy M) and the radial velocity $u^{\hat{r}}$ are used as phase-space coordinates. Each plane $(r_s, u^{\hat{r}})$ is a two-dimensional slice taken from the three-dimensional phase space by setting the angular momentum j (cf. eqn.(2)) equal to a constant. Here $j = 0.5M^2$, which represents the angular momentum of a "typical star" in this cluster ($0 \leq j \lesssim M^2$ for $f \neq 0$). Lines of constant f are shown, equally spaced between 0 and its maximum value in this slice (Note that since $j \neq 0$, the matter in this slice can never reach $r_s = 0$). In the final plot, the entire mass of the cluster has collapsed inside an event horizon located at $r_s = 2M$.



We now focus our attention on the unstable cluster with $Z_c = 0.52$. In a few dynamical times, this cluster collapses to a black hole, which is identified in our numerical code by the appearance of an event horizon and a region of trapped surfaces. The unique character of our method is revealed in figure 2, which shows how this collapse proceeds in *phase-space*. The coordinates used here are the Schwarzschild areal radius $r_s = Ar$ and the radial velocity $u^{\hat{r}} = u_r/(A\alpha u^0)$ measured by a *normal* observer. The reason for this choice is that r_s and $u^{\hat{r}}$ are *freezing variables*, i.e. they become constant (with t) at late times wherever the lapse of proper time in the normal observer's reference frame goes to zero (cf. ST4). Therefore we expect the distribution function expressed in these variables to exhibit a steady configuration at late coordinate time, wherever $\alpha \to 0$. This is indeed what we find: once all the matter in the cluster has collapsed inside the horizon, the distribution function very slowly evolves towards a final steady structure, symmetric with respect to $\pm u^{\hat{r}}$.

Figure 3. Fractional binding energy E_b/M_0 versus central redshift Z_c for the sequence of clusters constructed by Kochanek *et al.* (1987) as a possible relativistic end point to the gravothermal catastrophe. Note the absence of turning point in the curve, indicating stability.



STABLE RELATIVISTIC CLUSTERS WITH ARBITRARILY LARGE CENTRAL REDSHIFTS?

All relativistic clusters studied in the past by semi-analytic methods or particle simulations exhibit a behavior similar to that of truncated isothermal clusters, i.e. they become unstable to spherical perturbations once their central redshift becomes greater than about 0.5 (Ipser 1969, ST2). The onset of instability always appears to coincide with the first maximum along the sequence of the *fractional binding energy* $E_b/M_0 \equiv 1 - M/M_0$, where M is the total mass-energy of the cluster, and M_0 is its total rest mass. This is known to be true for all spherical *fluid* configurations, but could never be proved for collisionless systems. However it was proved that clusters located along the ascending branch of E_b/M_0, in an appropriately constructed one-parameter sequence of models, are stable (Ipser 1980).

In the process of studying the stability properties of several one-parameter sequences of relativistic clusters, we discovered one which does not conform at all to the above picture. It was constructed by Kochanek *et al.* (1987) as a possible relativistic generalization to the self-similar structures that constitute the end point of the gravothermal catastrophe. Our dynamical calculations indicate that all clusters along this sequence remain stable to spherical perturbations, even when their central redshifts become much larger than 0.5. This is in agreement with the fact that the fractional binding energy of these clusters appears to increase monotonically with central redshift (Figure 3).

The existence of stable relativistic star clusters with arbitrarily large central redshifts was already postulated by Bisnovatyi-Kogan and Thorne (1970). However, their clusters had rather unphysical properties, like infinite radii and central densities. On the contrary, the clusters we studied, even though centrally condensed, have finite radius and densities. More on this exciting discovery will be published elsewhere (Rasio, Shapiro, and Teukolsky 1988b).

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