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ORBITAL EVOLUTION OF THE PSR1257+12 PLANETARY SYSTEM

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ABSTRACT The detection of orbital perturbation effects in the PSR 1257 +12 timing data would provide irrefutable confirmation that planets are indeed orbiting the pulsar. Here we give an overview of how perturbation effects are expected to affect the orbital elements of the two planets over the next few years. In particular, we give a simple calculation of resonant perturbations, including the nonlinear effects which could be important if $\sin i \lesssim 0.1$. We also present a new analysis of the effects of close encounters between the two planets and we discuss their detectability.

RESONANT PERTURBATIONS

One of the most intriguing, and potentially important, features of the putative planetary system around PSR1257+12 is the near-commensurability between the mean angular velocities (n_i) of the two planets. Wolszczan and Frail (1992) report orbital periods of $P_1 = 66.563 \pm 0.009$ days and $P_2 = 98.234 \pm 0.021$ days, or $n_1/n_2 = 1.4758 \pm 0.0004 \simeq 3/2$. Rasio *et al.* (1992) showed that, as a consequence of this near-commensurability, the orbital periods, eccentricities (e_i), and longitudes of pericenter (ω_i) should exhibit relatively large periodic perturbations. For masses near the minimum values of $m_1 = 3.4M_\oplus$ and $m_2 = 2.8M_\oplus$ (which correspond to $\sin i = 1$, i.e., orbits viewed edge-on), the period of these perturbations is $2\pi/(2n_1 - 3n_2) = 5.56\,\mathrm{yr}$, while the amplitudes are proportional to the planet masses.

If the masses of the planets are appreciably larger than the minimum values (i.e., if $\sin i \ll 1$), then it is possible that the planets are actually "locked" in the 3:2 resonance, and the dynamical evolution of the system is qualitatively different from that described by Rasio *et al.* (1992). This was first pointed out by Malhotra *et al.* (1992), who found that resonant locking was likely for values of $(\sin i)^{-1} \gtrsim 15$, or masses $m_i \gtrsim 45M_\oplus$. In this situation, the period and amplitude of the perturbations increase, and the perturbations themselves cease to show a simple sinusoidal character. This is illustrated in Fig. I, which shows the results of numerical integrations of the PSR1257+12 system for decreasing values of $\sin i$.

We present here a simplified dynamical model of the resonance which preserves the principal features of the full solution, while avoiding some of its complexity. The principal simplification is to treat the inner planet as a massless

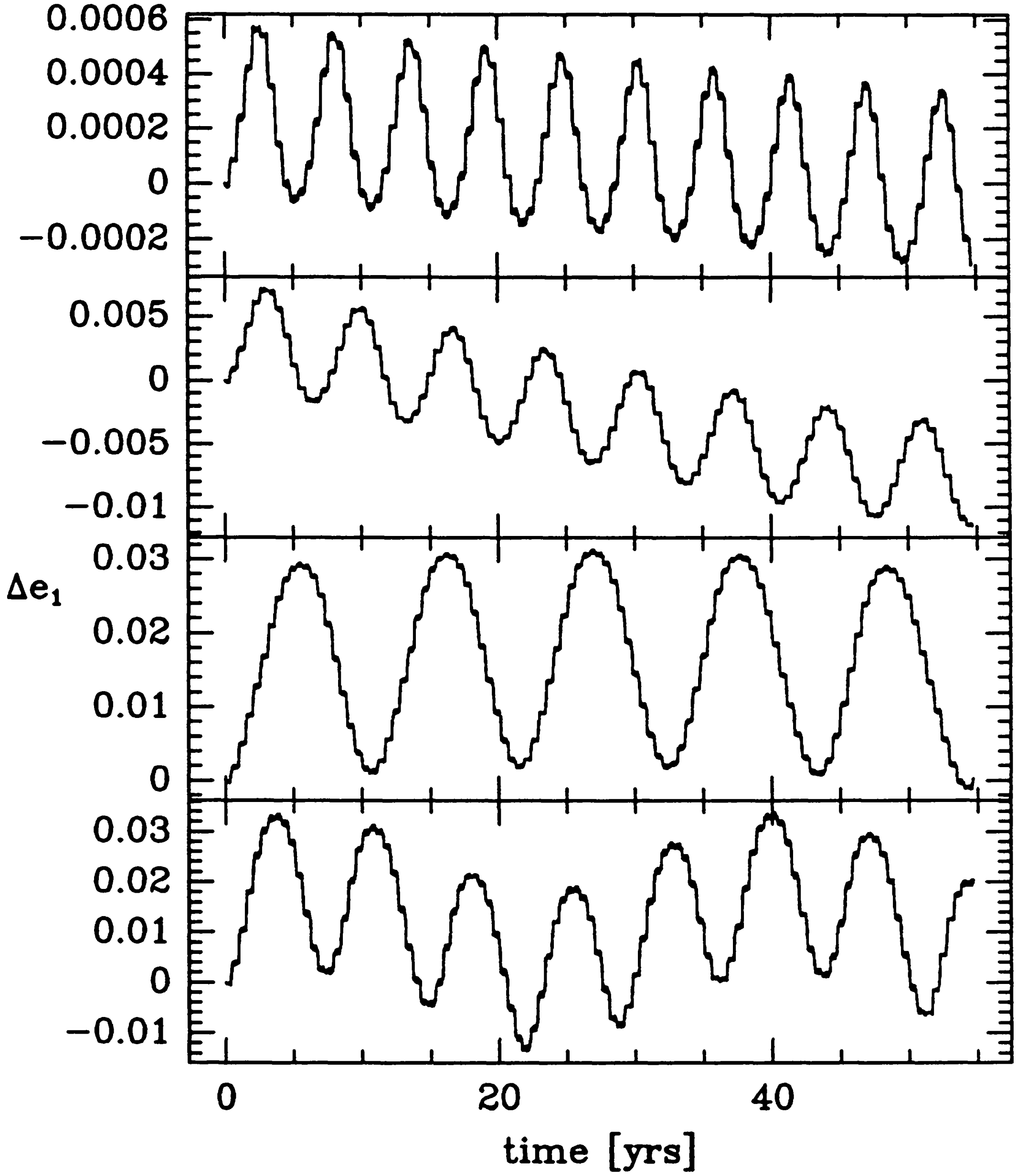


FIGURE I Numerical integration of the PSR1257+12 system, showing the changes in the eccentricity of the inner planet for (top to bottom) $(\sin i)^{-1} = 1$, 10, 20, and 40.

"test particle" perturbed by the outer planet, which is assumed to move on an unperturbed Keplerian orbit about the neutron star. We further assume that $e_i \ll 1$ and $m_2/M \ll 1$, where M is the mass of the neutron star, and that the planetary orbits are coplanar. Under these assumptions, and to lowest order in e_i and m_2/M, the Lagrange perturbation equations governing the variations in the orbital elements of the inner planet in the vicinity of the $m : m-1$ resonance may be written (Danby 1988)

$$\dot{n}_1 = -3\,(m-1)\,e_1 n_1^2\,\beta \sin\phi_1, \tag{1}$$

$$\dot{e}_1 = -n_1\,\beta \sin\phi_1, \tag{2}$$

$$\dot{\omega}_1 = -n_1\,\beta \cos\phi_1\,/e_1, \tag{3}$$

where the "resonance variable" ϕ_1 is defined by

$$\phi_1 \equiv (m-1)\,\lambda_1 - m\,\lambda_2 + \omega_1, \tag{4}$$

λ_i are the mean longitudes of the two planets, and the dimensionless parameter β which governs the amplitude of the perturbations is given by (Brouwer and Clemence 1961)

$$\beta \equiv \frac{m_2}{M}\alpha \left[m + \frac{1}{2}\alpha\frac{d}{d\alpha}\right] b^m_{1/2}(\alpha). \tag{5}$$

The function $b^m_{1/2}(\alpha)$ is a Laplace coefficient, which depends only on the ratio of semi-major axes $\alpha = a_1/a_2$. For the 3:2 resonance, $m = 3$, $\alpha \simeq (2/3)^{2/3} = 0.763$ and we find that $\beta = 1.546\,(m_2/M)$. For the PSR1257+12 system, $\beta = 0.937 \times 10^{-5} M_{1.4}^{-1/3}(\sin i)^{-1}$, where $M_{1.4} \equiv M/(1.4 M_\odot)$.

In writing the above equations, we have implicitly averaged over many short period, small amplitude perturbations, to single out the principal, slowly-varying resonant term. Since the averaged equations depend on the planet longitudes and pericenters only through ϕ_1, we may simplify the problem by replacing eqn. (3) by

$$\dot{\phi}_1 = (m-1)\,n_1 - m\,n_2 + \dot{\omega}_1 \tag{6}$$

$$= \xi - n_1\,\beta \cos\phi_1\,/e_1, \tag{7}$$

where ξ is the "slow" frequency $(m-1)\,n_1 - m\,n_2$. Henceforth, we will drop the subscript 1, except where clarity dictates otherwise. For the general problem, an analogous set of equations to (1)–(3) and (6) governs the evolution of n_2, e_2, ω_2 and the second resonance variable $\phi_2 = (m-1)\,\lambda_1 - m\,\lambda_2 + \omega_2$.

For sufficiently small perturbations, we see that ϕ circulates at a mean angular frequency ξ ($\simeq 2\pi/5.56\,$yr for the PSR1257+12 system) and suffers small periodic perturbations, while both e and n show similar periodic perturbations about their mean Keplerian values. In the limit where we may treat e and n on the right-hand sides of eqns. (1-3) as approximately constant, and assume that $|\dot{\omega}| < |\xi|$, we obtain the approximate solutions

$$n(t) \simeq \bar{n} + \frac{3\,(m-1)\,\bar{n}^2\bar{e}}{\xi}\,\beta \cos[\xi(t-t_0)], \tag{8}$$

$$e(t) \simeq \bar{e} + \frac{\bar{n}}{\xi}\,\beta\cos[\xi(t-t_0)], \tag{9}$$

and

$$\omega(t) \simeq \omega_0 - \frac{\bar{n}}{\xi\bar{e}}\,\beta\sin[\xi(t-t_0)], \tag{10}$$

where $\bar{n}$ and $\bar{e}$ are the mean angular velocity and eccentricity, and ω_0 is the pericenter longitude at time $t = t_0$. These are equivalent to the expressions derived by Rasio *et al.* (1992) and compared with numerical integrations for $\sin i = 1$.

For larger masses, or for sufficiently small ξ (i.e., n_1/n_2 very close to 1.5), these approximations are no longer valid and a full solution of eqns. (1), (2) and (6) is necessary. This is simplified by noting that

$$\frac{dn}{de} = 3\,(m-1)\,ne, \tag{11}$$

which may be integrated immediately to give the relation

$$\ln(n/n_*) = \frac{3}{2}(m-1)\,e^2, \tag{12}$$

where n_* is a constant of the motion, equal to the angular velocity when $e = 0$. As the original differential equations were accurate only for small e, we may replace this by the approximate relation

$$n \simeq n_*\left[1 + \frac{3}{2}(m-1)\,e^2\right]. \tag{13}$$

The quantity n_* is related to the conserved action $J_2 \equiv n^{-1/3}\left[1 + \frac{1}{2}(m-1)e^2\right]$ which arises in a Hamiltonian treatment of the resonance dynamics (Malhotra 1988). Using eqn. (11), and introducing the *constant* quantity $\nu \equiv (m-1)\,n_* - m\,n_2$, eqn. (6) becomes

$$\dot{\phi} = \nu + \frac{3}{2}(m-1)^2e^2n_* - \frac{n}{e}\,\beta\cos\phi, \tag{14}$$

which may be solved simultaneously with eqn. (2) for $\dot{e}$. The quantity ν may be thought of as the "distance from exact resonance" (Malhotra 1988).

Eqns. (2) and (14) admit of equilibrium solutions with constant n, e and ϕ for $\phi = 0$ or $\phi = \pi$ only, and with e given by the roots of

$$\nu + \frac{3}{2}(m-1)^2e^2n_* - \frac{n}{e}\,\beta\cos\phi = 0. \tag{15}$$

For $e \ll \beta^{1/3}$, we may ignore the second term and the fixed point is given approximately by

$$e_F \simeq \frac{n\beta}{\nu}\cos\phi_F, \tag{16}$$

so that $\phi_F = 0$ for $\nu > 0$ and $\phi_F = \pi$ for $\nu < 0$. For $e \gg \beta^{1/3}$, fixed points exist only for $\nu < 0$, with

$$e_F \simeq \left[\frac{-2\nu}{3(m-1)^2n_*}\right]^{1/2} \tag{17}$$

and $\phi_F = 0$ or π. Note that the large-e fixed point is almost independent of β, depending only on the distance from resonance, ν.

In general, eqn. (15) has a single root for $\nu > 0$, and either one or three roots for $\nu < 0$, depending on whether ν is greater or less than a critical value, ν_c given by

$$\nu_c \simeq -\frac{3}{2} n_* \left[3(m-1)^2\beta^2\right]^{1/3} . \tag{18}$$

The forced eccentricity corresponding to ν_c is

$$e_c = \left[\frac{\beta}{3(m-1)^2}\right]^{1/3} . \tag{19}$$

In Fig. II, we plot the equilibrium solutions $(e\cos\phi)_F$ vs. ν for $m = 3$ and $\beta = 10^{-5}$, appropriate to PSR1257+12 with $\sin i = 1$.

For small displacements from the fixed points, we find that

$$\frac{d^2\delta\phi}{dt^2} \simeq -3(m-1)^2\left[1 + (e_c/e_F)^3\cos\phi_F\right] n_*^2 \beta e_F \cos\phi_F\, \delta\phi. \tag{20}$$

For $\phi_F = 0$ the fixed points are stable for all values of e_F, while for $\phi_F = \pi$ the fixed points are stable only for $e_F < e_c$. The stable and unstable fixed points merge at $\nu = \nu_c$ and $e = e_c$. The angular frequency of librations about the stable fixed points is given in the limiting cases of eqns. (16) and (17) by

$$\omega_L \simeq \begin{cases} \nu & \text{for } e \ll e_c \\ \left[-6(m-1)^2\nu\beta^2 n_*^3\right]^{1/4} & \text{for } e \gg e_c. \end{cases} \tag{21}$$

The nature of the solutions in the neighborhood of the resonance is best visualized in the form of a series of polar plots of (e, ϕ) for fixed values of n_*. Fig. III shows such a set of plots for $\beta = 10^{-5}$ and increasing values of n_*/n_2. The critical frequency $\nu_c \simeq -0.0024$, corresponding to $n_*/n_2 = \frac{3}{2} + \frac{1}{2}\nu = 1.4988$. For $n_*/n_2 < 1.4988$, two stable and one unstable fixed points are found, as predicted above and shown in Fig. II. For $n_*/n_2 > 1.4988$, only a single stable fixed point is present, at $\phi = 0$.

Three types of motion can be distinguished in these plots for $\nu < \nu_c$, or $n_*/n_2 < 1.4988$:

1. clockwise circulation about the small-e fixed point at $\phi = \pi$;
2. anticlockwise circulation at large-e encompassing all fixed points;
3. anticlockwise libration on bean-shaped trajectories about the fixed point at $\phi = 0$.

The trajectory passing through the unstable fixed point at $\phi = \pi$, or separatrix, (shown best in Fig. IIIb, for $n_*/n_2 = 1.497$) separates the type 3 orbits from those of types 1 and 2. Only type 3 trajectories are strictly classified as "resonant". For $\nu > \nu_c$, or $n_*/n_2 > 1.4988$, the type 1 trajectories and their enclosing separatrix cease to exist, and the motions of type 2 (large-e anticlockwise circulation) and type 3 (anticlockwise libration about the single fixed point)

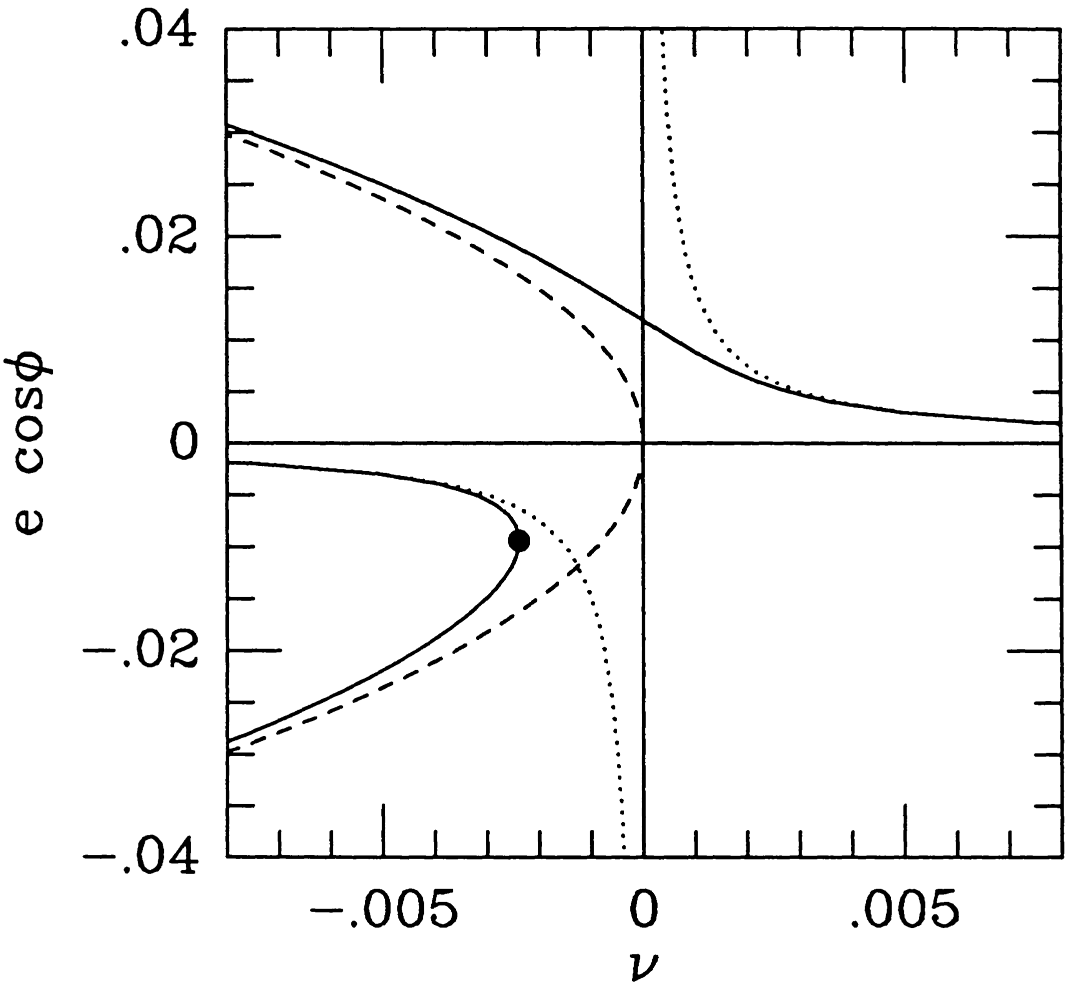


FIGURE II Equilibrium solutions $(e\cos\phi)_F$ vs. ν for $m = 3$ and $\beta = 10^{-5}$ (solid lines). Dotted and dashed curves show the asymptotic expressions of eqns. (16) and (17) for the small-e and large-e fixed points, respectively. The critical point, (ν_c, e_c) is indicated by the filled circle.

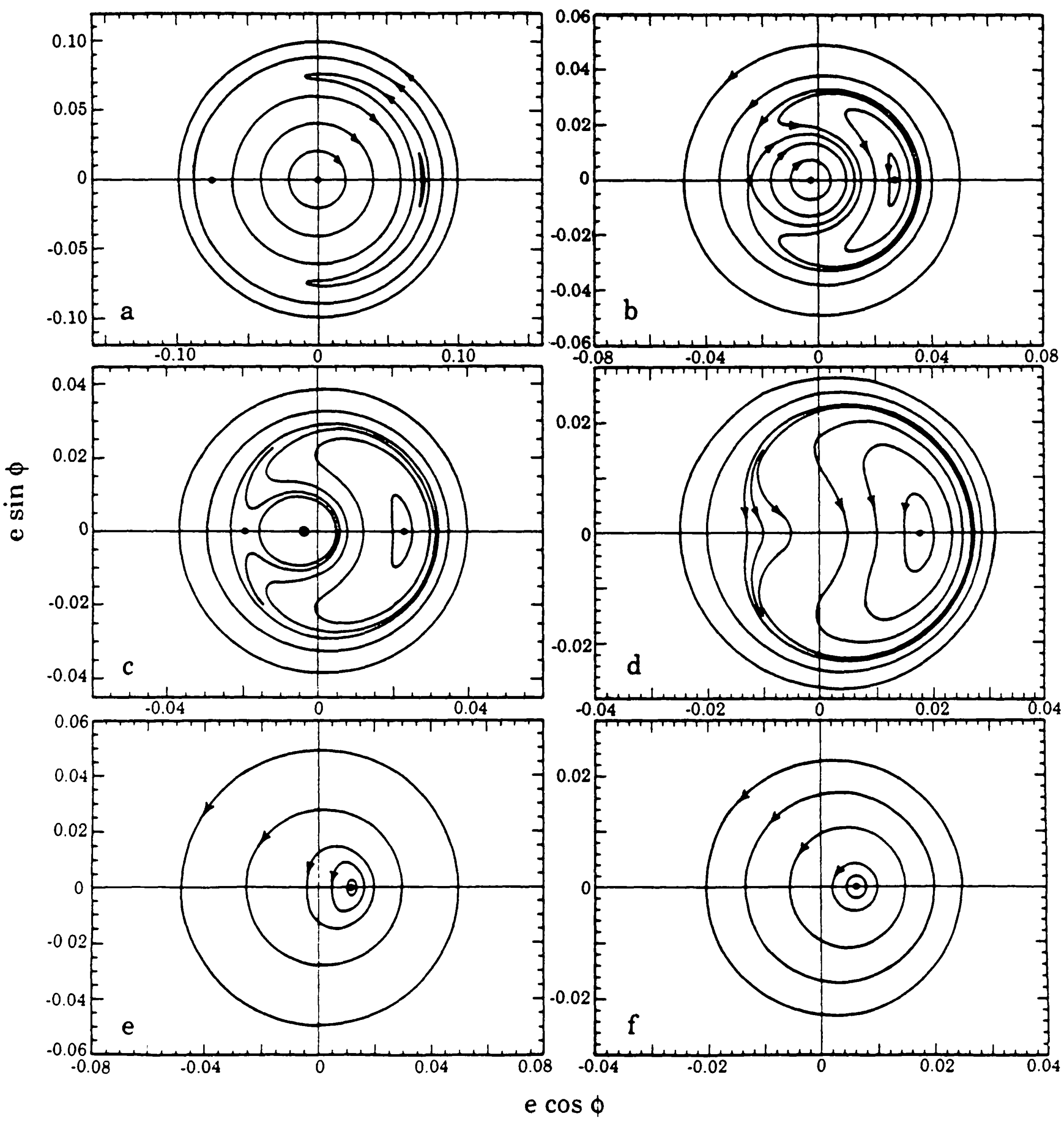


FIGURE III Phase space trajectories for $m = 3$, $\beta = 10^{-5}$ and $n_*/n_2 = 1.475$ (a), 1.497 (b), 1.498 (c), 1.499 (d), 1.500 (e), and 1.501 (f). The fixed points calculated from eqn. (15) are indicated by filled circles. Arrows indicate the direction of evolution.

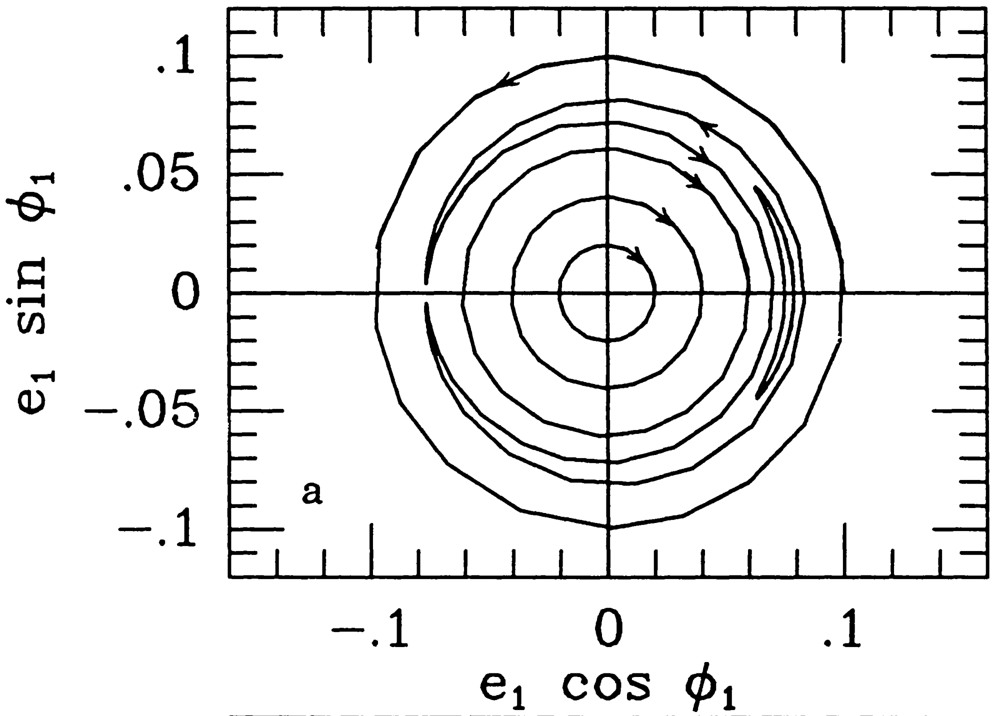
merge into a single family of orbits. Note that strictly resonant motion, or resonance "locking", can occur for all values of $\nu < \nu_c$, but for $\nu \ll \nu_c$ is restricted to a very narrow range of eccentricities close to the stable equilibrium value of e_F given by eqn. (17), as shown in Fig. IIIa.

The principal observational significance of these solutions lies in the large variations in eccentricity that accompany resonant librations for small values of $|\nu|$, and the oscillatory variations in the pericenter, ω, that result from the non-uniform motion in ϕ. The above simplified model, which neglects the perturbations of the inner planet on the outer, does not, of course, provide a good quantitative description of PSR1257+12 (A more accurate model may be obtained by a straightforward extension of the above equations to include simultaneous perturbations of both planets). Nevertheless, the qualitative predictions of this model are in reasonable agreement with numerical integrations of the full three-body problem. From the observed values of n_1, n_2 and $e_1 = 0.022$, we find that $n_* = 1.4737\, n_2$, so that $\nu = 2n_* - 3n_2 = -0.0527\, n_2$. The critical point is at $\nu_c = -0.0024\, \beta_5^{2/3} n_2$ and $e_c = 0.0094\, \beta_5^{1/3}$, where $\beta_5 \equiv 10^5 \beta$.

For $\sin i = 1$, $\beta_5 \simeq 1$ and we find that $\nu \ll \nu_c$. The corresponding phase space trajectories are shown in Fig. IVa, and are similar to those of Fig. IIIa. There are two possible stable fixed points: the small-e point at $(e \cos\phi)_F \simeq -0.0003$ and the large-e solution at $(e\cos\phi)_F \simeq +0.0765$. The region of resonant librations is restricted to a narrow range of eccentricity about the latter value ($0.070 \lesssim e \lesssim 0.085$), and well above the observed eccentricity of 0.022. The motion for $\sin i \simeq 1$ is thus of type 1 above: small-e circulation about the fixed point at $e_F \simeq 0.0003$. The resulting fractional eccentricity variations are $\sim e_F/e \simeq 1.4\%$, comparable to that seen in the numerical integrations in Fig. I and predicted by Rasio *et al.* (1992).

For larger values of $(\sin i)^{-1} \simeq \beta_5$, $|\nu_c|$ increases and the phase space portrait more closely resembles that of Fig. IIIb or IIIc. Fig. IVb shows the situation for $(\sin i)^{-1} = \beta_5 = 40$, corresponding to $\nu_c = -0.028\, n_2$ and $e_c = 0.0322$. Since $\nu < \nu_c$, there are again two stable fixed points, at $(e\cos\phi)_F \simeq -0.0114$ and $+0.0765$, as well as an unstable point and a separatrix. The domain of resonant motion is now much larger than for $\sin i = 1$, as indicated in the figure. In this simple test-particle model, the nominal elements of the inner planet ($e_1 = 0.022$ and $\phi_1 = 354°$ at JD 244 8040) would place the system just outside the resonant region. Numerical integrations of the full system, however, show that both ϕ_1 and the corresponding argument for the outer planet, $\phi_2 = 2\lambda_1 - 3\lambda_2 + \omega_2$ are in resonant libration for $(\sin i)^{-1} = 40$, as illustrated in Fig. V. The critical value of $(\sin i)^{-1}$ necessary for the system to be in resonance is $\simeq 15$ (Malhotra *et al.* 1992), the exact value depending on the still uncertain eccentricities and pericenters of the two orbits.

Unlike the situation for $(\sin i)^{-1} \simeq 1$, the variations in e and the departures of $\dot\phi$ and $\dot\omega$ from their average values will become large and nonsinusoidal for $(\sin i)^{-1} \gtrsim 10$, as illustrated in Fig. I. As pointed out by Malhotra *et al.* (1992), this would make the perturbations much easier to detect. However, to observe the system with its orbital plane so nearly face-on seems rather unlikely. Indeed, for random orientations, the probability $P(i < i_o) = \frac{1}{2}(1 - \cos i_o)$, so that $P(\sin i < 0.1) \simeq 0.25\%$.



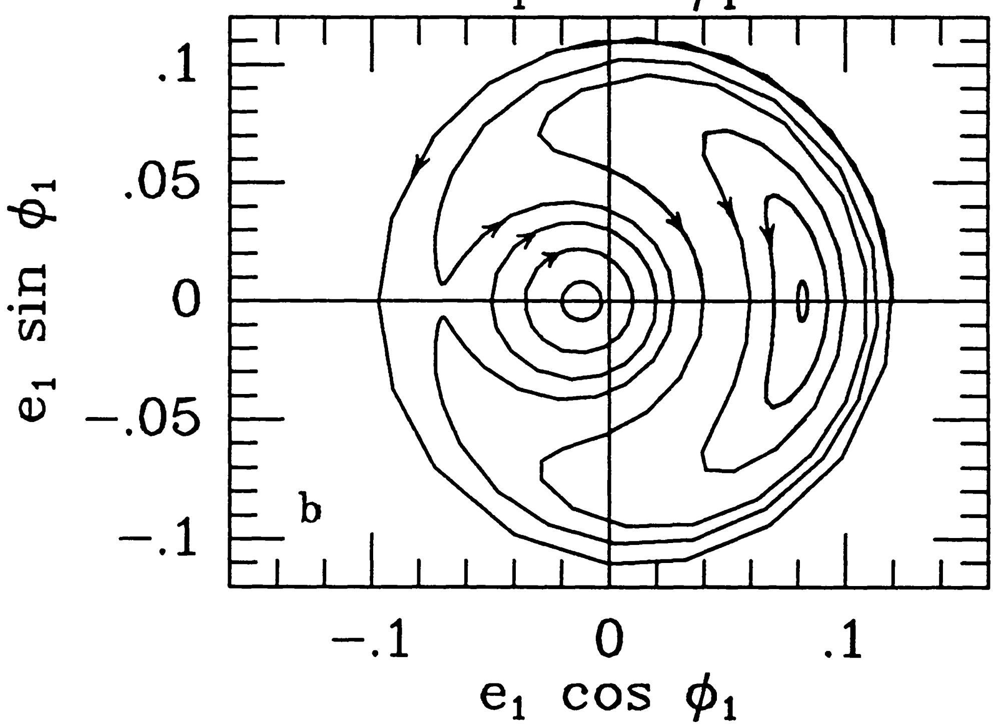


FIGURE IV Phase space trajectories for a test particle model of the inner planet in the PSR1257+12 system, for (a) $\beta = 10^{-5}$ ($\sin i = 1$) and (b) $\beta = 4 \times 10^{-4}$ ($\sin i = 1/40$).

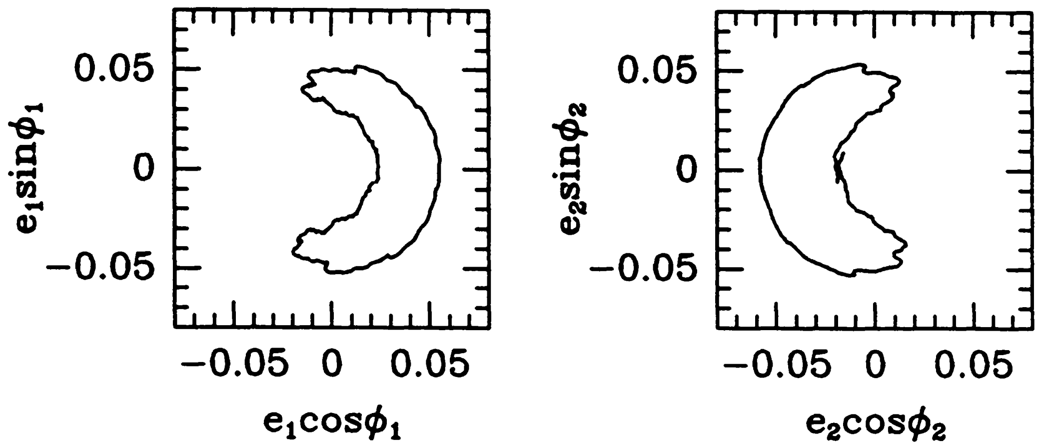


FIGURE V Phase-space trajectories for $(\sin i)^{-1} = 40$ obtained from a numerical integration as shown in he bottom panel of Fig. I. Both resonant variables ϕ_1 and ϕ_2 are clearly librating, with consequent large variations in e_1 and e_2 as seen in Fig. I. Compare the (e_1, ϕ_1) trajectory with the analytical model trajectories in Fig. IVb.

EFFECTS OF INDIVIDUAL CLOSE ENCOUNTERS

Quite independent from the presence of a near-resonance in the system, the fact that $(a_2 - a_1)/a_2 \ll 1$ implies that the two planets can perturb each other quite significantly during close encounters (near conjunctions). These close encounters occur every $2\pi/(n_1 - n_2) \simeq 200$ days in the PSR1257+12 system. Fig. VI shows the result of a numerical integration of the system for $M = 1.4 M_\odot$ and $\sin i = 1$. During close encounters, the instantaneous (osculating) orbital period of the inner planet increases briefly, while that of the outer planet decreases. These changes have amplitudes $|\Delta P_i / P_i| \sim 10^{-4}$, and take place over a timescale of about a month. In contrast, the long-term variations over a period $\simeq 5.5$ ¡yr are caused by the near 3:2 resonance (cf. previous section).

The orbital perturbations corresponding to close encounters are easy to calculate analytically. Here, for simplicity, we present this calculation for the case of two planets in coplanar circular orbits. As in the previous section, we treat the inner planet as a test particle perturbed by its interaction with the outer planet. We assume that a conjunction takes place at $t = 0$. Following Danby (1988), we write the perturbation equation for a_1 as

$$\dot{a}_1 = \frac{2 n_1 a_1^3}{GM} B_{12}, \tag{22}$$

where B_{12} is the tangential component of the perturbing force on m_1,

$$B_{12} = \frac{-G m_2 a_2 \sin(n_1 - n_2)t}{[a_1^2 + a_2^2 - 2 a_1 a_2 \cos(n_1 - n_2)t]^{3/2}}. \tag{23}$$

Integrating eqn. (22) from $-t_o$ to t, where $t_o \equiv \pi/(n_1 - n_2)$ is the time of

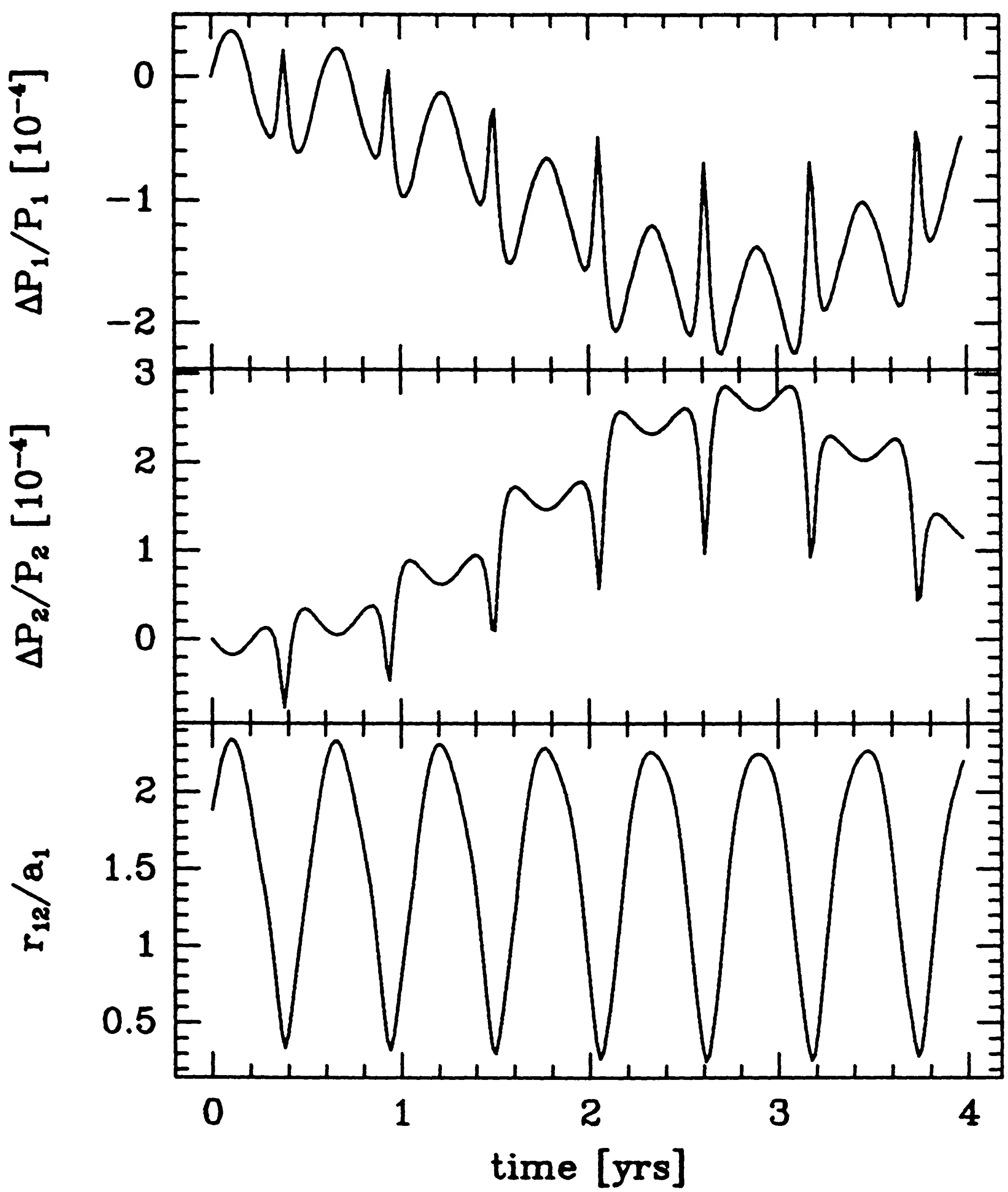


FIGURE VI Effects of close encounters (conjunctions) between the two planets in PSR1257+12. Changes in the osculating orbital periods of the two planets are shown as a function of time. Also shown is the separation r_{12} between the two planets. The large spikes of amplitude $\Delta P_i/P_i \sim 10^{-4}$ and width ~ 1 month occur at conjunctions.

opposition, we find

$$\frac{\Delta a_1(t)}{a_1} = -2\alpha \left(\frac{1}{1+\alpha} - \frac{1}{[1+\alpha^2 - 2\alpha\cos(n_1-n_2)t]^{1/2}} \right) \left(\frac{n_1}{n_1-n_2} \right) \left(\frac{m_2}{M} \right). \tag{24}$$

The maximum amplitude of the change is obtained by setting $t = 0$ in this expression, giving

$$\left| \frac{\Delta a_1}{a_1} \right|_{max} = 4 \frac{\alpha^2}{1-\alpha^2} \left(\frac{n_1}{n_1-n_2} \right) \left(\frac{m_2}{M} \right), \tag{25}$$

and the FWHM of the peak is

$$\Delta t_{\rm FWHM} = \left(\frac{2}{n_1-n_2} \right) \cos^{-1} \left[\frac{\alpha}{2}(3-\alpha^2) \right]. \tag{26}$$

Note that the semimajor axis $a_1(t)$ returns to its unperturbed value after the encounter ($\Delta a_1(t_o) = 0$ and expression (24) is symmetric in t). The same is true for the osculating orbital period P_1 and mean angular frequency $n_1 = 2\pi/P_1$, since $\dot{n}_1 = -(3/2)(n_1/a_1)\dot{a}_1$. However, the mean longitude of the planet, which is given by $\lambda_1 = const + \int n_1\, dt$ does *not* return to its unperturbed value. The total net shift in mean longitude from one opposition to the next is

$$\Delta\lambda_1 = \int_{-t_o}^{+t_o} \Delta n_1(t)\, dt = -6\alpha \left[2K(\alpha) - \frac{\pi}{1+\alpha} \right] \left(\frac{n_1}{n_1-n_2} \right)^2 \left(\frac{m_2}{M} \right) \tag{27}$$

where $K(\alpha) \equiv \int_0^{\pi/2} (1-\alpha^2\sin^2 x)^{-1/2}\, dx$ is the complete elliptic integral of the first kind.

The net shift in mean longitude is negative for the inner planet (and positive for the outer one). When $\alpha \ll 1$ (and $n_2 \ll n_1$), we have $\Delta\bar{n}_1 \equiv \Delta\lambda_1/(2t_o) \simeq -3\alpha^2 n_1 (m_2/M)$. This is the $\mathcal{O}(\alpha^2)$ correction to the mean angular velocity of the inner planet corresponding to the *tangential* component of the perturbing force. Note that, in this limit, the leading, $\mathcal{O}(\alpha)$ correction corresponds to the *radial* force of a distant ring of radius $a_2 \gg a_1$ and mass $m_2 \gg m_1$. In the opposite limit where $|\alpha - 1| \ll 1$, i.e., when the two orbits are very close to each other, the tangential component of the perturbing force becomes dominant, causing step-like shifts in the planet's longitudes occurring near conjunctions over a time short compared to the time between conjunctions.

For the PSR1257+12 system we have $\alpha \simeq (P_1/P_2)^{2/3} \simeq 0.77$ (giving $K(\alpha) \simeq 1.95$) and $n_1/(n_1 - n_2) \simeq 3.1$, independent of M and $\sin i$. From eqns. (25) and (26) we get an amplitude $|\Delta P_1/P_1|_{max} = \frac{3}{2}|\Delta a_1/a_1|_{max} \simeq 1.7 \times 10^{-4}\, M_{1.4}^{-1/3} (\sin i)^{-1}$ and width $\Delta t_{\rm FWHM} \simeq 25\,$days, in good agreement with the results of our numerical integrations (Fig. VI). In reality, the orbital eccentricities produce slight variations in the amplitude $|\Delta P_i/P_i|_{max}$ from one encounter to the next. The predicted net angular shift $\Delta\lambda_1 = -2.6'\, M_{1.4}^{-1/3} (\sin i)^{-1}$. This corresponds to a shift in the time of zero longitude of $(-\Delta\lambda_1/2\pi)P_1 = +678\,{\rm s}\, M_{1.4}^{-1/3} (\sin i)^{-1}$ after each conjunction. While this is probably too small to be detectable at present (assuming $\sin i \simeq 1$), the effects of a sufficiently large number of these close encounters could become measurable after several years. One approach would be to compare orbital fits to timing data sets obtained during individual $\sim$ 200 day segments between successive conjunctions (see Table I), but excluding data from the $\sim$ 3 week periods around conjunctions. By combining fits to many such data sets, the signal-to-noise ratio might be increased to the point where the long-term average mean motions of the planets can be distinguished from the mean motion between conjunctions.

TABLE I Predicted dates of planet conjunctions.

JD 244 8040.31	=	05/28.81/1990
JD 244 8245.48	=	12/19.98/1990
JD 244 8450.77	=	07/13.27/1991
JD 244 8656.85	=	02/04.35/1992
JD 244 8863.79	=	08/29.29/1992
JD 244 9071.77	=	03/25.27/1993
JD 244 9280.21	=	10/19.71/1993
JD 244 9587.53	=	05/15.03/1994
JD 244 9693.83	=	12/07.33/1994

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REFERENCES

Brouwer, D., and Clemence, G. M. 1961, *Methods of Celestial Mechanics* (Academic, New York).

Danby, J. M. A. 1988, *Fundamentals of Celestial Mechanics* (Willmann-Bell, Richmond).

Malhotra, R. 1988, *Ph.D. Thesis*, Cornell University.

Malhotra, R. *et al.* 1992, *Nature*, **356**, 583.

Rasio, F. A. *et al.* 1992, *Nature*, **355**, 325.

Wolszczan, A., and Frail, D. A. 1992, *Nature*, **355**, 145.

Section 4:

Evolutionary Scenarios: Neutron Star and Planet Formation