

INTERACTING BINARIES WITH ECCENTRIC ORBITS. SECULAR ORBITAL EVOLUTION DUE TO CONSERVATIVE MASS TRANSFER

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ABSTRACT

We investigate the secular evolution of the orbital semi-major axis and eccentricity due to mass transfer in eccentric binaries, assuming conservation of total system mass and orbital angular momentum. Assuming a delta function mass transfer rate centered at periastron, we find rates of secular change of the orbital semi-major axis and eccentricity which are linearly proportional to the magnitude of the mass transfer rate at periastron. The rates can be positive as well as negative, so that the semi-major axis and eccentricity can increase as well as decrease in time. Adopting a delta-function mass-transfer rate of $10^{-9}M_{\odot}\text{yr}^{-1}$ at periastron yields orbital evolution timescales ranging from a few Myr to a Hubble time or more, depending on the binary mass ratio and orbital eccentricity. Comparison with orbital evolution timescales due to dissipative tides furthermore shows that tides cannot, in all cases, circularize the orbit rapidly enough to justify the often adopted assumption of instantaneous circularization at the onset of mass transfer. The formalism presented can be incorporated in binary evolution and population synthesis codes to create a self-consistent treatment of mass transfer in eccentric binaries.

Subject headings: Celestial mechanics, Stars: Binaries: Close, Stars: Mass Loss

1. INTRODUCTION

Mass transfer between components of close binaries is a common evolutionary phase for many astrophysically interesting binary systems. Indeed, mass ejection and/or accretion is responsible for many of the most recognizable phenomena associated with close binaries, such as persistent or transient X-ray emission, neutron star spin-up, and orbital contraction or expansion. Theoretical considerations of these and other associated phenomena in the literature probe these systems quite effectively, yet they often do not consider the effects of any eccentricity associated with the binary orbit. This can be of particular importance for binaries containing a neutron star or a black hole, where mass loss and natal kicks occurring during compact object formation may induce a significant eccentricity to the binary (e.g. Hills 1983; Brandt & Podsiadlowski 1995; Kalogera 1996). After the formation of the compact object, tides tend to circularize the orbit on a timescale which strongly depends on the ratio of the radius of the compact object's companion to the orbital semi-major axis. Because of this, orbits are usually assumed to circularize instantaneously when a binary approaches or begins a mass transfer phase.

Despite our generally well developed understanding of tidal interactions in close binaries, quantitative uncertainties in tidal dissipation mechanisms propagate into the determination of circularization timescales. For example, Meibom & Mathieu (2005) have shown that current theories of tidal circularization cannot explain observed degrees of circularization of solar-type binaries in open clusters. Circularization of high-mass binaries, on the other hand, is currently thought to be driven predominantly by resonances between dynamic tides and free oscillation modes, but initial conditions play an important role and an extensive computational survey of relevant parts of the initial parameter space has yet to be undertaken (Witte & Savonije 1999, 2001; Willems et al.

2003).

Furthermore, assumptions of instantaneous circularization immediately before or at the onset of mass transfer are in clear contrast with observations of eccentric mass transferring systems. In the most recent catalog of eccentric binaries with known apsidal-motion rates compiled by Petrova & Orlov (1999), 26 out of the 128 listed systems are semi-detached or contact binaries. Among these mass-transferring systems, 9 have measured eccentricities greater than 0.1. In addition, many high-mass X-ray binaries are known to have considerable orbital eccentricities (Raguzova & Popov 2005). While mass transfer in these systems is generally thought to be driven by the stellar wind of a massive O- or B-star, it has been suggested that some of them may also be subjected to atmospheric Roche-lobe overflow at each periastron passage of the massive donor (e.g. Petterson 1978).

Huang (1956), Kruszewski (1964), and Piotrowski (1964) were the first to study the effects of mass transfer on the orbital elements of eccentric binaries. However, their treatment was restricted to perturbations of the orbital motion caused by the variable component masses. Matese & Whitmire (1983, 1984) extended these early pioneering studies to include the effects of linear momentum transport from one star to the other, as well as any other possible perturbations caused by the mass transfer stream in the system. However, these authors derived the equations governing the motion of the binary components with respect to a reference frame with origin at the mass center of the binary, which is not an inertial frame. Their equations therefore do not account for the accelerations of the binary mass center caused by the mass transfer (see § 3.3).

More recent work on mass transfer in eccentric binaries has mainly focused on smoothed particle hydrodynamics calculations of the mass transfer stream over the course of a few orbits, without any consideration of the long-term

evolution of the binary (Layton et al. 1998; Regös et al. 2005).

Hence, there is ample observational and theoretical motivation to revisit the study of eccentric mass-transferring binaries. In this paper, our aim is to derive the equations governing the evolution of the orbital semi-major axis and eccentricity in eccentric mass-transferring binaries, assuming conservation of total system mass and orbital angular momentum. In a subsequent paper, we will incorporate the effects of mass and orbital angular momentum losses from the system.

Our analysis is based on the seminal work of Hadjidemetriou (1969b) who was the first to derive the equations of motion of the components of eccentric mass-transferring binaries while properly accounting for the effects of the variable component masses on the stars' mutual gravitational attraction, the transport of linear momentum from one star to the other, the accelerations of the binary mass center due to the redistribution of mass in the system, and the perturbations of the orbital motion caused by the mass-transfer stream. While the equations of motion derived by Hadjidemetriou (1969b) are valid for orbits of arbitrary eccentricity, the author restricted the derivation of the equations governing the evolution of the semi-major axis and eccentricity to orbits with small initial eccentricities.

The paper is organized as follows. In § 2 and § 3 we present the basic assumptions relevant to the investigation and derive the equations governing the motion of the components of an eccentric mass-transferring binary under the assumption of conservative mass transfer. The associated equations governing the rates of change of the semi-major axis and the orbital eccentricity are derived in § 4, while numerical results for the timescales of orbital evolution due to mass transfer as a function of the initial binary mass ratio and orbital eccentricity are presented in § 5. For comparison, timescales of orbital evolution due to dissipative tidal interactions between the binary components are presented in § 6. § 7 is devoted to a summary of our main results and a discussion of future work. In the appendices, lastly, we derive an equation for the position of the inner Lagrangian point in eccentric binaries with non-synchronously rotating component stars (Appendix A), and present an alternative derivation for the equations governing the secular evolution of the orbital semi-major axis and eccentricity assuming instantaneous mass transfer between two point masses (Appendix B).

2. BASIC ASSUMPTIONS

We consider a binary system consisting of two stars in an eccentric orbit with period P_{orb} , semi-major axis a , and eccentricity e . We let the component stars rotate with angular velocities $\vec{\Omega}_1$ and $\vec{\Omega}_2$ parallel to the orbital angular velocity $\vec{\Omega}_{\text{orb}}$, and assume the rotation rates to be uniform throughout the stars. We also note that the magnitude of $\vec{\Omega}_{\text{orb}}$ varies periodically in time for eccentric binaries, but its direction remains fixed in space. Because of this, the stars cannot be synchronized with the orbital motion at all times.

At some time t , one of the stars is assumed to fill its Roche lobe and begins transferring mass to its companion through the inner Lagrangian point L_1 . We assume

this point to lie on the line connecting the mass centers of the stars, even though non-synchronous rotation may cause it to oscillate in the direction perpendicular to the orbital plane with an amplitude proportional to the degree of asynchronism (Matese & Whitmire 1983). Since the donor's rotation axis is assumed to be parallel to the orbital angular velocity, we can safely assume that the transferred mass remains confined to the orbital plane.

We furthermore assume that all mass lost from the donor is accreted by its companion, and that any orbital angular momentum transported by the transferred mass is immediately returned to the orbit. The mass transfer thus conserves both the total system mass and the orbital angular momentum.

We also neglect any perturbations to the orbital motion other than those due to mass transfer. At the lowest order of approximation, these additional perturbations (e.g., due to tides, magnetic braking, or gravitational radiation) are decoupled from those due to mass transfer, and can thus simply be added to obtain the total rates of secular change of the orbital elements.

3. EQUATIONS OF MOTION

3.1. Absolute Motion of the Binary Components

Following Hadjidemetriou (1969b), we derive the equations of motion of the components of an eccentric mass-transferring binary with respect to a right handed inertial frame of reference $OXYZ$ which has an arbitrary position and orientation in space (see Fig. 1). We let M_i be the mass of star i at some time t at which mass is transferred from the donor to the accretor, and $M_i + \delta M_i$ the mass of the same star at some time $t + \delta t$, where $\delta t > 0$ is a small time interval. With these notations, $\delta M_i < 0$ corresponds to mass loss, and $\delta M_i > 0$ to mass accretion. We furthermore denote the point on the stellar surface at which mass is lost or accreted by A_i . For the donor star, A_i corresponds to the inner Lagrangian point L_1 , while for the accretor, A_i can be any point on the star's equator. For the remainder of the paper, we let $i = 1$ correspond to the donor and $i = 2$ to the accretor.

Because of the mass loss/gain, the center of mass of star i at time $t + \delta t$ is shifted from where it would have been had no mass transfer taken place. To describe this perturbation, we introduce an additional right-handed coordinate frame $O_i X_i Y_i Z_i$ with a spatial velocity such that its origin follows the unperturbed orbit of star i , i.e., the origin of $O_i X_i Y_i Z_i$ follows the path the center of mass of star i would have taken had no mass transfer occurred. Thus, at time t , the center of mass of star i lies at the origin of O_i , while at time $t + \delta t$ it has a non-zero position vector with respect to O_i . We furthermore let the Z_i -axis of the $O_i X_i Y_i Z_i$ frame point in the direction of the orbital angular momentum vector, and let the frame rotate synchronously with the unperturbed orbital angular velocity of the binary in the absence of mass transfer. The direction and orientation of the X_i -axes are then chosen such that at time t the X_i -axis points along the direction from the mass center of star i to the mass center of its companion.

To describe the shift in the mass center of star i due to the mass loss/gain, we denote the position vector of O_i at times t and $t + \delta t$ with respect to the inertial frame by \vec{R}_i and \vec{R}'_i , respectively. The position vector of the

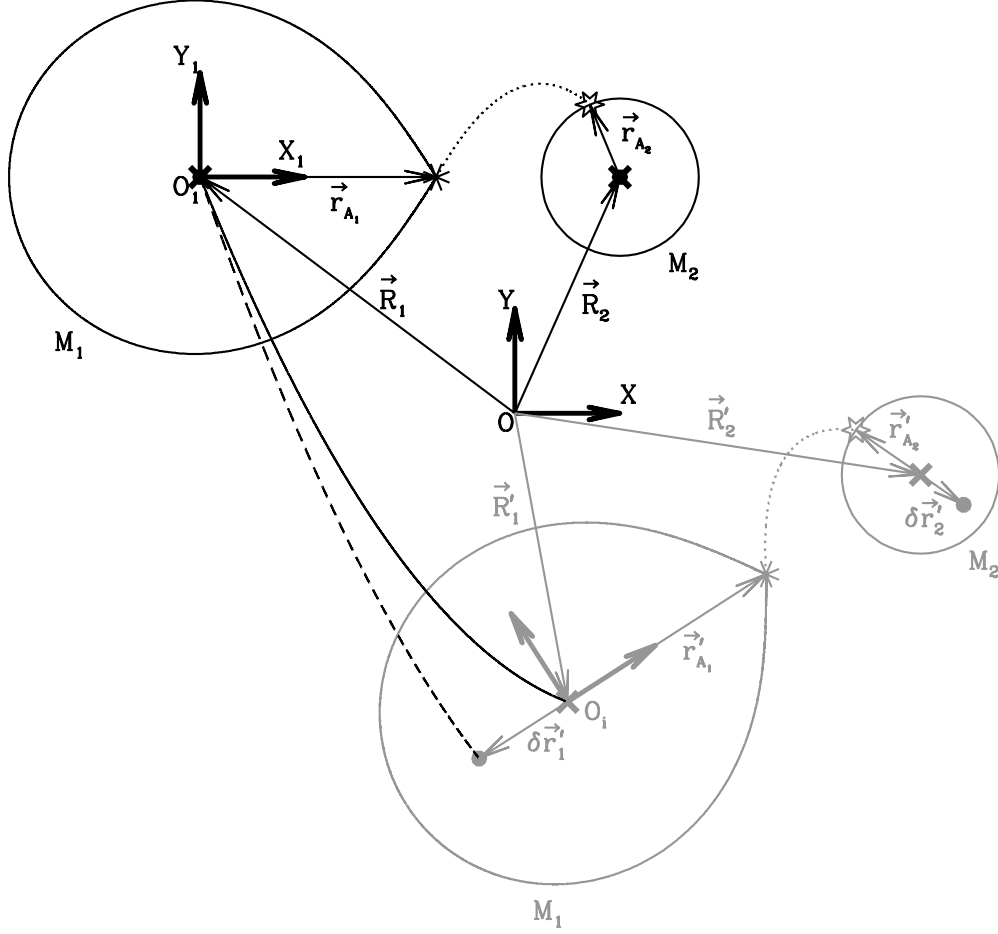


FIG. 1.— Schematic representation of the reference frames and position vectors adopted in the derivation of the equations of motion of the components of an eccentric mass-transferring binary. The Z - and Z_1 -axes of the $OXYZ$ and $O_1X_1Y_1Z_1$ frames are perpendicular to the plane of the page and are therefore not drawn. The geometry of the system at time t is shown in black, while the geometry at time $t + \delta t$ is shown in gray. The solid line connecting the origin of the $O_1X_1Y_1Z_1$ frame at time t to the origin of the frame at time $t + \delta t$ represents the path the donor would have taken had no mass transfer occurred. The dashed line, on the other hand, represents the perturbed motion of the donor's mass center (small solid circles) due to the mass transfer. A similar perturbation is imparted to the motion of the accretor. For clarity, this perturbed motion and the $O_2X_2Y_2Z_2$ reference frame connected to the accretor (see text) are omitted from the figure. The dotted line illustrates a possible path of a mass element transferred from the donor to the accretor. The element leaves the donor at the inner Lagrangian point L_1 (asterisk) and accretes onto the companion at the point A_2 (open star).

center of mass of star i at times t and $t + \delta t$ is then given by \vec{R}_i and $\vec{R}'_i + \delta\vec{r}'_i$, where $\delta\vec{r}'_i$ is the position vector of the center of mass of star i at time $t + \delta t$ with respect to O_i . Moreover, we denote by \vec{r}_{A_i} and \vec{r}'_{A_i} the vectors from O_i to the position where the point A_i on the stellar surface would be at times t and $t + \delta t$, respectively, had no mass been lost/accreted. The various position vectors at time $t + \delta t$ are related by

$$(M_i + \delta M_i) (\vec{R}'_i + \delta\vec{r}'_i) = M_i \vec{R}'_i + \delta M_i (\vec{R}'_i + \vec{r}'_{A_i}), \quad (1)$$

which, at the lowest order of approximation in δM_i and $\delta\vec{r}'_i$, yields

$$\delta\vec{r}'_i = \frac{\delta M_i}{M_i} \vec{r}'_{A_i}. \quad (2)$$

As expected, the displacement of the center of mass of star i due to the mass loss/gain is directed along the line connecting the center of mass of the star and the mass ejection/accretion point.

We furthermore denote with $\vec{\rho}'_{A_i}$ the vector from the center of mass of star i at time $t + \delta t$ to the position where the point A_i would be at time $t + \delta t$, had no mass been transferred between the binary components, and with $\delta\vec{\rho}'_{A_i}$ the perturbation of this vector caused by the mass transfer. It then follows that $\vec{\rho}'_{A_i} = \vec{r}'_{A_i} - \delta\vec{r}'_i$ and thus, by definition, $\delta\vec{\rho}'_{A_i} = \delta\vec{r}'_i$. At the lowest order of approximation in δM_i and $\delta\vec{r}'_i$, equation (2) therefore also yields

$$\delta\vec{\rho}'_{A_i} = \frac{\delta M_i}{M_i} \vec{r}'_{A_i}. \quad (3)$$

The definitions of and the relations between these various position vectors are illustrated schematically in Fig. 1.

Next, we denote the absolute velocity of the center of mass of star i with respect to the inertial frame of reference at times t and $t + \delta t$ by \vec{V}_i and \vec{V}'_i , respectively, and the absolute velocity of the ejected/accreted mass element by $\vec{W}_{\delta M_i}$. The linear momentum \vec{Q}_1 of star 1 at

time t is then given by

$$\vec{Q}_1 = M_1 \vec{V}_1, \quad (4)$$

and the total linear momentum \vec{Q}'_1 of star 1 and the ejected mass element at time $t + \delta t$ by

$$\vec{Q}'_1 = (M_1 + \delta M_1) \vec{V}'_1 - \delta M_1 \vec{W}_{\delta M_1}. \quad (5)$$

Similarly, the total linear momentum \vec{Q}_2 of star 2 and the mass element to be accreted at time t is given by

$$\vec{Q}_2 = M_2 \vec{V}_2 + \delta M_i \vec{W}_{\delta M_i}, \quad (6)$$

and the linear momentum \vec{Q}'_2 of star 2 at time $t + \delta t$ by

$$\vec{Q}'_2 = (M_2 + \delta M_2) \vec{V}'_2. \quad (7)$$

At time $t + \delta t$ the velocity of the center of mass of star i can be written as

$$\vec{V}'_i = \vec{V}'_{O_i} + (\vec{\Omega}'_{\text{orb}} + \delta \vec{\Omega}'_{\text{orb}}) \times \delta \vec{r}'_i, \quad (8)$$

where \vec{V}'_{O_i} is the absolute velocity of the origin of $O_i X_i Y_i Z_i$ at time $t + \delta t$, $\vec{\Omega}'_{\text{orb}}$ is the orbital angular velocity of the binary at time $t + \delta t$ in the absence of mass transfer, and $\delta \vec{\Omega}'_{\text{orb}}$ is the perturbation of the orbital angular velocity at time $t + \delta t$ due to the mass loss/gain of the binary components.

In the limit of small δt , taking the difference between the linear momenta \vec{Q}'_i and \vec{Q}_i , dividing the resulting equation by δt , and noting that the absolute velocity \vec{V}_{O_i} of the origin of $O_i X_i Y_i Z_i$ at time t is equal to \vec{V}_i , yields

$$M_i \frac{d\vec{V}_{O_i}}{dt} = \vec{F}_i + \dot{M}_i \vec{U}_{\delta M_i}. \quad (9)$$

Here, $\vec{F}_i = d\vec{Q}_i/dt$ is the sum of all external forces acting on star i , $\dot{M}_i = dM_i/dt$ is the mass loss/accretion rate of star i , and

$$\vec{U}_{\delta M_i} = \vec{W}_{\delta M_i} - \vec{V}_{O_i} - \vec{\Omega}_{\text{orb}} \times \vec{r}_{A_i} \quad (10)$$

is the relative velocity of the ejected/accreted mass element with respect to the ejection/accretion point A_i . In the derivation of equation (10), we have made use of equations (2) and (8) and restricted ourselves to first-order terms in the small quantities δM_i and $\delta \vec{\Omega}'_{\text{orb}}$.

The absolute acceleration of the center of mass of star i with respect to the inertial frame $OXYZ$ is given by

$$\frac{d^2 \vec{R}_i}{dt^2} = \vec{\gamma}_{O_i} + \vec{\gamma}_{\text{rel},i} + \vec{\gamma}_{\text{cor},i}, \quad (11)$$

where $\vec{\gamma}_{O_i} = d\vec{V}_{O_i}/dt$ is the acceleration of the origin of $O_i X_i Y_i Z_i$ with respect to $OXYZ$, $\vec{\gamma}_{\text{rel},i} = (\dot{M}_i/M_i) \vec{r}_{A_i}$ is the relative acceleration of the center of mass of star i with respect to O_i , and $\vec{\gamma}_{\text{cor},i} = 2(\dot{M}_i/M_i)(\vec{\Omega}_{\text{orb}} \times \vec{r}_{A_i})$ is the Coriolis acceleration of the center of mass of star i with respect to O_i ¹. The expressions for $\vec{\gamma}_{\text{rel},i}$ and $\vec{\gamma}_{\text{cor},i}$ follow from the observation that $d\vec{\rho}'_{A_i}/dt = (\dot{M}_i/M_i) \vec{r}_{A_i}$, which one obtains by dividing equation (3) by δt in the limiting case of small δt . The equation of motion for the mass center of star i with respect to the inertial frame $OXYZ$ then becomes

$$M_i \frac{d^2 \vec{R}_i}{dt^2} = \vec{F}_i + \dot{M}_i \left(\vec{U}_{\delta M_i} + 2 \vec{\Omega}_{\text{orb}} \times \vec{r}_{A_i} \right) + \ddot{M}_i \vec{r}_{A_i}. \quad (12)$$

¹ The centrifugal acceleration does not play a role since it is proportional to $\delta \vec{r}'_i$ which vanishes for small δt .

3.2. Relative Motion of the Binary Components

We can now obtain the equation describing the relative motion of the accretor (star 2) with respect to the donor (star 1) by taking the difference of the equations of motion of the stars with respect to the inertial frame of reference. For convenience, we first decompose the sum of the external forces acting on each star as

$$\vec{F}_i = -\frac{G M_1 M_2}{|\vec{r}|^2} \frac{\vec{R}_i}{|\vec{R}_i|} + \vec{f}_i, \quad (13)$$

where G is the Newtonian constant of gravitation, and \vec{f}_i the total gravitational force exerted on star i by the particles in the mass-transfer stream. It follows that

$$\begin{aligned} \frac{d^2 \vec{r}}{dt^2} = & -\frac{G(M_1 + M_2)}{|\vec{r}|^3} \vec{r} + \frac{\vec{f}_2}{M_2} - \frac{\vec{f}_1}{M_1} \\ & + \frac{\dot{M}_2}{M_2} \left(\vec{v}_{\delta M_2} + \vec{\Omega}_{\text{orb}} \times \vec{r}_{A_2} \right) \\ & - \frac{\dot{M}_1}{M_1} \left(\vec{v}_{\delta M_1} + \vec{\Omega}_{\text{orb}} \times \vec{r}_{A_1} \right) \\ & + \frac{\ddot{M}_2}{M_2} \vec{r}_{A_2} - \frac{\ddot{M}_1}{M_1} \vec{r}_{A_1}, \end{aligned} \quad (14)$$

where $\vec{r} = \vec{R}_2 - \vec{R}_1$ is the position vector of the accretor with respect to the donor, and $\vec{v}_{\delta M_i} = \vec{W}_{\delta M_i} - \vec{V}_i$ is the velocity of the ejected/accreted mass element with respect to the mass center of the mass losing/gaining star.

Equation (14) can be written in the form of a perturbed two-body problem as

$$\frac{d^2 \vec{r}}{dt^2} = -\frac{G(M_1 + M_2)}{|\vec{r}|^3} \vec{r} + S \hat{x} + T \hat{y} + W \hat{z}, \quad (15)$$

where \hat{x} is a unit vector in the direction of \vec{r} , \hat{y} is a unit vector in the orbital plane perpendicular to \vec{r} in the direction of the orbital motion, and \hat{z} is a unit vector perpendicular to the orbital plane parallel to and in the same direction as $\vec{\Omega}_{\text{orb}}$. The functions S , T , and W are found by taking the dot product of the perturbing force arising from the mass transfer between the binary components and the unit vector in the \hat{x} , \hat{y} , and \hat{z} directions, respectively. These vector components are

$$S = \frac{f_{2,x}}{M_2} - \frac{f_{1,x}}{M_1} + \frac{\dot{M}_2}{M_2} \left(v_{\delta M_2,x} - |\vec{\Omega}_{\text{orb}}| |\vec{r}_{A_2}| \sin \phi \right) - \frac{\dot{M}_1}{M_1} v_{\delta M_1,x} + \frac{\ddot{M}_2}{M_2} |\vec{r}_{A_2}| \cos \phi - \frac{\ddot{M}_1}{M_1} |\vec{r}_{A_1}|, \quad (16)$$

$$T = \frac{f_{2,y}}{M_2} - \frac{f_{1,y}}{M_1} + \frac{\dot{M}_2}{M_2} \left(v_{\delta M_2,y} + |\vec{\Omega}_{\text{orb}}| |\vec{r}_{A_2}| \cos \phi \right) - \frac{\dot{M}_1}{M_1} \left(v_{\delta M_1,y} + |\vec{\Omega}_{\text{orb}}| |\vec{r}_{A_1}| \right) + \frac{\ddot{M}_2}{M_2} |\vec{r}_{A_2}| \sin \phi, \quad (17)$$

$$W = \frac{f_{2,z}}{M_2} - \frac{f_{1,z}}{M_1}, \quad (18)$$

where ϕ is the angle between \hat{x} and the vector from the center of mass of the accretor to the mass accretion point A_2 , and the subscripts x , y , and z denote vector components in the \hat{x} , \hat{y} , and \hat{z} directions, respectively. In

working out the vector products $\vec{\Omega}_{\text{orb}} \times \vec{r}_{A_2}$, we assumed that A_2 is located on the equator of the accreting star². The terms contributing to the perturbed orbital motion can be categorized as follows: (i) term proportional to \vec{f}_i represent gravitational perturbation on the binary components caused by mass elements in the mass-transfer stream; (ii) terms proportional to \dot{M}_i represent linear momentum exchange between the mass donor and accretor; and (iii) terms proportional to \ddot{M}_i represent shifts in the position of the mass centers of the mass donor and accretor due to the non-spherical symmetry of the mass loss or gain. In the limiting case where both stars are treated as point masses ($|\vec{r}_{A_1}| \rightarrow 0$ and $|\vec{r}_{A_2}| \rightarrow 0$), the only non-zero terms in the perturbed equations of motion are those due to gravitational perturbations of the mass transfer stream and the transport of linear momentum.

3.3. Comparison with Previous Work

The most recent study on the orbital evolution of eccentric mass-transferring binaries has been presented by Matese & Whitmire (1983, 1984, hereafter MW83 and MW84, respectively). These authors extended the work of Huang (1956), Kruszewski (1964), and Piotrowski (1964) by accounting for the effects of linear momentum transport between the binary components, as well as possible perturbations to the orbital motion caused by the mass transfer stream. However, they also derived the equations describing the motion of the binary components with respect to a frame of reference with origin at the mass center of the binary, which, for mass transferring systems, is not an inertial frame of reference. The equations therefore do not account for the accelerations of the binary mass center caused by the mass transfer. Here, we demonstrate that if the procedure adopted by Matese & Whitmire is developed with respect to an inertial frame of reference that is not connected to the binary, the resulting equations are in agreement with those derived in §3.2.

The core of Matese & Whitmire's derivation is presented in Section II of MW83. While the authors choose to adopt a reference frame with origin at the binary mass center early on in the investigation, the choice of the frame does not affect the derivation of the equations of motion up to and including their equation (24). In particular, equation (13) in MW83, which, in our notation, reads

$$\vec{p}_i = M_i \dot{\vec{R}}_i - \dot{M}_i \vec{r}_{A_i}, \quad (19)$$

is valid with respect to any inertial frame of reference with arbitrary position and orientation in space. The same applies to equation (1) in MW84:

$$\dot{\vec{p}}_i = -GM_1 M_2 \frac{\vec{R}_i - \vec{R}_{3-i}}{|\vec{R}_i - \vec{R}_{3-i}|^3} + \vec{f}_i + \vec{\Psi}_i. \quad (20)$$

In these equations, \vec{p}_i is the linear momentum of star i , and $\vec{\Psi}_i = \dot{M}_i(\vec{R}_i + \vec{v}_{\delta M_i})$ is the amount of linear momentum transported by the transferred mass per unit

time (see equation (3) of MW84). Substitution of equation (19) into equation (20) then yields

$$M_i \ddot{\vec{R}}_i = -GM_{3-i} \frac{\vec{R}_i - \vec{R}_{3-i}}{|\vec{R}_i - \vec{R}_{3-i}|^3} + \frac{\vec{f}_i}{M_i} + \frac{\dot{M}_i}{M_i} (\vec{v}_{\delta M_i} + \dot{\vec{r}}_{A_i}) + \frac{\ddot{M}_i}{M_i} \vec{R}_{A_i}, \quad (21)$$

and thus

$$\begin{aligned} \frac{d^2 \vec{r}}{dt^2} = & -\frac{G(M_1 + M_2)}{|\vec{r}|^3} \vec{r} + \frac{\vec{f}_2}{M_2} - \frac{\vec{f}_1}{M_1} \\ & + \frac{\dot{M}_2}{M_2} (\vec{v}_{\delta M_2} + \dot{\vec{r}}_{A_2}) - \frac{\dot{M}_1}{M_1} (\vec{v}_{\delta M_1} + \dot{\vec{r}}_{A_1}) \\ & + \frac{\ddot{M}_2}{M_2} \vec{r}_{A_2} - \frac{\ddot{M}_1}{M_1} \vec{r}_{A_1}, \end{aligned} \quad (22)$$

where $\vec{r} = \vec{R}_2 - \vec{R}_1$. Setting $\dot{\vec{r}}_{A_i} = \vec{\Omega}_{\text{orb}} \times \vec{r}_{A_i}$, this equation is in perfect agreement with equation (14) derived in §3.2.

In MW83 and MW84, the authors incorrectly set $\vec{R}_1 = -M_2 \vec{r}/(M_1 + M_2)$ and $\vec{R}_2 = M_1 \vec{r}/(M_1 + M_2)$ in equation (21), which is valid only when the origin of the frame of reference coincide with the mass center of the binary. For a mass-transferring binary, such a frame is, however, not an inertial frame and can therefore not be used for the derivation of the equations of motion of the binary components. Instead of equation (22), Matese & Whitmire therefore find equations (7)–(8) in MW84, which lack the terms associated with the acceleration of the binary mass center due to the mass transfer.

4. ORBITAL EVOLUTION EQUATIONS

4.1. Secular Variation of the Orbital Elements

In the classical framework of the theory of osculating elements, the equations governing the rate of change of the orbital semi-major axis a and eccentricity e due to mass transfer are obtained from the perturbing functions S and T as (see, e.g., Sterne 1960; Brouwer & Clemence 1961; Danby 1962; Fitzpatrick 1970)

$$\frac{da}{dt} = \frac{2}{n(1-e^2)^{1/2}} [Se \sin \nu + T(1 + e \cos \nu)], \quad (23)$$

$$\begin{aligned} \frac{de}{dt} = & \frac{(1-e^2)^{1/2}}{na} \\ & \times \left\{ S \sin \nu + T \left[\frac{2 \cos \nu + e(1 + \cos^2 \nu)}{1 + e \cos \nu} \right] \right\}, \end{aligned} \quad (24)$$

where $n = 2\pi/P_{\text{orb}}$ is the mean motion and ν the true anomaly. These equations are independent of the perturbing function W which solely appears in the equations governing the rates of change of the orbital inclination, the longitude of the ascending node, and the longitude of the periastron.

After substitution of equations (16) and (17) for S and T into equations (23) and (24), the equations governing the rates of change of the semi-major axis and eccentricity contain periodic as well as secular terms. Here we are mainly interested in the long-term secular evolution

² For brevity, we refer to the point A_2 as lying on the stellar surface. Though, in practice, it can lie at any point near the star where the transferred mass can be considered to be part of the accretor. For instance, if an accretion disk has formed around the accretor, it would be equally valid to write A_2 as the point where the transferred mass impacts the outer edge of the accretion disk.

of the orbit, and so we remove the periodic terms by averaging the equations over one orbital period:

$$\left\langle \frac{da}{dt} \right\rangle_{\text{sec}} \equiv \frac{1}{P_{\text{orb}}} \int_{-P_{\text{orb}}/2}^{P_{\text{orb}}/2} \frac{da}{dt} dt, \quad (25)$$

$$\left\langle \frac{de}{dt} \right\rangle_{\text{sec}} \equiv \frac{1}{P_{\text{orb}}} \int_{-P_{\text{orb}}/2}^{P_{\text{orb}}/2} \frac{de}{dt} dt. \quad (26)$$

The integrals in these definitions are most conveniently computed in terms of the true anomaly, ν . We therefore make a change of variables using

$$dt = \frac{(1 - e^2)^{3/2}}{n(1 + e \cos \nu)^2} d\nu. \quad (27)$$

For binaries with eccentric orbits, the resulting integrals can be calculated analytically only for very specific functional prescriptions of the mass-transfer rate \dot{M}_1 (e.g., when \dot{M}_1 is approximated by a Dirac delta function centered on the periastron, see § 5). In general, the integrals must be computed numerically.

4.2. Conservation of Orbital Angular Momentum

Since the perturbing functions S and T depend on the properties of the mass transfer stream, calculation of the rates of secular change of the orbital semi-major axis and eccentricity, in principle, requires the calculation of the trajectories of the particles in the stream (cf. Hadjidemetriou 1969a). As long as no mass is lost from the system, such a calculation automatically incorporates the conservation of total angular momentum in the system. Special cases of angular momentum conservation can, however, be used to bypass the calculation of detailed particle trajectories. Here, we adopt such a special case and assume that any orbital angular momentum carried by the particles in the mass-transfer stream is always immediately returned to the orbit, so that the orbital angular momentum of the binary is conserved.

The orbital angular momentum of a binary with a semi-major axis a and eccentricity e is given by

$$J_{\text{orb}} = M_1 M_2 \left[\frac{Ga(1 - e^2)}{M_1 + M_2} \right]^{1/2}, \quad (28)$$

so that

$$\frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} = \frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} - \frac{1}{2} \frac{\dot{M}_1 + \dot{M}_2}{M_1 + M_2} + \frac{1}{2} \frac{\dot{a}}{a} - \frac{e \dot{e}}{1 - e^2}, \quad (29)$$

where a dot indicates the time derivative.

In the case of eccentric orbits, substitution of equations (23) and (24) into equation (29) leads to

$$\begin{aligned} \frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} &= \frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} - \frac{1}{2} \frac{\dot{M}_1 + \dot{M}_2}{M_1 + M_2} \\ &+ \frac{(1 - e^2)^{1/2}}{n a (1 + e \cos \nu)} T. \end{aligned} \quad (30)$$

As we shall see in the next section, by setting $\dot{M}_1 + \dot{M}_2 = 0$ and $\langle \dot{J}_{\text{orb}}/J_{\text{orb}} \rangle_{\text{sec}} = 0$ and substituting equation (17) for T , equation (30) allows us to calculate the \hat{y} -component of the final velocities of the accreting particles as a function of their initial velocities without resorting to the computation of the ballistic trajectories of the mass transfer stream.

In the limiting case of a circular orbit, equation (29) is usually used to derive the rate of change of the semi-major axis of circular binaries under the assumption of conservation of both total mass ($\dot{M}_1 = -\dot{M}_2$) and orbital angular momentum ($\dot{J}_{\text{orb}} = 0$):

$$\frac{da}{dt} = 2a \left(\frac{M_1}{M_2} - 1 \right) \frac{\dot{M}_1}{M_1}. \quad (31)$$

The assumption of orbital angular momentum conservation over secular timescales ($\langle \dot{J}_{\text{orb}}/J_{\text{orb}} \rangle_{\text{sec}} = 0$) is a standard assumption in nearly all investigations of conservative mass transfer in binary systems (e.g., Soberman et al. 1997; Pribulla 1998), which is valid over long timescales provided there is no significant storage of angular momentum in the spins of the component stars, the accretion flow, and/or the accretion disk. In future work, we will investigate the consequences of both mass and orbital angular momentum losses from the binary on the evolution of the orbital elements.

5. ORBITAL EVOLUTION TIMESCALES

In order to assess the timescales of orbital evolution due to mass transfer in eccentric binaries, we observe that, for eccentric binaries, mass transfer is expected to occur first at the periastron of the relative orbit, where the component stars are closest to each other. We therefore explore the order of magnitude of the timescales assuming a delta function mass transfer profile centered at the periastron of the binary orbit

$$\dot{M}_1 = \dot{M}_0 \delta(\nu), \quad (32)$$

where $\dot{M}_0 < 0$ is the instantaneous mass transfer rate, and $\delta(\nu)$ is the Dirac delta function.

We calculate the rates of secular change of the orbital semi-major axis and eccentricity from equations (23)–(27), and neglect any gravitational attractions exerted by the particles in the mass-transfer stream on the component stars. Hence, we set

$$f_{1,x} = f_{2,x} = 0, \quad (33)$$

$$f_{1,y} = f_{2,y} = 0. \quad (34)$$

Substituting equations (17), (27), and (32)–(34) into equation (30) for $\langle \dot{J}_{\text{orb}}/J_{\text{orb}} \rangle_{\text{sec}} = 0$ then yields a relationship between the initial and final \hat{y} -component of the velocities of the transferred mass and the initial and final positions of the transferred mass given by

$$\begin{aligned} qv_{\delta M_2, y} + v_{\delta M_1, y} &= na(1 - q) \left(\frac{1 + e}{1 - e} \right)^{1/2} - |\vec{\Omega}_{\text{orb}, P}| |\vec{r}_{A_1, P}| \\ &- q |\vec{\Omega}_{\text{orb}, P}| |\vec{r}_{A_2}| \cos \phi_P \left(1 - \left. \frac{d\phi}{d\nu} \right|_{\nu=0} \right), \end{aligned} \quad (35)$$

where the subscript P indicates quantities evaluated at the periastron of the binary orbit, $q = M_1/M_2$ is the binary mass ration, and we have used the relation $d\delta(\nu)/d\nu = -\delta(\nu)/\nu$. Assuming the transferred mass elements are ejected by star 1 at the L_1 point with a velocity $\vec{v}_{\delta M_1}$ equal to the star's rotational velocity at L_1 , we write

$$v_{\delta M_1, x} = 0, \quad (36)$$

$$v_{\delta M_1, y} = -|\vec{\Omega}_{\text{orb}, P}| |\vec{r}_{A_1, P}|. \quad (37)$$

Moreover, under the assumption that each periastron passage of the binary components give rise to an extremum of $\phi(\nu)$, the derivative $d\phi/d\nu|_{\nu=0}$ is equal to zero in equation (35), so that

$$v_{\delta M_2, y} = |\vec{\Omega}_{\text{orb}, P}| \left[|\vec{r}_{A_1, P}| \frac{(1-q)}{q} + |\vec{r}_{A_2}| \cos \phi_P \right]. \quad (38)$$

For a binary with orbital period $P_{\text{orb}} = 1$ day, eccentricity $e = 0.2$, and component masses $M_1 = 2 M_{\odot}$ and $M_2 = 1.44 M_{\odot}$, with $|\vec{r}_{A_1, P}|$ the distance from star 1 to L_1 (See Appendix A), and $|\vec{r}_{A_2}| \cos \phi_P \approx 2.9 \times 10^5$ km (the circularization radius around a compact object for these binary parameters; see Frank, King, & Raine (2002)), the accreting matter has a \hat{y} -velocity component of the order of $\sim -45 \text{ km s}^{-1}$.

After substitution of equation (27) and equations (33)–(35), the integrals in equations (25) and (26) for the rates of secular change of the orbital semi-major axis and eccentricity can be solved analytically to obtain

$$\left\langle \frac{da}{dt} \right\rangle_{\text{sec}} = \frac{a \dot{M}_0}{\pi M_1 (1-e^2)^{1/2}} \left[q e \frac{|\vec{r}_{A_2}|}{a} \cos \phi_P + e \frac{|\vec{r}_{A_1, P}|}{a} + (q-1)(1-e^2) \right], \quad (39)$$

$$\left\langle \frac{de}{dt} \right\rangle_{\text{sec}} = \frac{(1-e^2)^{1/2} \dot{M}_0}{2\pi M_1} \left[q \frac{|\vec{r}_{A_2}|}{a} \cos \phi_P + \frac{|\vec{r}_{A_1, P}|}{a} + 2(1-e)(q-1) \right]. \quad (40)$$

We note that for a delta-function mass transfer rate given by equation (32) the \hat{x} -component of the velocities $\vec{v}_{\delta M_1}$ and $\vec{v}_{\delta M_2}$ does not enter into the derivation of these equations due to the $\sin \nu$ term in equations (23) and (24). Furthermore, in the limiting case of a circular orbit, equation (39) reduces to equation (31), provided that \dot{M}_1 in that equation is interpreted as the secular mean mass transfer rate $\langle \dot{M}_1 \rangle_{\text{sec}} = \dot{M}_0 / (2\pi)$. In Appendix B, we present an alternative derivations to equations (39) and (40) in the limiting case where the stars are treated as point masses.

The rates of secular change of the semi-major axis and orbital eccentricity are thus linearly proportional to the magnitude of the mass transfer rate at periastron. Besides the obvious dependencies on a , e , q , and M_1 , the rates also depend on the ratio of the donor's rotational angular velocity Ω_1 to the orbital angular velocity $\Omega_{\text{orb}, P}$ at periastron through the position of the L_1 point, \vec{r}_{A_1} . A fitting formula for the position of the L_1 point accurate to better than 4% over a wide range of q , e , and $\Omega_1/\Omega_{\text{orb}, P}$ is given by equation (A15) in Appendix A. While the fitting formula can be used to obtain fully analytical rates of secular change of the semi-major axis and eccentricity, we here use the exact solutions for the position of L_1 obtained by numerically solving equation (A13) in Appendix A. For a detailed discussion of the properties of the L_1 point in eccentric binaries, we refer the interested reader to Sepinsky et al. (2007).

To explore the effects of mass transfer on the orbital elements of eccentric binaries, we calculate the rates of secular change of the semi-major axis and eccentricity and determine the characteristic timescales $\tau_a = a/\dot{a}$ and

$\tau_e = e/\dot{e}$. While the actual timescales are given by the absolute values of τ_a and τ_e , we here allow the timescales to be negative as well as positive in order to distinguish negative from positive rates of secular change of the orbital elements. We also note that since $|\vec{r}_{A_1, P}| \propto a$ (see Appendix A), the timescales do not explicitly depend on the orbital semi-major axis a except through the ratio $|\vec{r}_{A_2}|/a$ of the radius of the accretor to the semi-major axis. For convenience, we therefore assume the accretor to be a compact object with radius $|\vec{r}_{A_2}| \ll a$. The timescales are found to be insensitive to terms containing $|\vec{r}_{A_2}|/a$ in equation (39) and (40). Varying $|\vec{r}_{A_2}|$ from 0 to $0.01a$ changes the timescales by less than 10%. In what follows, we therefore set $|\vec{r}_{A_2}| = 0$. An implicit dependence on a may then still occur through the amplitude \dot{M}_0 of the mass transfer rate at periastron. Since incorporating such a dependence in the analysis requires detailed modeling of the evolution of the donor star, which is beyond the scope of this investigation, we here restrict ourselves to exploring the timescales of orbital evolution for a constant \dot{M}_0 . The linear dependence of $\langle \dot{a} \rangle_{\text{sec}}$ and $\langle \dot{e} \rangle_{\text{sec}}$ on \dot{M}_0 in any case allows for any easy rescaling of our results to different mass transfer rates.

In Fig. 2, we show the variations of τ_a and τ_e as functions of q for $\dot{M}_0 = -10^{-9} M_{\odot} \text{ yr}^{-1}$ and $e = 0.0, 0.1, \dots, 0.9$. In all cases, the donor is assumed to rotate synchronously with the orbital angular velocity at the periastron, and the accretor is assumed to be a neutron star of mass $M_2 = 1.44 M_{\odot}$. The timescales of the secular evolution of the semi-major axis show a strong dependence on q , and a milder dependence on e , unless $e \gtrsim 0.7$. The timescales for the secular evolution of the orbital eccentricity always depend strongly on both q and e . These timescales can furthermore be positive as well as negative, so that the semi-major axis and eccentricity can increase as well as decrease under the influence of mass transfer at the periastron of the binary orbit.

From Fig. 2, as well as equations (39) and (40), it can be seen that, for a given ratio of the donor's rotational angular velocity to the orbital angular velocity at periastron, the line dividing positive from negative rates of secular change of the orbital elements is a function of q and e . This is illustrated further in Fig. 3 where the timescales of orbital evolution are displayed as contour plots in the (q, e) -plane. The thick black line near the center of the plots marks the transition values of q and e where the rates of secular change of a and e transition from being positive (to the left of the thick black line) to negative (to the right of the thick black line). Varying $\Omega_1/\Omega_{\text{orb}, P}$ between 0.5 and 1.5 changes the position of the transition line by less than 10% in comparison to the $\Omega_1/\Omega_{\text{orb}, P} = 1$ case displayed in Figures 2 and 3.

In the limiting case of a circular orbit, the orbit expands when $q < 1$ and shrinks when $q > 1$, in agreement with the classical result obtained from equation (31). For non-zero eccentricities, the critical mass ratio separating positive from negative values of $\langle \dot{a} \rangle_{\text{sec}}$ decreases with increasing orbital eccentricities. This behavior can be understood by substituting the fitting formula for the position of the L_1 point given by equation (A15) in Appendix A into equation (39) and setting $\langle \dot{a} \rangle_{\text{sec}} = 0$. However, we can fit the critical mass ratio separating expand-

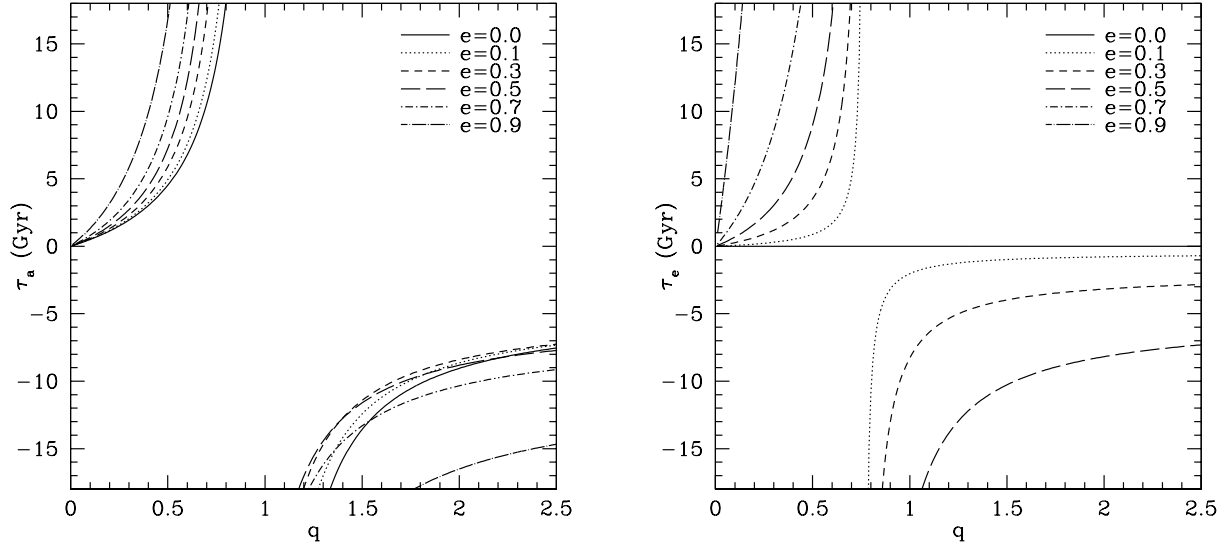


FIG. 2.— Orbital evolution timescales for a delta function mass transfer profile centered at the periastron of the binary orbit with an instantaneous mass transfer rate of $\dot{M}_0 = -10^{-9} M_\odot \text{yr}^{-1}$. The timescales are calculated under the assumption that the donor rotates synchronously with the orbital angular velocity at periastron, and that the accretor is a $1.44 M_\odot$ neutron star. Shown at left (right) are the timescales for the evolution of the semi-major axis (eccentricity) as a function of the mass ratio, q , for a range of eccentricities, e . Regimes where the timescale is negative correspond to a decrease of the semi-major axis (eccentricity), while regimes where the timescale is positive correspond to an increase of the semi-major axis (eccentricity).

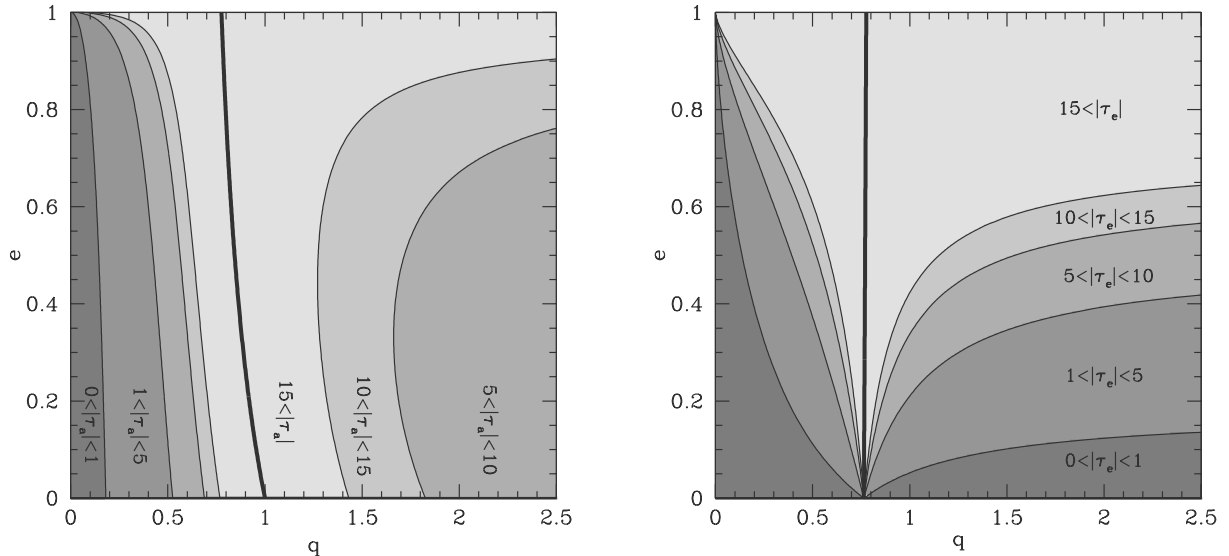


FIG. 3.— Contour plot of the orbital evolution timescales for the semi-major axis and eccentricity in the (q, e) -plane for the same set of assumptions as adopted in figure 2. Timescales for the evolution of the semi-major axis are shown on the left; timescales for the evolution of the orbital eccentricity on the right. The different shades of gray designate regions of the (q, e) parameter space with timescales (in Gyr) in the ranges labeled in the plots. The thick black line near the center of each plot designates the transition point where the rate of change of the semi-major axis or orbital eccentricity changes from positive (to the left of the thick black line) to negative (to the right of the thick black line).

ing from shrinking orbits with a simpler formula given by

$$q_{\text{crit}} \simeq 1 - 0.4e + 0.18e^2. \quad (41)$$

The critical mass ratio separating positive from negative values of $\langle \dot{e} \rangle_{\text{sec}}$ is largely independent of e . Proceeding in a similar fashion as for the derivation of equation (41), we derive the critical mass ratio separating increasing from decreasing eccentricities to be approximately given by

$$q_{\text{crit}} \simeq 0.76 + 0.012e. \quad (42)$$

Last, we note that a more quantitative numerical comparison between the above approximation formulae for q_{crit} and the exact numerical solutions shows that equations (41) and (42) are accurate to better than 1%.

6. TIDAL EVOLUTION TIMESCALES

A crucial question for assessing the relevance of the work presented here is how the derived orbital evolution timescales compare to the corresponding timescales associated with other orbital evolution mechanisms such as tides. In Fig. 4, we show the secular evolution timescales

of the semi-major axis and orbital eccentricity of a mass-transferring binary due to tidal dissipation in the donor star as a function of q , for different values of the eccentricity, e . The timescales are strong functions of $|\vec{r}_{A_1}|/a$ and are determined as in Hurley et al. (2002)³ (see also Zahn 1977, 1978; Hut 1981). The radius $|\vec{r}_{A_1}|$ is determined by assuming the donor is on the zero-age main sequence and that the orbital separation is then obtained by equating the radius of the donor (given by Tout et al. 1996) to the volume-equivalent radius of its Roche lobe at the periastron of the binary orbit (see Sepinsky et al. 2007). As before, we assume the donor rotates synchronously with the orbital motion at periastron and that the accretor is a $1.44 M_\odot$ neutron star.

The timescales of orbital evolution due to tides range from a few Myr to more than a Hubble time, depending on the binary mass ratio and the orbital eccentricity. The discontinuity in the timescales at $q \simeq 0.87$ corresponds to the transition from donor stars with convective envelopes ($M_1 \lesssim 1.25 M_\odot$) to donor stars with radiative envelopes ($M_1 \gtrsim 1.25 M_\odot$) which are subject to different tidal dissipation mechanisms. It follows that tides do not necessarily lead to rapid circularization during the early stages of mass transfer, especially for orbital eccentricities $e \gtrsim 0.3$. Furthermore, for the adopted system parameters, the orbital eccentricity always decreases, while the orbital semi-major axis can either increase or decrease. Hence, in some regions of the parameter space, the effects of tides and mass transfer are additive, while in other regions they are competitive. This is illustrated in more detail in figure 5 where we show the orbital evolution timescales due to the combined effect of tides and mass transfer. In the calculations of the timescales, we have assumed that, at the lowest order of approximation, the effects of tides and mass transfer are decoupled. The total rate of change of the orbital elements is then given by the sum of the rate of change of the orbital elements due to tides and mass transfer.

When $q \lesssim 0.87$ and $e \gtrsim 0.4$, the effects of tides and mass transfer on the orbital semi-major axis are always opposed, with the orbital expansion due to mass transfer dominating the orbital shrinkage due to tides. In the case of the orbital eccentricity, the increase of the eccentricity due to mass transfer dominates the decrease due to tides for mass ratios smaller than some critical mass ratio which depends strongly on the orbital eccentricity. Since the timescales of orbital evolution due to mass transfer are inversely proportional to the magnitude \dot{M}_0 of the mass-transfer rate at periastron, the parameter space where and the extent to which mass transfer dominates increases with the rate of mass transfer at periastron.

In figure 6, we show the total orbital evolution timescale due to the sum of tidal and mass transfer effects as a contour plot in the (q, e) -plane. The thick black lines indicate the transitions from positive (left of the thick black line) to negative (right of the thick

black line) rates of change of the semi-major axis and eccentricity. The white dividing line near $q \approx 0.87$ corresponds to the transition between tidal dissipation mechanisms in stars with convective envelopes ($M_1 \lesssim 1.25 M_\odot$) and stars with radiative envelopes ($M_1 \gtrsim 1.25 M_\odot$). It follows that there are large regions of parameter space where the combined effects of mass transfer and tidal evolution do not rapidly circularize the orbit. In particular, for $q > 0.87$ and $e \gtrsim 0.75$ orbital circularization always takes longer than 10 Gyr, while for q to the left of the thick black line the orbital eccentricity grows rather than shrinks. For a given q left of the thick black line, the timescales for eccentricity growth increase with increasing e though, so that there is no runaway eccentricity growth. Hence, for small q , mass transfer at the periastron of eccentric orbits may provide a means for inducing non-negligible eccentricities in low-mass binary or planetary systems. The orbital semi-major axis, on the other hand, always increases when $e \lesssim 0.55$, but can increase as well as decrease when $e \gtrsim 0.55$, depending on the binary mass ratio q . We recall that both the tidal and mass transfer orbital evolution time scales depend on the ratio of the donor's rotational angular velocity Ω_1 to the orbital angular velocity $\Omega_{\text{orb},P}$ and that we have set $\Omega_1/\Omega_{\text{orb},P} = 1$ in all figures shown.

7. CONCLUDING REMARKS

We developed a formalism to calculate the evolution of the semi-major axis and orbital eccentricity due to mass transfer in eccentric binaries, assuming conservation of total system mass and orbital angular momentum. Adopting a delta-function mass-transfer profile centered at the periastron of the binary orbit yields rates of secular change of the orbital elements that are linearly proportional to the magnitude \dot{M}_0 of the mass-transfer rate at the periastron. For $\dot{M}_0 = 10^{-9} M_\odot \text{yr}^{-1}$, this yields timescales of orbital evolution ranging from a few Myr to a Hubble time or longer. Depending on the initial binary mass ratio and orbital eccentricity, the rates of secular change of the orbital semi-major axis and eccentricity can be positive as well as negative, so these orbital elements can increase as well as decrease with time.

Comparison of the timescales of orbital evolution due to mass transfer with the timescales of orbital evolution due to tidal dissipation shows that the effects can either be additive or competitive, depending on the binary mass ratio, the orbital eccentricity, and the magnitude of the mass-transfer rate at the periastron. Contrary to what is often assumed in even the most state-of-the-art binary evolution and population synthesis codes, tides do not always lead to rapid circularization during the early stages of mass transfer. Thus, phases of episodic mass transfer may occur at successive periastron passages and may persist for long periods of time. As a first approximation, the evolution of the orbital semi-major axis and eccentricity due to mass transfer in eccentric binaries can be incorporated into binary evolution and population synthesis codes by means of equations (39) and (40) in which the mass-transfer rate is approximated by a delta-function of amplitude \dot{M}_0 centered at the periastron of the binary orbit.

In future papers, we will relax the assumption of conservation of total system mass and orbital angular mo-

³ Note that there is a typo in equation (42) of Hurley et al. (2002). The correct equation for k/T for stars with radiative envelopes is (J. Hurley, Private Communication)

$$(k/T)_r = 1.9782 \times 10^4 (MR^2/a^5)^{1/2} (1 + q_2)^{5/6} E_2 \text{yr}^{-1}.$$

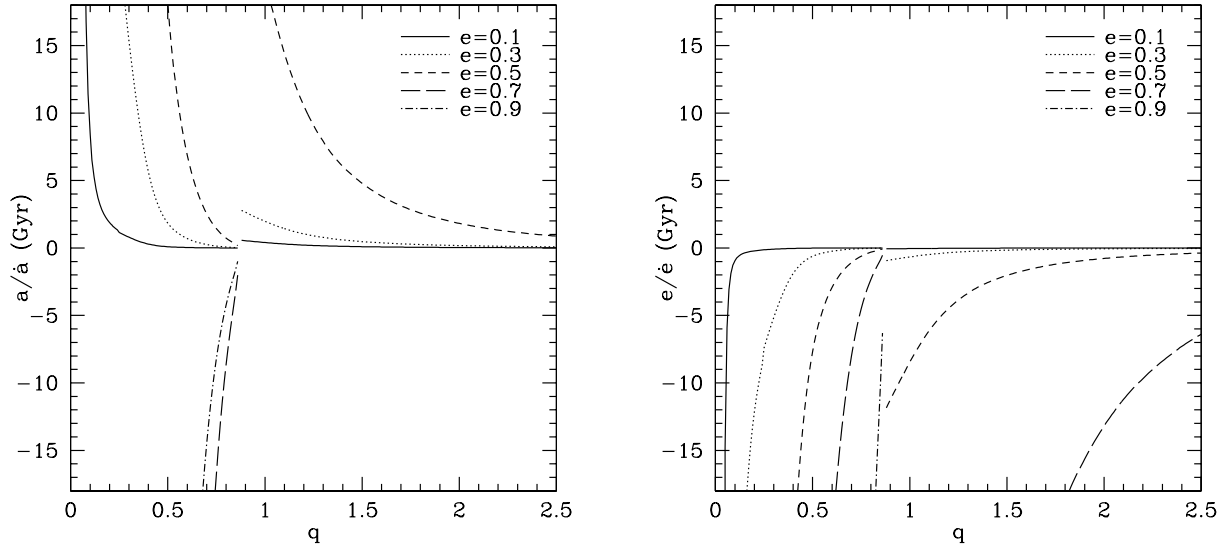


FIG. 4.— Timescales of orbital evolution due to tidal dissipation in a Roche-lobe filling component of a close binary under the assumption that the donor is a zero-age main-sequence star rotating synchronously with the orbital angular velocity at the periastron, and the accretor is a $1.44 M_{\odot}$ neutron star. Shown at left (right) are the timescales for the evolution of the semi-major axis (eccentricity) as a function of the mass ratio, q , for different orbital eccentricities e . Regimes where the timescales are negative correspond to a decrease of the semi-major axis (eccentricity), while regimes where the timescales are positive correspond to an increase of the semi-major axis (eccentricity). The discontinuity at $q \simeq 0.87$ corresponds to the transition from donor stars with convective envelopes ($M_1 \lesssim 1.25 M_{\odot}$) to donor stars with radiative envelopes ($M_1 \gtrsim 1.25 M_{\odot}$).

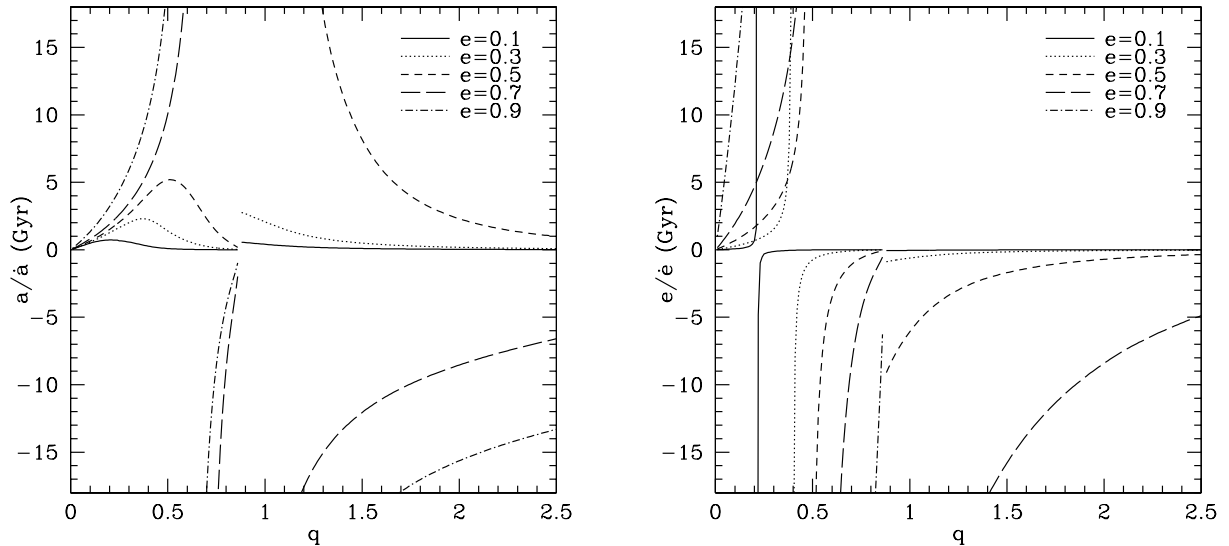


FIG. 5.— Orbital evolution timescales due to the combined effects of tidal dissipation in a Roche Lobe filling component of a close binary system and a delta function mass transfer with an amplitude $\dot{M}_0 = -10^{-9} M_{\odot} \text{ yr}^{-2}$ centered at the periastron of the orbit under the assumption that the donor is rotating synchronously with the orbital angular velocity at periastron, and the accretor is a $1.44 M_{\odot}$ neutron star. The contribution to the orbital evolution timescales due to tides is determined under the assumption that the donor is zero-age main-sequence star. Shown at left (right) are the timescales for the evolution of the semi-major axis (eccentricity) as a function of the mass ratio, q , for a range of eccentricities, e . Regimes where the timescale is negative correspond to a decrease in the semi-major axis (eccentricity), while regimes where the timescale is positive correspond to an increase in the semi-major axis (eccentricity). The discontinuity at $q \simeq 0.87$ corresponds to a transition of the dominant tidal dissipation mechanism which is different for donor stars with convective envelopes ($M_1 \lesssim 1.25 M_{\odot}$) than for donor stars with radiative envelopes ($M_1 \gtrsim 1.25 M_{\odot}$).

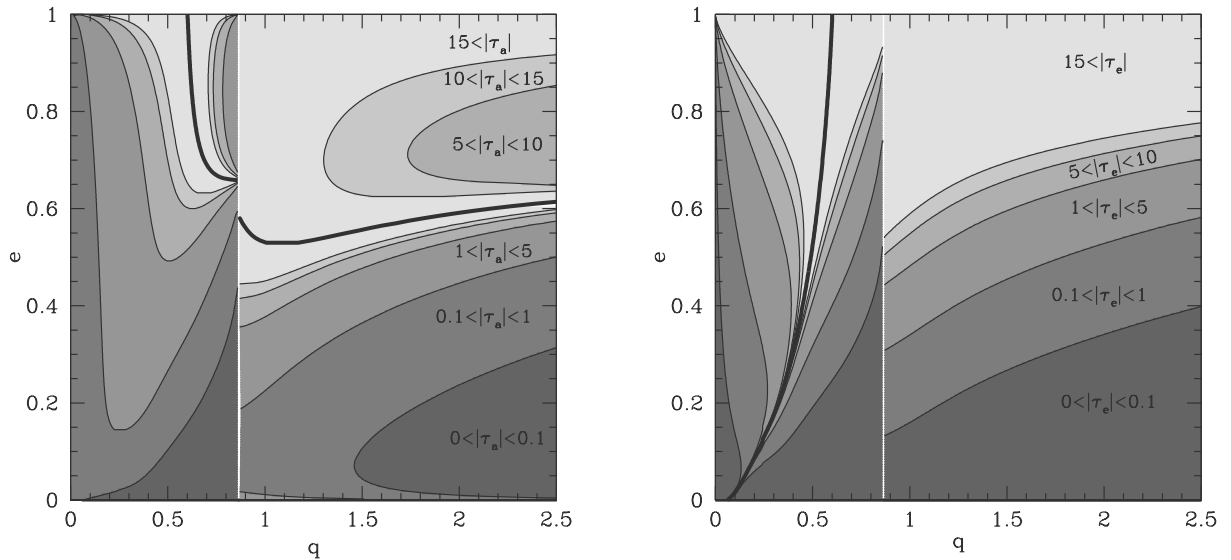


FIG. 6.— Contour plots of the total orbital evolution timescales in the (q, e) -plane due to the combined effects of tidal dissipation and mass transfer for the same set of assumptions adopted in figure 5. Timescales for the evolution of the semi-major axis are shown at left; timescales for the evolution of the orbital eccentricity are shown at right. The different shades of gray designate regions of the (q, e) parameter space with timescales (in Gyr) in the ranges labeled in the plots. The thick black line designates the transition point where the rate of change of the semi-major axis or orbital eccentricity changes from positive (to the left of the thick black line) to negative (to the right of the thick black line). The vertical white line at $q \simeq 0.87$ corresponds to a transition of the dominant tidal dissipation mechanism which is different for donor stars with convective envelopes ($M_1 \lesssim 1.25 M_\odot$) than for donor stars with radiative envelopes ($M_1 \gtrsim 1.25 M_\odot$).

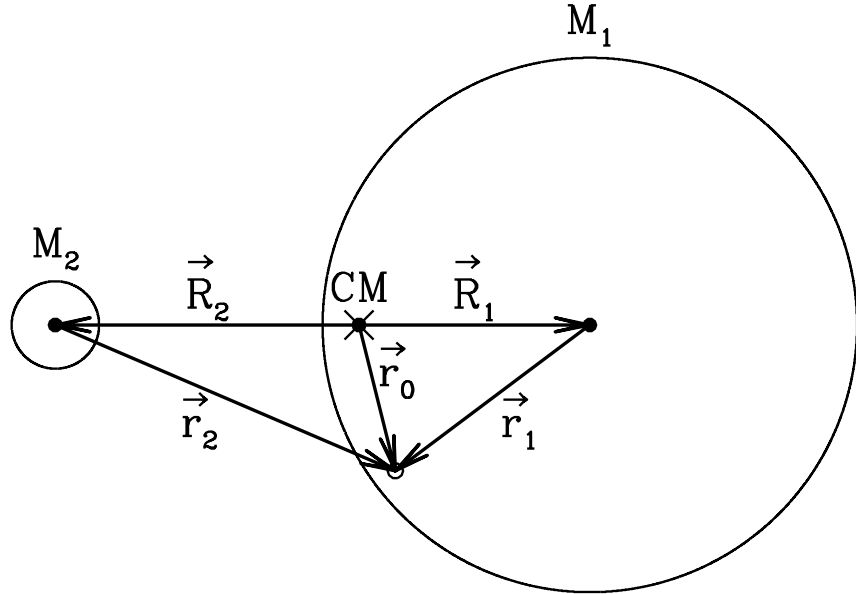


FIG. 7.— Schematic diagram showing the vectors pertinent to the derivation of the equipotential surfaces for a non-synchronous, eccentric binary. The center of mass of the system is shown as a cross near the center of the diagram and the center of mass of each star is shown as a filled circle at the star’s center. The vectors \vec{R}_1 and \vec{R}_2 begin at the center of mass of the system.

mentum, and examine the effects of non-conservative mass transfer on the orbital elements of eccentric binaries. We also intend to study the onset of mass transfer in eccentric binaries in more detail, adopting realistic mass-transfer rates appropriate for atmospheric Roche-lobe

overflow in interacting binaries as discussed by Ritter (1988). We will consider individual binary systems that are known to be eccentric and transferring mass during periastron passage, as well as populations of eccentric mass-transferring binaries and their descendants.

APPENDIX

EQUIPOTENTIAL SURFACES AND THE INNER LAGRANGIAN POINT IN ECCENTRIC BINARIES

A crucial element in the description of mass transfer in any binary system is the location of the inner Lagrangian point (L_1) through which matter flows from the donor to the accretor. While a solution for the location of L_1 is not analytic, the case of circular orbits with synchronized components can be approximated by the formula

$$X_{L_1} = 0.5 + 0.22 \log q, \quad (\text{A1})$$

where X_{L_1} is the distance of the L_1 point from the mass center of the donor star in units of the distance between the stars, and $q = M_1/M_2$ is the mass ratio of the binary defined as the ratio of the donor mass M_1 to the accretor mass M_2 . For mass ratios in the range $0.1 < q < 15$, this formula is valid to an accuracy of better than 2% (e.g. Drobyshevski & Reznikov 1974). The above formula has been generalized by Pratt & Strittmatter (1976) to include

the effect of a non-synchronously rotating donor star:

$$X_{L_1} = \left[0.53 - 0.03 \left(\frac{\Omega_1}{\Omega_{\text{orb}}} \right)^2 \right] \left(1 + \frac{4}{9} \log q \right). \quad (\text{A2})$$

Here, Ω_1 is the rotational angular velocity of the donor, and Ω_{orb} the orbital angular velocity of the binary. For $0 < \Omega_1/\Omega_{\text{orb}} < 1$ and $0.2 < q < 10$, the formula is valid to an accuracy of better than 3%. It is to be noted though that Pratt & Strittmatter (1976) derived the position of the L_1 point under the assumption that the orbital and rotational periods of the binary and its component stars are much longer than the dynamical timescale of the donor. Despite the better than 3% accuracy of the fit given by equation (A2), the formula may therefore still break down because the underlying formalism is no longer valid.

In this paper, it is necessary to generalize the formula for the position of the L_1 point even further to account for a non-zero eccentricity as well as non-synchronous rotation. For this purpose, we determine the equipotential surfaces describing the shape of the components of a non-synchronous, eccentric binary. Our procedure is a generalization of the steps outlined by Limber (1963) for the derivation of the equipotentials of a non-synchronous, circular binary.

Here, we consider an eccentric binary system where the stars are considered to be centrally condensed and spherically symmetric, and thus can be well described by Roche models of masses M_1 and M_2 . Their orbit is assumed to be Keplerian with semi-major axis a and eccentricity e . Star 1 is furthermore assumed to rotate uniformly with a constant angular velocity $\vec{\Omega}_1$ parallel to the orbital angular velocity $\vec{\Omega}_{\text{orb}}$. We note that for an unperturbed Keplerian orbit, the magnitude of $\vec{\Omega}_{\text{orb}}$ is a function of time, while the direction and orientation remain fixed in space.

As is customary, we determine the equipotential surfaces with respect to a Cartesian coordinate frame $OXYZ$ with origin O at the mass center of star 1, and with the Z -axis pointing along $\vec{\Omega}_1$. The X - and Y -axes are co-rotating with the star at angular velocity Ω_1 . Since non-synchronously rotating binary components are inevitably subjected to time-dependent tides invoked by their companion, the mass elements will oscillate with frequencies determined by the difference between the rotational and orbital frequencies. Here, we neglect these tidally induced oscillations as well as any other type of bulk motion of matter due to, e.g., convection. This approximation is valid as long as the characteristic timescale associated with the motion of the mass elements is sufficiently long compared to the star's dynamical timescale (more extended discussions on the validity of the approximation can be found in Limber (1963), Savonije (1978), and Sepinsky et al. (2007)). Star 1 is then completely stationary with respect to the co-rotating frame of reference.

With respect to the $OXYZ$ frame, the equation of motion of a mass element at the surface of star 1 is given by

$$\ddot{\vec{r}}_1 = \ddot{\vec{r}}_0 - \ddot{\vec{R}}_1 - \vec{\Omega}_1 \times (\vec{\Omega}_1 \times \vec{r}_1) - 2\vec{\Omega}_1 \times \dot{\vec{r}}_1, \quad (\text{A3})$$

where \vec{r}_0 and \vec{r}_1 are the position vectors of the mass element with respect to the mass center of the binary and star 1, respectively, and \vec{R}_1 is the position vector of the center of mass of star 1 with respect to the center of mass of the binary, and $-\vec{\Omega}_1 \times (\vec{\Omega}_1 \times \vec{r}_1)$ and $-2\vec{\Omega}_1 \times \dot{\vec{r}}_1$ are the centrifugal and Coriolis acceleration with respect to the rotating coordinate frame.

Under the assumption that, in an inertial frame of reference, the only forces acting on the considered mass element are those resulting from the pressure gradients in star 1 and the gravitational attractions of star 1 and its companion, the acceleration of the mass element with respect to the binary's center of mass is given by

$$\ddot{\vec{r}}_0 = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \left(-G \frac{M_1}{|\vec{r}_1|} - G \frac{M_2}{|\vec{r}_2|} \right), \quad (\text{A4})$$

where G is the Newtonian constant of gravitation, ρ the mass density, P the pressure, and \vec{r}_2 the position vector of the mass element with respect to the mass center of star 2. The gradient in equation (A4) and subsequent equations in this Appendix are taken with respect to X , Y , and Z .

The acceleration of the center of mass of star 1 with respect to the binary mass center, on the other hand, is given by

$$\ddot{\vec{R}}_1 = -\frac{GM_2}{D^2} \frac{\vec{R}_1}{|\vec{R}_1|}, \quad (\text{A5})$$

where $D(t)$ (simplified to D in what follows) is the time-dependent distance between star 1 and star 2.

Substituting equations (A4) and (A5) into equation (A3), we obtain

$$\ddot{\vec{r}}_1 = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \left(-G \frac{M_1}{|\vec{r}_1|} - G \frac{M_2}{|\vec{r}_2|} \right) + \frac{GM_2}{D^2} \frac{\vec{R}_1}{|\vec{R}_1|} - \vec{\Omega}_1 \times (\vec{\Omega}_1 \times \vec{r}_1) - 2\vec{\Omega}_1 \times \dot{\vec{r}}_1. \quad (\text{A6})$$

When the X -axis coincides with the line connecting the mass centers of the two stars, the third and fourth term in the right-hand member of equation (A6) can be written as the gradient of a potential function as

$$\frac{GM_2}{D^2} \frac{\vec{R}_1}{|\vec{R}_1|} = -\vec{\nabla} \left(\frac{GM_2}{D^2} X \right), \quad (\text{A7})$$

$$\vec{\Omega}_1 \times (\vec{\Omega}_1 \times \vec{r}_1) = -\vec{\nabla} \left[\frac{1}{2} |\vec{\Omega}_1|^2 (X^2 + Y^2) \right], \quad (\text{A8})$$

where X and Y are the Cartesian coordinates of the mass element under consideration. With these transformations, equation (A6) becomes

$$\ddot{\vec{r}}_1 = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} V_1 - 2\vec{\Omega}_1 \times \dot{\vec{r}}_1, \quad (\text{A9})$$

where

$$V_1 = -G \frac{M_1}{|\vec{r}_1|} - G \frac{M_2}{|\vec{r}_2|} - \frac{1}{2} |\vec{\Omega}_1|^2 (X^2 + Y^2) + \frac{GM_2}{D^2} X. \quad (\text{A10})$$

Since star 1 is assumed to be static in the rotating frame, $\dot{\vec{r}}_1 = \ddot{\vec{r}}_1 = 0$, so that equation (A9) reduces to

$$\vec{\nabla} P = -\rho \vec{\nabla} V_1. \quad (\text{A11})$$

This equation governs the instantaneous shape of the equipotential surfaces at the instant when the X -axis coincides with the line connecting the mass centers of the two stars. However, since this instant is entirely arbitrary, the equation is generally valid and can be used to determine the shape of the star at any phase of the orbit, with appropriate redefinition of the X -axis at every instant. A similar equation was derived by Avni (1976) and Wilson (1979).

Next, we look for the critical equipotential surface for which matter starts to flow from star 1 to star 2 through the inner Lagrangian point L_1 . For circular binaries with synchronously rotating component stars, the L_1 point is always located on the line connecting the mass centers of the two stars. Matese & Whitmire (1983) have shown that non-synchronous rotation combined with a misalignment between the spin axis of star 1 and the orbital plane may cause the L_1 point to oscillate in the Z -direction. As a first approximation, we here neglect these oscillations and assume the L_1 point to be located on the line connecting the mass centers of the two stars. The position of the L_1 point is then obtained by setting $dV_1/dX = 0$. The resulting equation for the position of L_1 is

$$G \frac{M_1}{X^2} - G \frac{M_2}{(X-D)^2} - |\vec{\Omega}_1|^2 X + G \frac{M_2}{D^2} = 0, \quad (\text{A12})$$

which can be rewritten in dimensionless form as

$$\frac{q}{X_{L_1}^2} - \frac{1}{(X_{L_1} - 1)^2} - f^2 X_{L_1} (1+q)(1+e) + 1 = 0. \quad (\text{A13})$$

Here, $X_{L_1} = X/D$ is the position of the L_1 point on the X -axis in units of the distance between the two stars, $q = M_1/M_2$ is the binary mass ratio, $f = \Omega_1/\Omega_{\text{orb},P}$ is the ratio of the star 1's rotational angular velocity to the orbital angular velocity at periastron, and $\Omega_{\text{orb},P}^2 = G(M_1 + M_2)(1+e)/D_P^3$. In the latter expression, $D_P = a(1-e)$ is the distance between the stars at periastron. Note that in the particular case of a circular orbit, equation (A13) reduces to the equation for the position of the L_1 point derived by Kruszewski (1963) and Pratt & Strittmatter (1976).

Equation (A13) must be solved numerically for the location of L_1 . Since we have made no explicit assumptions about the relative location of the two star in the orbit, and we have let the distance between the stars $D = a(1 - e^2)/(1 + e \cos \nu)$, with ν the true anomaly, be an explicit function of time, this equation characterizes the position of the L_1 point at any position in the orbit. Figure 8 shows the relative change

$$\Delta_{X_{L_1}} = \frac{X_{L_1}(\nu) - X_{L_1}(\nu = 0)}{X_{L_1}(\nu = 0)} \quad (\text{A14})$$

in the position of the L_1 point as a function of the true anomaly for $q = 1$, $f = 1$, and a variety of eccentricities. The position of the L_1 point over the course of a single orbit varies by less than 15% when $e \lesssim 0.1$, and by 30–40% when $e \gtrsim 0.2$. At apastron, $\Delta_{X_{L_1}}$ increases with increasing eccentricity when $e \lesssim 0.5$, and decreases with increasing eccentricity when $e \gtrsim 0.5$. This turn-around at $e \simeq 0.5$ is caused by the increasing deviations from synchronous rotation at apastron with increasing orbital eccentricity.

We have also derived a fitting formula for the position of the L_1 point at the periastron of the binary orbit as a function of q , e , and f :

$$X_{L_1} = 0.529 + 0.231 \log q - f^2(0.031 + 0.025e)(1 + 0.4 \log q). \quad (\text{A15})$$

This fitting formula is accurate to better than 4% for mass ratios $0.1 \leq q \leq 10$, eccentricities $0 \leq e \leq 0.9$, and ratios of stellar rotational angular velocities to orbital angular velocities at periastron $0 \leq f \leq 1.5$. In the limiting case of a circular, synchronized binary, equation (A15) reduces to equation (A1), while for a circular, non-synchronized binary, equation (A15) reduces to equation (A2). A detailed discussion of the properties of all five Lagrangian points and the volume of the critical Roche Lobe in eccentric binaries is presented in Sepinsky et al. (2007)

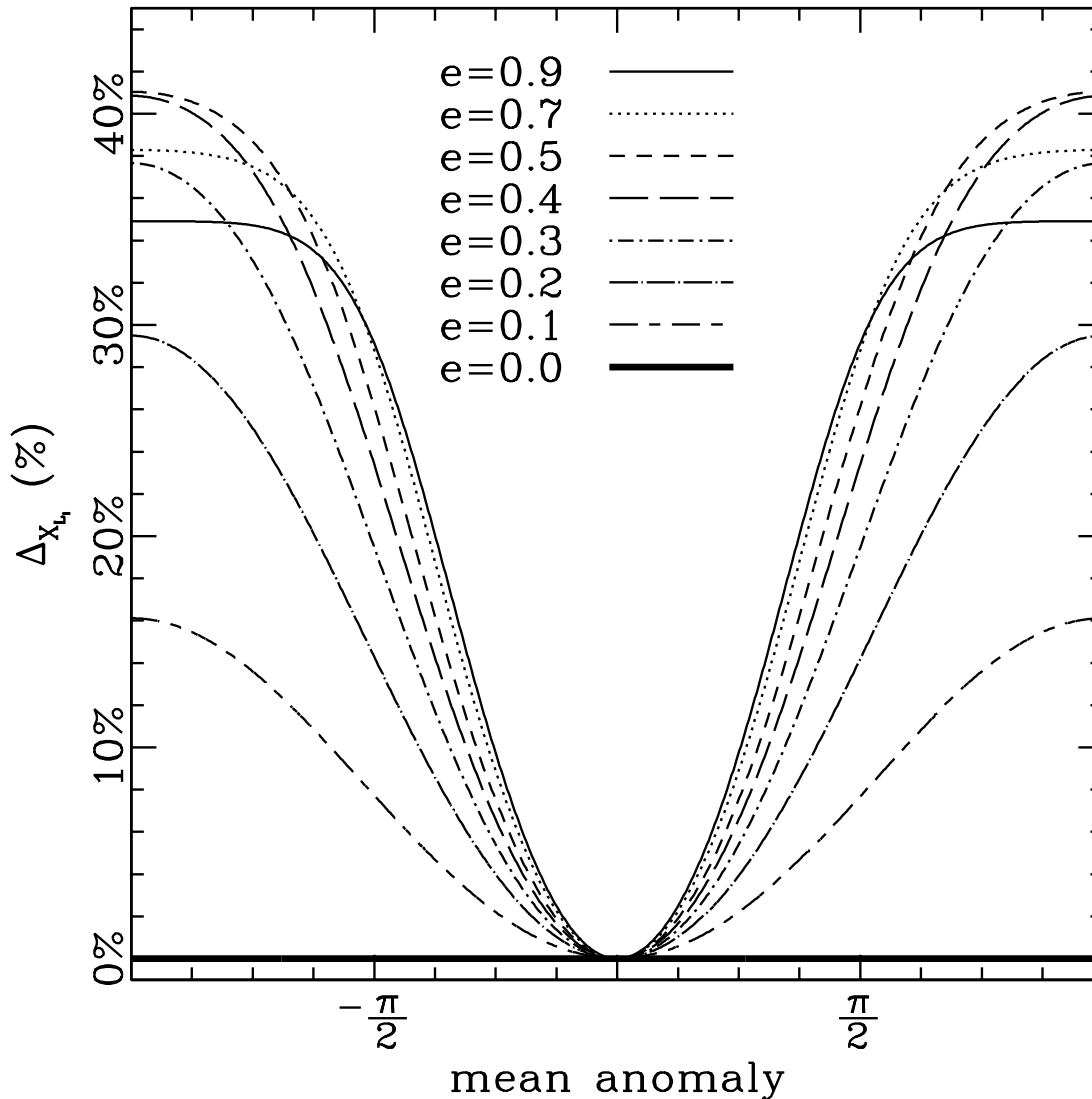


FIG. 8.— The relative change in the position of the L_1 point as a function of true anomaly ν , for $q = 1$, $f = 1$, and a range of eccentricities.

ORBITAL EVOLUTION DUE TO INSTANTANEOUS MASS TRANSFER BETWEEN TWO POINT MASSES

When the components of a mass-transferring binary are treated as point masses and mass transfer is assumed to be instantaneous, an alternative derivation of the equations governing the secular change of the orbital semi-major axis and eccentricity may be obtained from the equations for the specific orbital energy and angular momentum. This method is commonly used to relate the post- to pre-supernova orbital parameters of binaries in which one of the components undergoes an instantaneous supernova explosion (e.g. Hills 1983; Brandt & Podsiadlowski 1995; Kalogera 1996). As in Section 5, we assume conservation of total system mass and describe the mass transfer by a delta function mass-transfer rate centered at the periastron of the binary orbit [see equation (32)].

The relative orbital velocity, V_{rel} , of two stars with masses M_1 (the donor star) and M_2 (the accreting star) in a binary with orbital semi-major axis a and eccentricity e is given by

$$V_{\text{rel}}^2 = G(M_1 + M_2) \left(\frac{2}{|\vec{r}|} - \frac{1}{a} \right), \quad (\text{B1})$$

where G is the Newtonian constant of gravitation, and \vec{r} is the relative position vector of the accretor with respect to the donor. Under the assumption that mass elements are transferred instantaneously between the two stars, the distance $|\vec{r}|$ is the same right before and right after mass transfer, so that the rate of change of the orbital semi-major axis due to instantaneous mass transfer is given by

$$\frac{da}{dt} = \frac{2a^2 V_{\text{rel}}}{G(M_1 + M_2)} \frac{dV_{\text{rel}}}{dt}. \quad (\text{B2})$$

Next, the specific orbital angular momentum of the binary written in terms of the binary component masses and orbital elements is given by

$$|\vec{r} \times \vec{V}_{\text{rel}}|^2 = G(M_1 + M_2)a(1 - e^2). \quad (\text{B3})$$

Noting that $|\vec{r} \times \vec{V}_{\text{rel}}| = |\vec{r}||\vec{V}_{\text{rel}}|$ at periastron, and proceeding in a similar way as for the derivation of equation (B2), we derive the rate of change of the orbital eccentricity due to instantaneous mass transfer to be

$$\frac{de}{dt} = \frac{2a(1 - e)|\vec{V}_{\text{rel}}|}{G(M_1 + M_2)} \frac{d|\vec{V}_{\text{rel}}|}{dt}. \quad (\text{B4})$$

Finally, we eliminate V_{rel} from equations (B2) and (B4) to obtain equations for the rates of changes of the orbital semi-major axis and eccentricity in terms of the binary orbital elements and component masses. For this purpose, we use the conservation of linear momentum to write the rate of change of the relative orbital velocity due to mass-transfer as

$$\frac{d\vec{V}_{\text{rel}}}{dt} = \frac{\dot{M}_2}{M_2} \vec{v}_{\delta M_2} - \frac{\dot{M}_1}{M_1} \vec{v}_{\delta M_1}, \quad (\text{B5})$$

where $\vec{v}_{\delta M_1}$ and $\vec{v}_{\delta M_2}$ are the relative velocities of the transferred mass elements with respect to the mass center of the donor and accretor, respectively. To obtain an equation for the change in the magnitude of \vec{V}_{rel} , we write the vector components of \vec{V}_{rel} , $\vec{v}_{\delta M_1}$, and $\vec{v}_{\delta M_2}$ with respect to the \hat{x} , \hat{y} , and \hat{z} unit vectors introduced in Section 3.2, and take the dot product of equation (B5) with \vec{V}_{rel} . Since, $\vec{V}_{\text{rel}} = V_{\text{rel}} \hat{y}$ at the periastron of the binary orbit, it follows that

$$\frac{dV_{\text{rel}}}{dt} = -\frac{\dot{M}_1}{M_1} (qv_{\delta M_2,y} + v_{\delta M_1,y}). \quad (\text{B6})$$

Assuming conservation of orbital angular momentum it then follows from equations (29) and (B2)–(B6) that

$$qv_{\delta M_2,y} + v_{\delta M_1,y} = na(1 - q) \left(\frac{1 + e}{1 - e} \right)^{1/2}, \quad (\text{B7})$$

where we have used Kepler's 3rd law and that $|\vec{V}_{\text{rel}}| = (G(M_1 + M_2)/a)^{1/2} ((1 + e)/(1 - e))^{1/2}$ at the periastron of the binary orbit. This equation is identical to equation (35) in the limiting case where $|\vec{r}_{A1}| \rightarrow 0$ and $|\vec{r}_{A2}| \rightarrow 0$. Substituting equations (eq-C-vdm2) and (eq-C-vrel) into equations (B2) and (B4) and averaging over the orbital period, we find

$$\left\langle \frac{da}{dt} \right\rangle_{\text{sec}} = \frac{a \dot{M}_0}{\pi M_1} (1 - e^2)^{1/2} (q - 1) \quad (\text{B8})$$

$$\left\langle \frac{de}{dt} \right\rangle_{\text{sec}} = \frac{1 \dot{M}_0}{\pi M_1} (1 - e^2)^{1/2} (1 - e)(q - 1). \quad (\text{B9})$$

These equations are identical to equations (25) and (26) in the limiting case where $|\vec{r}_{A1}| \rightarrow 0$ and $|\vec{r}_{A2}| \rightarrow 0$.

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