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SECULAR DYNAMICS IN THREE-BODY SYSTEMS

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ABSTRACT

The secular approximation for the evolution of hierarchical triple configurations has proven to be very useful in many astrophysical contexts, from planetary to triple-star systems. In this approximation the orbits may change shape and orientation but the semimajor axes are constant. For example, for highly inclined systems, the Kozai-Lidov mechanism can produce large-amplitude oscillations of the eccentricities. Here we re-derive the secular evolution equations including both *quadrupole* and *octupole* orders using Hamiltonian perturbation theory. Our new derivation corrects an error in previous treatments of the problem. Already to quadrupole order, our results disagree with the previous "standard" treatment; they agree only in the test-particle limit where one of the bodies in the inner binary has negligible mass compared to that of the outer perturber. Assuming, as done in previous treatments, that the z-component of the inner orbit's angular momentum (perpendicular to the invariable plane) is conserved can produce erroneous results for various astrophysical systems of interest.

1. INTRODUCTION

Triple star systems are believed to be very common (e.g., Tokovinin 1997; Eggleton et al. 2007). From dynamical stability arguments these must be hierarchical triples, in which the (inner) binary is orbited by a third body on a much wider orbit. Probably more than 50% of bright stars are at least double (Tokovinin 1997; Eggleton et al. 2007). Given the selection effects against finding faint and distant companions we can be reasonably confident that the proportion is actually substantially greater. Tokovinin (1997) showed that 40% of binary stars with period < 10 d in which the primary is a dwarf ($0.5-1.5\,M_\odot$) have at least one additional companion. He found that the fraction of triples and higher multiples among binaries with period ($10-100$ d) is $\sim 10\%$. Moreover, Pribulla & Rucinski (2006) have surveyed a sample of contact binaries, and noted that among 151 contact binaries brighter than 10 mag., $42\pm5\%$ are at least triple.

Many close stellar binaries with two compact objects are likely produced through triple evolution. Secular effects (i.e., coherent interactions on timescales long compared to the orbital period), and specifically Kozai-Lidov cycling (Kozai 1962; Lidov 1962, see below), have been proposed as an important element in the evolution of triple stars (e.g. Harrington 1969; Mazeh & Shaham 1979; Kiseleva et al. 1998; Fabrycky & Tremaine 2007; Perets & Fabrycky 2009). In addition, Kozai-Lidov cycling has been suggested to play an important role in both the growth of black holes at the centers of dense star clusters and the formation of short-period binary black holes (Wen 2003; Miller & Hamilton 2002; Blaes et al. 2002). Recently, Ivanova et al. (2010) showed that the most important formation mechanism for black hole XRBs in globular clusters may be triple-induced mass transfer in a black hole-white dwarf binary.

Secular perturbations in triple systems also play an important role in planetary system dynamics. Kozai (1962) studied the effects of Jupiter's gravitational perturbation on an inclined asteroid in our own solar system. In the assumed hierarchical configuration, treating the asteroid as a test particle, Kozai (1962) found that its inclination and eccentricity fluctuate on timescales much larger than its orbital period. Jupiter, assumed to be in a circular orbit, carries most of the angular momentum of the system. Due to Jupiter's circular orbit and the negligable mass of the asteroid, the system's potential is axisymmetric and thus the component of the inner orbit's angular momentum along the total angular momentum is conserved during the evolution. Kozai (1979) also showed the importance of secular interactions for the dynamics of comets (see also Quinn et al. 1990; Bailey et al. 1992; Thomas & Morbidelli 1996). The evolution of the orbits of binary minor planets is dominated by the secular gravitational perturbation from the sun (Perets & Naoz 2009); properly accounting for the resulting secular effects—including Kozai cycling—accurately reproduces the binary minor planet orbital distribution seen today (Naoz et al. 2010; Grundy et al. 2011). In addition Kinoshita & Nakai (1991), Vashkov'yak (1999), Carruba et al. (2002), Nesvorný et al. (2003), Ćuk & Burns (2004) and Kinoshita & Nakai (2007) suggested that secular interactions may explain the significant inclinations of gas giant satellites and Jovian irregular satellites.

Similar analyses have been applied to the orbits of extrasolar planets (e.g., Innanen et al. 1997; Wu & Murray 2003; Fabrycky & Tremaine 2007; Wu et al. 2007; Naoz et al. 2011; Correia et al. 2011). Naoz et al. (2011) considered the secular evolution of a triple system consisting of an inner binary containing a star and a Jupiter-like planet at several AU, orbited by a distant Jupiter-like planet or brown-dwarf companion. Perturbations from the outer body can drive Kozai-like cycles in the inner binary, which, when planet-star tidal effects are incorporated, can lead to the capture of the inner planet onto a close, highly-inclined or even retrograde orbit, similar to the orbits of the observed retrograde "hot Jupiters." Many other studies of exoplanet dynamics have consid-

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ered similar systems, but with a very distant stellar binary companion acting as perturber. In such systems, the outer star completely dominates the orbital angular momentum, and the problem reduces to test-particle evolution in the lowest level of approximation, which leads to the conservation of the z-component of the inner orbit's angular momentum (e.g. Wu & Murray 2003; Wu et al. 2007; Fabrycky & Tremaine 2007; Takeda et al. 2008).

In early studies of high-inclination secular perturbations (Kozai 1962; Lidov 1962), the outer orbit was circular and again dominated the orbital angular momentum of the system. In this situation, the component of the inner orbit's angular momentum along the z-axis is conserved. In many later studies the assumption that the z-component of the inner orbit's angular momentum is constant was built into the equations (e.g. Eggleton et al. 1998; Mikkola & Tanikawa 1998; Zdziarski et al. 2007; Takeda et al. 2008). In fact these studies are only valid in the limit of a test particle forced by a perturber on a circular orbit. To leading order in the ratio of semimajor axes, the double averaged potential of the outer orbit is axisymmetric (even for an eccentric outer perturber), thus if taken to the test particle limit, this results in a conservation of the z-component of the inner orbit's angular momentum. We refer to this limit as the "standard" treatment of Kozai oscillations, i.e. quadrupole-level approximation in the test particle limit (test particle quadrupole, hereafter TPQ).

In this paper we show that a common mistake in the Hamiltonian treatment of these secular systems can lead to the erronious conclusion that the z-component of the inner orbit's angular momentum is constant outside the TPQ limit; in fact, the z-component of the inner orbit's angular momentum is only conserved by the evolution in the test-particle limit and to quadrupole order. To demonstrate the error we focus on the quadrupole (non-test-particle) approximation in the main body of the paper, but we include the full octupole-order equations of motion in an appendix.

This paper is organized as follows. We first present the general framework (§2); we then derive the complete formalism for the quadrupole-level approximation and the equations of motion (§3), we also develop the octupole-level approximation equations of motion in §4. We discuss a few of the most important implications of the correct formalism in §5. We also compare our results with those of previous studies (§6) and offer some conclusions in §7.

2. HAMILTONIAN PERTURBATION THEORY FOR HIERARCHICAL TRIPLE SYSTEMS

Many gravitational triple systems are in a hierarchical configuration—two objects orbit each other in a relatively tight inner binary while the third object is on a much wider orbit. If the third object is sufficiently distant, an analytic, perturbative approach can be used to calculate the evolution of the system. In the usual secular approximation (e.g., Marchal 1990), the two orbits torque each other and exchange angular momentum, but not energy. Therefore the orbits can change shape and orientation (on timescales much longer than their orbital periods), but not semimajor axes (SMA).

We first define our basic notations. The system consists of a close binary (bodies of masses m_1 and m_2) and

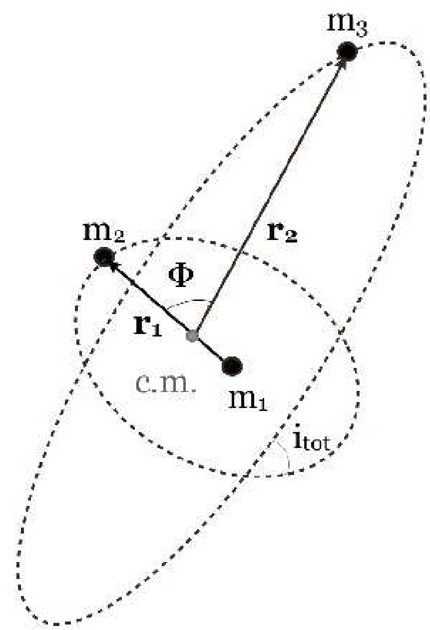

FIG. 1.— Coordinate system used to describe the hierarchical triple system (*not to scale*). Here 'c.m.' denotes the center of mass of the inner binary, containing objects of masses m_1 and m_2. The separation vector $\mathbf{r_1}$ points from m_1 to m_2; $\mathbf{r_2}$ points from 'c.m.' to m_3. The angle between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is Φ.

a third body (mass m_3). It is convenient to describe the orbits using Jacobi coordinates (Murray & Dermott 2000, p. 441-443). Let $\mathbf{r}_1$ be the relative position vector from m_1 to m_2 and $\mathbf{r}_2$ the position vector of m_3 relative to the center of mass of the inner binary (see fig. 1). Using this coordinate system the dominant motion of the triple can be divided into two separate Keplerian orbits: the relative orbit of bodies 1 and 2, and the orbit of body 3 around the center of mass of bodies 1 and 2. The Hamiltonian for the system can be decomposed accordingly into two Keplerian Hamiltonians plus a coupling term that describes the (weak) interaction between the two orbits. Let the SMAs of the inner and outer orbits be a_1 and a_2, respectively. Then the coupling term in the complete Hamiltonian can be written as a power series in the ratio of the semi-major axes $\alpha = a_1/a_2$ (e.g., Harrington 1968). In a hierarchical system, by definition, this parameter α is small.

The complete Hamiltonian expanded in orders of α is (e.g., Harrington 1968),

$$\mathcal{H} = \frac{k^2 m_1 m_2}{2a_1} + \frac{k^2 m_3 (m_1+m_2)}{2a_2} + \frac{k^2}{a_2}\sum_{j=2}^{\infty} \alpha^j M_j \left(\frac{r_1}{a_1}\right)^j \left(\frac{a_2}{r_2}\right)^{j+1} P_j(\cos\Phi) \ , \quad (1)$$

where k^2 is the gravitational constant, P_j are Legendre polynomials, Φ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ (see Figure 1) and

$$M_j = m_1 m_2 m_3 \frac{m_1^{j-1} - (-m_2)^{j-1}}{(m_1+m_2)^j} \ . \quad (2)$$

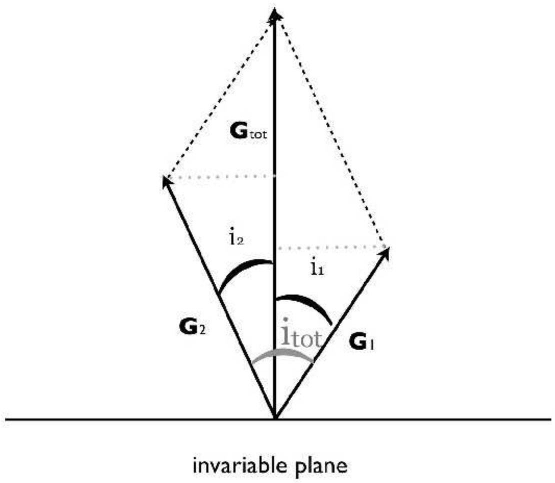

FIG. 2.— Geometry of the angular momentum vectors. We show the total angular momentum vector ($\mathbf{G}_{\rm tot}$), the angular momentum vector of the inner orbit ($\mathbf{G_1}$) with inclination i_1 with respect to $\mathbf{G}_{\rm tot}$ and the angular momentum vector of the outer orbit ($\mathbf{G_2}$) with inclination i_2 with respect to $\mathbf{G}_{\rm tot}$. The angle between $\mathbf{G_1}$ and $\mathbf{G_2}$ defines the mutual inclination $i_{\rm tot} = i_1 + i_2$. The invariable plane is perpendicular to $\mathbf{G}_{\rm tot}$.

Note that we have followed the convention of Harrington (1969) and chosen our Hamiltonian to be the negative of the total energy, so that $\mathcal{H} > 0$ for bound systems.

We adopt the canonical variables known as Delaunay's elements, which provide a particularly convenient dynamical description of our three-body system (e.g. Valtonen & Karttunen 2006). The coordinates are chosen to be the mean anomalies, l_1 and l_2, the longitudes of ascending nodes, h_1 and h_2, and the arguments of periastron, g_1 and g_2, where subscripts 1, 2 denote the inner and outer orbits, respectively. Their conjugate momenta are

$$L_1 = \frac{m_1 m_2}{m_1+m_2}\sqrt{k^2(m_1+m_2)a_1} \ , \tag{3}$$

$$L_2 = \frac{m_3(m_1+m_2)}{m_1+m_2+m_3}\sqrt{k^2(m_1+m_2+m_3)a_2} \ ,$$

$$G_1 = L_1\sqrt{1-e_1^2} \ , \quad G_2 = L_2\sqrt{1-e_2^2} \ , \tag{4}$$

and

$$H_1 = G_1 \cos i_1 \ , \quad H_2 = G_2 \cos i_2 \ . \tag{5}$$

Note that G_1 and G_2 are also the magnitudes of the angular momentum vectors ($\mathbf{G}_1$ and $\mathbf{G}_2$), and H_1 and H_2 are the z-components of these vectors. Figure 2 shows the resulting configuration of theses vectors. The following geometric relations between the momenta follow from the law of cosines:

$$\cos i_{\rm tot} = \frac{G_{\rm tot}^2 - G_1^2 - G_2^2}{2G_1G_2} \ , \tag{6}$$

$$H_1 = \frac{G_{\rm tot}^2 + G_1^2 - G_2^2}{2G_{\rm tot}} \ , \tag{7}$$

$$H_2 = \frac{G_{\rm tot}^2 + G_2^2 - G_1^2}{2G_{\rm tot}} \ , \tag{8}$$

where $\mathbf{G}_{\rm tot} = \mathbf{G}_1 + \mathbf{G}_2$ is the (conserved) total angular momentum, and the angle between $\mathbf{G_1}$ and $\mathbf{G_2}$ defines the mutual inclination $i_{\rm tot} = i_1 + i_2$. From eqs. (7) and (8) we find that the inclinations i_1 and i_2 are determined by the orbital angular momenta:

$$\cos i_1 = \frac{G_{\rm tot}^2 + G_1^2 - G_2^2}{2G_{\rm tot}G_1} \ , \tag{9}$$

$$\cos i_2 = \frac{G_{\rm tot}^2 + G_2^2 - G_1^2}{2G_{\rm tot}G_2} \ . \tag{10}$$

In addition to these geometrical relations we also have that

$$H_1 + H_2 = G_{\rm tot} = {\rm const} \ . \tag{11}$$

The canonical relations give the equations of motion:

$$\frac{dL_j}{dt} = \frac{\partial \mathcal{H}}{\partial l_j} \ , \quad \frac{dl_j}{dt} = -\frac{\partial \mathcal{H}}{\partial L_j} \ , \tag{12}$$

$$\frac{dG_j}{dt} = \frac{\partial \mathcal{H}}{\partial g_j} \ , \quad \frac{dg_j}{dt} = -\frac{\partial \mathcal{H}}{\partial G_j} \ , \tag{13}$$

$$\frac{dH_j}{dt} = \frac{\partial \mathcal{H}}{\partial h_j} \ , \quad \frac{dh_j}{dt} = -\frac{\partial \mathcal{H}}{\partial H_j} \ , \tag{14}$$

where $j = 1, 2$. Note that these canonical relations have the opposite sign relative to the usual relations (e.g., Goldstein 1950) because of the sign convention we have chosen for our Hamiltonian. Finally we write the Hamiltonian through second order in α as (e.g., Kozai 1962)

$$\mathcal{H} = \frac{\beta_1}{2L_1^2} + \frac{\beta_2}{2L_2^2} + \tag{15}$$
$$4\beta_3\left(\frac{L_1^4}{L_2^6}\right)\left(\frac{r_1}{a_1}\right)^2\left(\frac{a_2}{r_2}\right)^3(3\cos 2\Phi + 1) \ ,$$

where the mass parameters are

$$\beta_1 = k^2 m_1 m_2 \frac{L_1^2}{a_1} \ , \tag{16}$$

$$\beta_2 = k^2(m_1+m_2)m_3\frac{L_2^2}{a_2} \tag{17}$$

and

$$\beta_3 = \frac{k^4}{16}\frac{(m_1+m_2)^7 m_3^7}{(m_1m_2)^3(m_1+m_2+m_3)^3} \ . \tag{18}$$

3. SECULAR EVOLUTION EQUATIONS TO QUADRUPOLE ORDER

In this section, we derive the secular equations of motion to the quadrupole-level, where in Appendix A we develop the complete quadrupole-level secular approximation. The main difference between the derivation shown here (see also Appendix A) and those of previous studies lies in the "elimination of nodes" (e.g., Kozai 1962; Jefferys & Moser 1966). This is related to the transition to a coordinate system with the total angular momentum along the z-axis, which is known as the *invariable plane* (e.g., Murray & Dermott 2000). In this coordinate system (see Figure 2), the longitudes of the ascending nodes differ by π, i.e., $h_1 - h_2 = \pi$. Conservation of angular momentum implies that this relation holds at all times. Many previous works have exploited it to explicitly simplify the Hamiltonian. However, as we explain below (and see also §6.1), this substitution leads to the incorrect conclusion that $\dot{H}_1 = \dot{H}_2 = 0$ when the canonical equations of motion are applied; thus, some previous studies incorrectly concluded that the z-components of the orbital angular momenta are always constant. We will show that one can still use the (incorrect) Hamiltonian found in previous studies (e.g., Kozai 1962; Harrington 1969) as long as the evolution equations for the inclinations are derived from the total angular momentum conservation, instead of using the canonical relations. Of course, the correct evolution equations can also be calculated from the correct Hamiltonian, which we derive in this section.

We note that there are some other derivations of the secular evolution equations that avoid the elimination of the nodes (Farago & Laskar 2010; Laskar & Boué 2010; Mardling 2010; Katz & Dong 2011), and thus do not suffer from this error.

Previous studies made the substitution $h_1 - h_2 = \pi$ directly in the Hamiltonian (see §6.1). After the substitution, the Hamiltonian is independent of the longitudes of ascending nodes (h_1 and h_2), and thus gives an evolution where both H_1 and H_2 are constant. The substitution $h_1 - h_2 = \pi$ is incorrect at the Hamiltonian level because it unduly restricts variations in the trajectory of the system to those where $\delta h_1 = \delta h_2$ (see Appendix E). After deriving the equations of motion, however, we can exploit the relation $h_1 - h_2 = \Delta h = \pi$, which comes from the conservation of angular momentum and the fact that $\mathbf{G}_1 + \mathbf{G}_2 = \mathbf{G}_{\rm tot} = G_{\rm tot}\hat{z}$. This considerably simplifies the equations.

The secular Hamiltonian is given by the average over the rapidly-varying l_1 and l_2 in equation (15) (Appendix A for more details)

$$\begin{aligned}\mathcal{H}_2 = \frac{C_2}{8}\big\{&[1+3\cos(2i_2)]\big([2+3e_1^2][1+3\cos(2i_1)] \\ &+30e_1^2\cos(2g_1)\sin^2(i_1)\big)+3\cos(2\Delta h)[10e_1^2\cos(2g_1) \\ &\times\{3+\cos(2i_1)\}+4(2+3e_1^2)\sin(i_1)^2]\sin^2(i_2) \\ &+12\{2+3e_1^2-5e_1^2\cos(2g_1)\}\cos(\Delta h)\sin(2i_1)\sin(2i_2) \\ &+120e_1^2\sin(i_1)\sin(2i_2)\sin(2g_1)\sin(\Delta h) \\ &-120e_1^2\cos(i_1)\sin^2(i_2)\sin(2g_1)\sin(2\Delta h)\big\}\ ,\end{aligned} \quad (19)$$

where

$$C_2 = \frac{k^4}{16}\frac{(m_1+m_2)^7}{(m_1+m_2+m_3)^3}\frac{m_3^7}{(m_1m_2)^3}\frac{L_1^4}{L_2^3G_2^3}\ . \quad (20)$$

Making the usual (incorrect) transformation $\Delta h \to \pi$, we get the quadrupole-level Hamiltonian that has appeared in many previous works (see, e.g. Ford et al. 2000b):

$$\begin{aligned}\mathcal{H}_2(\Delta h\to\pi) = C_2\big\{&\left(2+3e_1^2\right)\left(3\cos^2 i_{\rm tot}-1\right) \\ &+15e_1^2\sin^2 i_{\rm tot}\cos(2g_1)\big\}\ ,\end{aligned} \quad (21)$$

where we have set $i_1 + i_2 = i_{\rm tot}$. Because this Hamiltonian is missing the longitudes of ascending nodes (h_1 and h_2), previous studies concluded that the z-component (i.e., vertical) angular momenta of the inner and outer orbits (i.e., H_1 and H_2) are constants.

We use the canonical relations [equations (12)] in order to derive the equations of motion from the Hamiltonian. In our treatment, both H_1 and H_2 evolve with time because the Hamiltonian is not independent of h_1 and h_2. From eq. (7), we see that

$$\dot{H}_1 = \frac{G_1}{G_{\rm tot}}\dot{G}_1 - \frac{G_2}{G_{\rm tot}}\dot{G}_2\ , \quad (22)$$

and from eq. (11) we see that $\dot{H}_1 = -\dot{H}_2$. The quadrupole-level Hamiltonian does not depend on g_2; thus the magnitude of the outer orbit's angular momentum, G_2, is constant[3], and therefore

$$\dot{H}_1 = \frac{G_1\dot{G}_1}{G_{\rm tot}}\ . \quad (23)$$

From relations (12-14) we have $\dot{H}_1 = \partial\mathcal{H}/\partial h_1$, and $\dot{G}_1 = \partial\mathcal{H}/\partial g_1$. The former gives

$$\dot{H}_1 = -30C_2e_1^2\sin i_2\sin i_{\rm tot}\sin(2g_1)\ . \quad (24)$$

and the latter evaluates to

$$\dot{G}_1 = -30C_2e_1^2\sin^2 i_{\rm tot}\sin(2g_1)\ . \quad (25)$$

Employing the law of sines, $G_{\rm tot}/\sin i_{\rm tot} = G_1/\sin i_2 = G_2/\sin i_1$, equation (24) can also be written as

$$\dot{H}_1 = -\frac{G_1}{G_{\rm tot}}30C_2e_1^2\sin^2 i_{\rm tot}\sin(2g_1)\ , \quad (26)$$

which satisfies the relation in eq. (23). The evolution of the arguments of periapse are given by

$$\begin{aligned}\dot{g}_1 = 6C_2\Big\{&\frac{1}{G_1}[4\cos^2 i_{\rm tot}+(5\cos(2g_1)-1) \\ &\times(1-e_1^2-\cos^2 i_{\rm tot})]+\frac{\cos i_{\rm tot}}{G_2}[2+e_1^2(3-5\cos(2g_1))]\Big\}\ ,\end{aligned} \quad (27)$$

and

$$\begin{aligned}\dot{g}_2 = 3C_2\Big\{&\frac{2\cos i_{\rm tot}}{G_1}[2+e_1^2(3-5\cos(2g_1))] \\ &+\frac{1}{G_2}[4+6e_1^2+(5\cos^2 i_{\rm tot}-3)(2+e_1^2[3-5\cos(2g_1)])\Big\}\ .\end{aligned} \quad (28)$$

Previous quadrupole-level calculations that made the substitution error in the Hamiltonian lack the $1/G_2$ term in the latter equation. The evolution of the longitudes of ascending nodes is given by

$$\dot{h}_1 = -\frac{3C_2}{G_1\sin i_1}\{2+3e_1^2-5e_1^2\cos(2g_1)\}\sin(2i_{\rm tot}) \quad (29)$$

and

$$\dot{h}_2 = -\frac{3C_2}{G_2\sin i_2}\{2+3e_1^2-5e_1^2\cos(2g_1)\}\sin(2i_{\rm tot})\ . \quad (30)$$

Using the law of sines, $G_1\sin i_1 = G_2\sin i_2$, from which we get $\dot{h}_1 = \dot{h}_2$, as required by the relation $h_1 - h_2 = \pi$. In many systems it is useful to calculate the time evolution of the eccentricity, obtained through the following relation:

$$\frac{de_j}{dt} = \frac{\partial e_j}{\partial G_j}\frac{\partial\mathcal{H}}{\partial g_i}\ , \quad (31)$$

[3] This conserved quantity is lost at higher orders of the approximation; see §4 and Appendix C.

In the quadrupole approximation $\dot{e}_2 = \dot{G}_2 = 0$ (which is not the case at higher order in α; see Appendix C). The eccentricity evolution for the inner orbit is given by

$$\dot{e}_1 = C_2 \frac{1-e_1^2}{G_1} 30 e_1 \sin^2 i_{\rm tot} \sin(2g_1) \ . \tag{32}$$

Another useful parameter is the inclination, which can be found through the z-component of the angular momentum:

$$\frac{d(\cos i_1)}{dt} = \frac{\dot{H}_1}{G_1} - \frac{\dot{G}_1}{G_1} \cos i_1 \ , \tag{33}$$

and similarly for i_2 (but note again that $\dot{G}_2 = 0$ to quadrupole order). In Appendix B we show that the quadrupole approximation leads to well-defined minimum and maximum eccentricity and inclination. The eccentricity of the inner orbit and the inner (and mutual) inclination oscillate. We also demonstrate in Appendix B that our formalism gives critical initial mutual inclination angles for large oscillations of $39.2^\circ \leq i_{\rm tot} \leq 140.8^\circ$ in the test-particle limit and with nearly-zero initial inner eccentricity, in agreement with Kozai (1962).

4. OCTUPOLE-LEVEL EVOLUTION

In Appendix C, we derive the secular evolution equations to octupole order. Many previous octupole-order derivations provided correct secular evolution equations for at least some of the elements, in spite of using the elimination of nodes substitution at the Hamiltonian level (e.g. Harrington 1968, 1969; Sidlichovsky 1983; Krymolowski & Mazeh 1999; Ford et al. 2000b; Blaes et al. 2002; Lee & Peale 2003b; Thompson 2010). This is because the evolution equations for e_2, g_2, g_1 and e_1 can be found correctly from a Hamiltonian that has had h_1 and h_2 eliminated by the relation $h_1 - h_2 = \pi$; the partial derivatives with respect to the other coordinates and momenta are not affected by the substitution. The correct evolution of H_1 and H_2 can then be derived, not from the canonical relations, but from *total* angular momentum conservation. We discuss in more details the comparison between this work and previous analyses in §6. The full consequences of allowing the H_i to be dynamical are explored here for the first time.

The octupole-level terms in the Hamiltonian can become important when the eccentricity of the outer orbit is non-zero, and if α is not negligible. We quantifying this by considering the ratio between the octupole to quadrupole-level coefficients, which is

$$\frac{C_3}{C_2} = \frac{15}{4}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)\left(\frac{a_1}{a_2}\right)\frac{1}{1-e_2^2} \ , \tag{34}$$

where C_3 is the octupole-level coefficient [eq. (C1)] and C_2 is the quadrupole-level coefficient [eq. (20)]. We define

$$\epsilon_M = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\left(\frac{a_1}{a_2}\right)\frac{e_2}{1-e_2^2} \ , \tag{35}$$

which gives the relative significance of the octupole-level approximation. This parameter has three important parts; first the eccentricity of the outer orbit (e_2), second, the mass difference of the inner binary (m_1 and m_2) and the SMA ratio[4]. In the test particle limit (i.e., $m_1 \gg m_2$) we find that ϵ_M is reduced to the octupole coefficient introduced in Lithwick & Naoz (2011) and Katz et al. (2011),

$$\epsilon = \left(\frac{a_1}{a_2}\right)\frac{e_2}{1-e_2^2} \ . \tag{36}$$

This coefficient measures the significance of the octupole contribution in the test particle limit. We will use here the general form (i.e., ϵ_M). We label the behavior of a system for which $\epsilon_M \ll 1$ is not satisfied as "eccentric Kozai-Lidov" mechanism.

The octupole terms vanish when $e_2 = 0$. Therefore if one artificially held $e_2 = 0$, in the test-particle limit the inner body's orbit would be given by the equations derived by Kozai (1962), i.e. by the test particle quadrupole equations. However, at octupole order the value of e_2 evolves in time if the inner body is massive. Furthermore, even if the inner body is massless, if the outer body has $e_2 > 0$ then the inner body's behavior will also be different than in Kozai's treatment. For example, Lithwick & Naoz (2011) and Katz et al. (2011) find that the inner orbit can flip orientation (see below) even in the test-particle, octupole limit. The octupole-level effects can change qualitatively the evolution of a system. Compared to the quadrupole-level behavior, the eccentricity of the inner orbit can sometimes reach a much higher value. In some cases these excursions to very high eccentricities can be accompanied by a "flip" of the orbit with respect to the total angular momentum, i.e., starting with $i_1 < 90^\circ$ the inner orbit can eventually reach $i_1 > 90^\circ$ (see Figures 6–9 for examples). Chaotic behavior is also possible at the octupole level (Lithwick & Naoz 2011; Katz et al. 2011), but not at the quadrupole-level (see Appendix B). In contrast to the octupole-level behavior, the quadrupole-level approximation leads to regular cycles in eccentricity and inclination, with well-defined maximum and minimum values, and it cannot produce flips for the inner orbit (again, see Appendix B).

Given the large, qualitative changes in behavior moving from quadrupole to octupole order in the Hamiltonian, is it possible that similar changes in the secular evolution may occur at even higher orders? Intuitively, the answer to this question lies in the elimination of G_2 as an integral of motion at octupole order, leaving only four integrals of motion: the energy of the system, and the three components of the total angular momentum. There are no more integrals of motion to be eliminated, and thus one might expect no more dramatic changes in the evolution when moving to even higher orders. It is possible to see this quantitatively for specific initial conditions through comparisons with direct n-body integrations. We compare our octupole equations with direct n-body integrations, using the `Mercury` software package (Chambers & Migliorini 1997). We used both Bulirsch-Stoer and symplectic integrators (Wisdom & Holman 1991) and found consistent results between the two. We present the results of a typical integration compared to the integration of the

[4] Note here that the subscripts "1" and "2" refer to the *inner* bodies in m_1 and m_2, but the subscript "2" refers to the *outer* body in e_2.

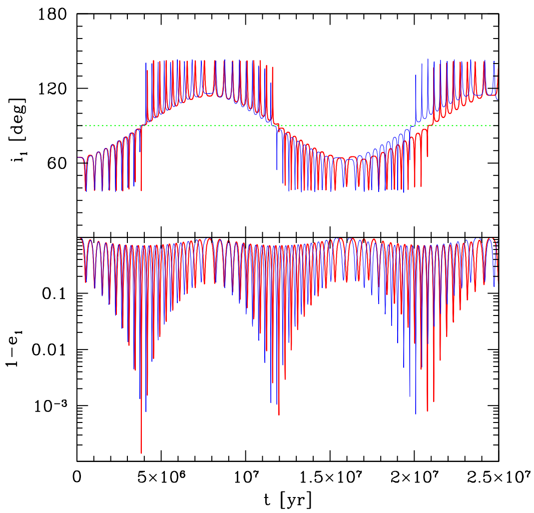

Fig. 3.— Comparison between a direct integration (using a B-S integrator) and the octupole-level approximation (see Appendix C). The red lines are from the integration of the octupole-level perturbation equations, while the blue lines are from the direct numerical integration of the three-body system. Here the inner binary contains a star of mass $1\,M_\odot$ and a planet of mass $1\,M_{\rm J}$, while the outer object is a brown dwarf of mass $40\,M_{\rm J}$. The inner orbit has $a_1 = 6\,$AU and the outer orbit has $a_2 = 100\,$AU. The initial eccentricities are $e_1 = 0.001$ and $e_2 = 0.6$ and the initial relative inclination $i_{\rm tot} = 65°$. The thin horizontal line in the top panel marks the 90° boundary, separating prograde and retrograde orbits. The initial mutual inclination of 65° corresponds to an inner and outer inclination with respect to the total angular momentum (parallel to z) of 64.7° and 0.3°, respectively. The arguments of pericenter of the inner and outer orbits are initially set to zero. The SMA of the two orbits (not shown) are nearly constant during the direct integration, varying by less than 0.02 percent. The agreement in both period and amplitude of oscillation between the direct integration and the octupole-level approximation is quite good.

octupole-level secular equations in Figure 3. The initial conditions (see caption) for this system are those of Naoz et al. (2011), Figure 1. We find good agreement between the direct integration and the secular evolution at octupole order. Both show a beat-like pattern of eccentricity oscillations, suggesting an interference between the quadrupole and octupole terms, and both methods show similar flips of the inner orbit.

5. IMPLICATIONS

The Kozai (1962) and Lidov (1962) equations of motions are correct to quadrupole order and for a test particle. Using them outside this limit can lead to incorrect results. Here we discuss a few examples and the importance of higher-order effects in various astrophysical settings.

5.1. *Massive Inner Object at the Quadrupole Level*

The danger with working in the wrong limit is apparent if we consider an inner object that is more massive then the outer object. While the standard formalism incorrectly assumes that the orbit of the outer body is fixed in the invariable plane, and therefore the inner body's vertical angular momentum is constant, the quadrupole-level equations presented in Section 3 do not.

We compare the two formalisms in Figure 4. We consider the triple system PSR B1620−26 located near the core of the globular cluster M4. The inner binary contains a millisecond radio pulsar of $m_1 = 1.4M_\odot$ and a companion of $m_2 = 0.3M_\odot$ (McKenna & Lyne 1988). From Ford et al. (2000a), we adopt parameters for the outer perturber of $m_3 = 0.01\,M_\odot$ and $e_2 = 0.45$. The inner binary separation is $a_1 = 5$ AU while $a_2 = 50$ AU. We initialize the system with $i_{\rm tot} = 70°$, $g_1 = 120°$ and $g_2 = 0°$ and $e_1 = 0.5$. Note that the actual measured inner binary eccentricity is $e_1 \sim 0.045$, however in order to illustrate the difference we adopt a higher value. There is no logical reason to assume the observed eccentricity as the initial eccentricity when modeling the formation of a system, since different physical processes can contribute to eccentricity damping (for example, tidal friction (Hut 1980) and mass transfer (Sepinsky et al. 2010)). The initial mutual inclination of 70° corresponds to an inner and outer inclination with respect to the total angular momentum (parallel to z) of 6.75° and 63.25°, respectively. Remember that we consider the evolution of the system to quadrupole order for comparison, even though there is no apriori reason to truncate the evolution at this order, especially since $\epsilon_M = 0.036$. We have verified, however, that octupole order effects do not qualitatively change the behavior. This is because the outer companion mass is low, and hence the inner orbit does not exhibit large amplitude oscillations[5].

For the comparison, we do not compare the (constant) H_1 from the TPQ formalism to the (varying) H_1 of the correct formalism. Instead, we compare the (constant) H_1 from the TPQ formalism with $G_1 \cos i_{\rm tot}$, which is the vertical angular momentum that would be inferred in our formalism *if the outer orbit were instantaneously in the invariable plane*, as assumed in the TPQ formalism.

In Figure 4, the mutual inclination oscillates between 106.7° to 57.5°, and thus crosses 90°. These oscillations are mostly due to the oscillations of the outer orbit's inclination, while i_1 does not change by more than $\sim 1°$ in each cycle. Clearly, the outer orbit does not lie in the fixed invariable plane! Figure 4, bottom panel, shows $\sqrt{1-e_1^2}\cos i_{\rm tot}$, which, in the TPQ limit, is the vertical angular momentum of the inner body.

We can evaluate analytically the error introduced by the application of the TPQ formalism to this situation. We compare the vertical angular momentum (H_1) as calculated here to $H_1^{TPQ} = L_1\sqrt{1-e_1^2}\cos i = \text{const.}$. The relative error between the formalisms is $H_1^{TPQ}/H_1 - 1$. In Figure 5 we show the ratio between the inner orbit's vertical angular momentum in the TPQ limit (i.e., $H_1^{TPQ} = G_1 \cos i$) and equation (24) as a function of the total angular momentum ratio, G_1/G_2, for various inclinations. Note that this error can be calculated without evolving the system by using angular momentum conservation, equation (6). The TPQ limit is only valid when $G_1/G_2 \lesssim 10^{-4}$.

5.2. *Planetary Dynamics*

Recent measurements of the sky-projected angle between the orbits of several hot Jupiters and the spins

[5] Unlike the test particle octupole-level approximation (Lithwick & Naoz 2011; Katz et al. 2011), backreaction of the outer orbit may suppress the eccentric Kozai effect. We address this in further detail in Teyssandier et al. in Prep.

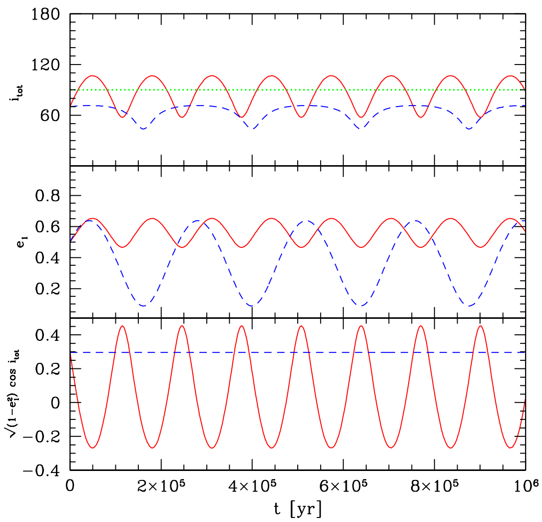

Fig. 4.— Comparison between the standard TPQ formalism (dashed blue lines) evolution and our methods (solid red lines) for the case of PSR B1620−26. Here the inner binary is a millisecond pulsar of mass $1.4\,M_\odot$ with a companion of $m_2 = 0.3 M_\odot$, and the outer body has mass $m_3 = 0.01 M_\odot$. The inner orbit has $a_1 = 5$ AU and the outer orbit has $a_2 = 50$ AU (Ford et al. 2000a). The initial eccentricities are $e_1 = 0.5$ and $e_2 = 0.45$ and the initial relative inclination $i_{\rm tot} = 70^\circ$. The thin horizontal line in the top panel marks the 90° boundary, separating prograde and retrograde orbits. The initial mutual inclination of 70° corresponds to an inner and outer inclination with respect to the total angular momentum (parallel to z) of 6.75° and 63.25°, respectively. The argument of pericenter of the inner orbit is initially set 120°, while the outer orbit's is set to zero. We consider, from top to bottom, the mutual inclination $i_{\rm tot}$, the inner orbit's eccentricity and $\sqrt{1-e_1^2}\cos i_{\rm tot}$, which the standard formalism assumes to be constant (dashed line).

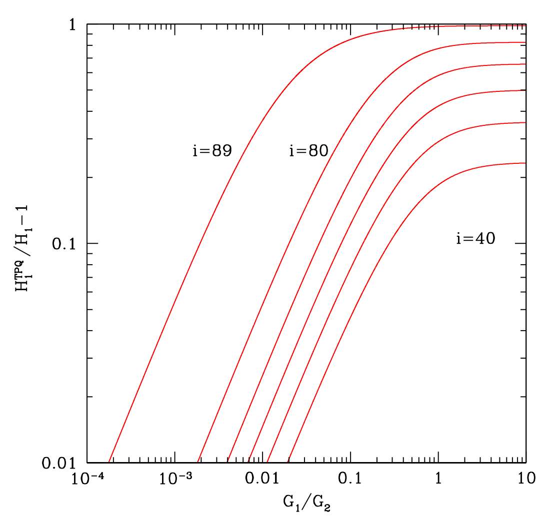

Fig. 5.— The ratio between the correct, changing z-component angular momentum, H_1, and the assumption often used in the literature which is $H_1^{TPQ} = G_1 \cos i$. This ratio was calculated analytically for various total angular momentum ratios, G_1/G_2, and inclinations. The curves, from bottom to top, have $i = 40, 50, 60, 70, 80$ and 89 degrees.

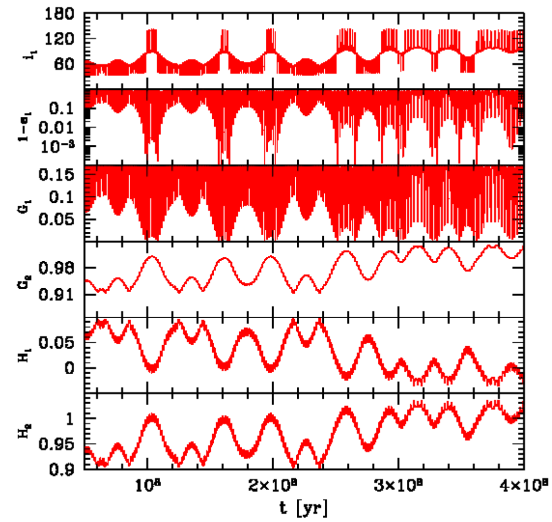

Fig. 6.— Evolution of a planetary system with $m_1 = 1\,M_\odot$, $m_2 = 1\,M_J$ and $m_3 = 2\,M_J$, with $a_1 = 4$ AU and $a_2 = 45$ AU. We initialize the system at $t = 0$ with $e_1 = 0.01$, $e_2 = 0.6$, $g_1 = 180^\circ$, $g_2 = 0^\circ$ and $i_{\rm tot} = 67^\circ$. For these initial conditions $i_1 = 57.92^\circ$ and $i_2 = 9.08^\circ$. The z-components of the orbital angular momenta, H_1 and H_2, are shown normalized to the total angular momentum of each orbit. The inner orbit flips quasi-periodically between prograde ($i_1 < 90^\circ$) and retrograde ($i_1 > 90^\circ$).

of their host stars have shown that roughly one in four is retrograde (Gaudi & Winn 2007; Triaud et al. 2010). If these planets migrated in from much larger distances through their interaction with the protoplanetary disk (Lin & Papaloizou 1986; Masset & Papaloizou 2003), their orbits should have low eccentricities and inclinations[6]. Disk migration scenarios therefore have difficulty accounting for the observed retrograde hot Jupiter orbits. An alternative migration scenario that can account for the retrograde orbits is the secular interaction between a planet and a binary stellar companion (Wu & Murray 2003; Fabrycky & Tremaine 2007; Wu et al. 2007; Takeda et al. 2008; Correia et al. 2011). For a very distant companion ($\epsilon_M \ll 1$) the quadrupole test-particle approximation applies, and $\sqrt{1-e_1^2}\cos i_1$ is nearly constant. Although this forbids orbits that are truly retrograde (with respect to the total angular momentum of the system), if the inner orbit begins highly inclined relative to the outer star's orbit and aligned with the spin of the inner star, then the star-planet spin-orbit angle can change by more than 90° during the secular evolution of the system, producing apparently retrograde orbits (Fabrycky & Tremaine 2007; Correia et al. 2011). Nonetheless, a difficulty with this "stellar Kozai" mechanism is that even with the most optimistic assumptions it can only produce $\lesssim 10\%$ of hot Jupiters (Wu et al. 2007).

Naoz et al. (2011) considered planet-planet secular interactions as a possible source of retrograde hot Jupiters. In this situation ϵ_M is not small, requiring computation

[6] This assumption can be invalid if there are significant magnetic interactions between the star and the protoplanetary disk (Lai et al. 2010) or if there is an episode of planet-planet scattering following planet formation (Chatterjee et al. 2008; Nagasawa et al. 2008) see also Merritt et al. (2009).

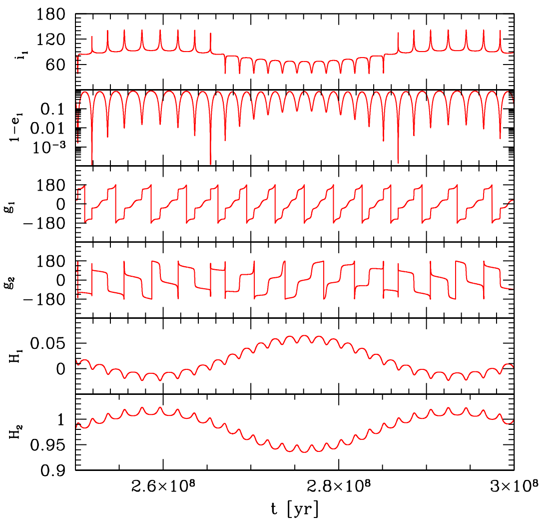

FIG. 7.— Zoom-in on part of the evolution of the point-mass planetary system in Figure 6. In this zoom-in, we can see that flips in the inner orbit—i_1 crossing 90°—are associated with excursions to very high eccentricity.

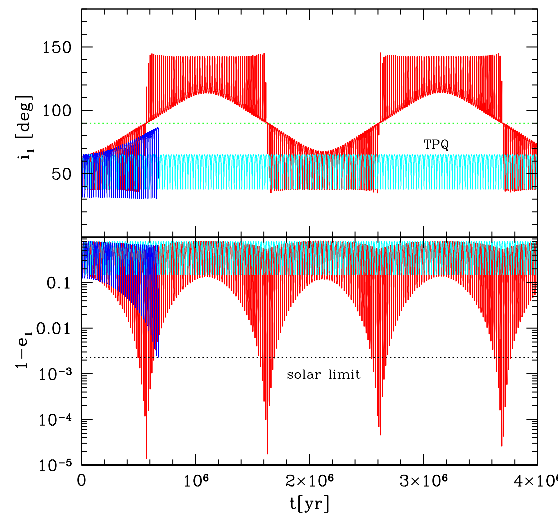

FIG. 8.— Evolution of an asteroid due to Jupiter secular gravitational perturbations (Kozai 1962). We consider $m_1 = 1\,M_\odot$, $m_2 \to 0$ and $m_3 = 1$ M$_J$, with $a_1 = 2$ AU and $a_2 = 5$ AU. We initialize the system at $t = 0$ with $e_1 = 0.2$, $e_2 = 0.05$, $g_1 = g_2 = 0°$ and $i_{\rm tot} = 65°$. We show the TPQ limit evolution (cyan lines) and the octupole equations (red lines). The thin horizontal dotted line in the top panel marks the 90° boundary, separating prograde and retrograde orbits. The inner orbit flips quasi-periodically between prograde ($i_1 < 90°$) and retrograde ($i_1 > 90°$). We also show the result of an N-body simulation (blue lines). The thin horizontal dotted line in the bottom panel marks the solar radius as $1 - e_1 = R_\odot/a_1$.

of the octupole-level secular dynamics. In Figures 6 and 7 we show the evolution of a representative configuration, where $m_1 = 1$ M$_\odot$, $m_2 = 1\,M_J$ and $m_3 = 2\,M_J$, with $a_1 = 4$ AU and $a_2 = 45$ AU. For this configuration, $\epsilon_M = 0.083$. Flips of the inner orbit are associated with evolution to very high eccentricity (see Figure 7).

5.3. *Solar System Dynamics*

Kozai (1962) studied the dynamical evolution of an asteroid due to Jupiter's secular perturbations. He assumed that Jupiter's eccentricity is strictly zero. However, Jupiter's eccentricity is ~ 0.05, and thus studying the evolution of a test particle in the asteroid belt ($a_1 \sim 2-3$ AU) places the evolution in a regime where the eccentric Kozai-Lidov effect could be significant, with $\epsilon_M = \epsilon = 0.03$ (Lithwick & Naoz 2011; Katz et al. 2011).

We considered the evolution of asteroid at 2 AU (assumed to be a test particle) due to Jupiter at 5 AU with eccentricity of $e_2 = 0.05$. We set $e_1 = 0.2$, $i_{\rm tot} = 65°$ and $g_1 = g_2 = 0°$ initially. The asteroid is a test particle and therefore $i_1 \approx i_{\rm tot}$. In Figure 8 we compare the evolution of an asteroid using the TPQ limit (e.g., Kozai 1962; Thomas & Morbidelli 1996; Kinoshita & Nakai 2007) and the octupole-level evolution discussed here. For this value of ϵ, the eccentric Kozai-Lidov effect significantly alters the evolution of the asteroid, even driving it to such high inclination that the orbit becomes retrograde. Though we deal only with point masses in this work, note that the eccentricity is so high that the inner orbit's pericenter lies well within the sun.

The value of ϵ here is mainly due to the relative high α in the problem. The system is very packed which raise questions with regards to the actual validity of the approximation in that regime. In fact, such high eccentricities drive the pericenter of the asteroid to collide with the sun and the apo-center of the asteroid to approach about 1 AU from Jupiter's orbit. To address this question we run a N-body simulation using `Mercury` software package (Chambers & Migliorini 1997). We used both Bulirsch-Stoer and symplectic integrators (Wisdom & Holman 1991). The results are depicted at Figure 8, which show that the TPQ limit is indeed in adequate for the system. In addition the octupole–level approximation has some deviations from the direct N-body integration, and does not follow the direct integration results in the high eccentricity regime. Note that the evolution of the asteroid in the direct integration had resulted with a collision with the Sun. In reality, highly eccentric asteroids do not live very long in the solar system, also due to planet-crossing. We also note that the assuming zero eccentricity for Jupiter results in consistent results (we tested the case for $a_1 = 2$ AU) between the secular evolution and the direct integration (Thomas & Morbidelli 1996, see also). Note also that Kozai mentions that the TPQ limit may not be correct, both due to the relatively large value of α and the non-negligable eccentricity of Jupiter, but proceeds anyway with the theory because it is analytically tractable.

5.4. *Triple Stars*

The evolution of triple stars has been studied by many authors using the standard (TPQ) formalism (e.g., Mazeh & Shaham 1979; Eggleton et al. 1998; Kiseleva et al. 1998; Mikkola & Tanikawa 1998; Eggleton & Kiseleva-Eggleton 2001; Fabrycky & Tremaine 2007; Perets & Fabrycky 2009). In some cases the corrected formalism derived here can give rise to qualitatively different results. We show that some of the previous studies should be repeated in order to account for the correct dynamical evolution, and give one example where the eccentric Kozi-Lidov mechanism

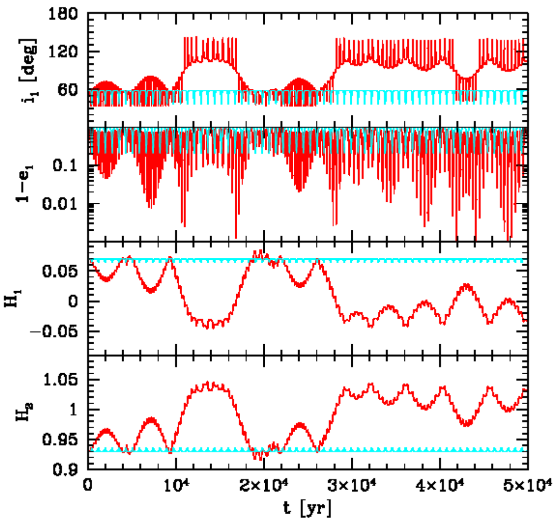

FIG. 9.— An example of dramatically different evolution between the quadruple and octupole approximations for a triple-star system. The system has $m_1 = 1M_\odot$, $m_2 = 0.1M_\odot$ and $m_3 = 0.4M_\odot$, with $a_1 = 2$ AU and $a_2 = 11$ AU. We initialize the system with $e_1 = 0.01$, $e_2 = 0.6$, $g_1 = 145°$, $g_2 = 0°$ and $i_{\rm tot} = 65°$ according to Fabrycky & Tremaine (2007) Monte-Carlo simulations. For these initial conditions $i_1 = 58.1°$ and $i_2 = 6.9°$. We show both the (correct) quadrupole-level evolution (light-blue lines) and the octupole-level evolution (red lines). H_1 and H_2, the z-components of the angular momenta of the orbits, are normalized to the total angular momentum. Note that the octupole-level evolution produces periodic transitions from prograde to retrograde inner orbits (relative to the total angular momentum), while at the quadrupole-level the inner orbit remains prograde.

dramatically changes the evolution.

Fabrycky & Tremaine (2007) studied the distribution of triple star properties using Monte-Carlo simulations. We choose a particular system from their triple-star suite of simulations to illustrate how the dynamics including the octupole order can be qualitatively different from what would be seen at quadrupole order, when ϵ_M is not negligible[7]. We adopt the following parameters: $m_1 = 1M_\odot$, $m_2 = 0.1M_\odot$ and $m_3 = 0.4M_\odot$, $a_1 = 2$ AU and $a_2 = 11$ AU. We initialize the system at $t = 0$ with $e_1 = 0.01$, $e_2 = 0.6$, $g_1 = 145°$, $g_2 = 0°$ and $i_{\rm tot} = 65°$, corresponding to $i_1 = 58.1°$, $i_2 = 6.9°$ and $\epsilon_M = 0.14$. The evolution of the system is shown in Figure 9. At octupole order, the inclination of the inner orbit oscillates between about 40° and 140°, often becoming retrograde (relative to the total angular momentum), while the quadrupole-order behavior is very different and the inner orbit remains always prograde. The octupole-order treatment also gives rise to much higher eccentricities (Krymolowski & Mazeh 1999; Ford et al. 2000b). The evolution shown in Figure 9 is for point-mass stars; in reality, these high-eccentricity excursions would actually drive the inner binary to its Roche limit, leading to mass transfer.

The possibility of forming blue stragglers through secular interactions in triple star systems has been suggested by Perets & Fabrycky (2009) and Geller et al. (2011). As shown in Krymolowski & Mazeh (1999); Ford et al. (2000b) and in the example above the minimum pericenter that the inner binary can reach can vary (from few percents to orders of magnitude depending on ϵ_M). Thus, it suggests that the correct formalism may increase the likelihood of such a formation mechanism for blue stragglers.

For many years CH Cygni was considered to be an interesting triple candidate because it exhibits two clear distinguishable periods (e.g. Donnison & Mikulskis 1995; Skopal et al. 1998; Mikkola & Tanikawa 1998; Hinkle et al. 1993). However, a triple system model based on the TPQ Kozai mechanism (Mikkola & Tanikawa 1998) did not reproduce the observed masses of the system (Hinkle et al. 1993, 2009). Applying the corrected formalism in this paper to the system parameters derived in Mikkola & Tanikawa (1998) gives a very different evolution than in the TPQ formalism[8]. Therefore, it seems likely that an analysis based on the formalism discussed in this paper would give a significantly different fit. In Figure 10 we illustrate the differences between the TPQ, correct quadrupole, and octupole evolution of the system. The best-fit parameters of the system from Mikkola & Tanikawa (1998) are as follows: $m_1 = 3.51M_\odot$, $m_2 = 0.5M_\odot$ and $m_3 = 0.909M_\odot$, $a_1 = 0.05$ AU and $a_2 = 0.21$ AU. We initialize the system at $t = 0$ with $e_1 = 0.32$, $e_2 = 0.6$, $g_1 = 145°$, $g_2 = 0°$ and $i_{\rm tot} = 72°$, corresponding to $i_1 = 57.01°$, $i_2 = 14.98°$ and $\epsilon_M = 0.14$. We allowed for a freedom in our choice of e_2, g_1, g_2 and $i_{\rm tot}$ since the the best fit was found using the TPQ limit, at which e_2 is fixed. Note that the choice of the inner eccentricity does not strongly influence the evolution while the choice of the outer orbit's eccentricity does.

6. COMPARISON WITH PREVIOUS STUDIES

Kozai (1962) studied the motion of an inclined asteroid due to perturbations from Jupiter. He derived the Hamiltonian for this system to high order in α , assuming a circular orbit for Jupiter. He then truncated the expansion at quadrupole order in α to derive the secular evolution equations for the asteroid; his equations thus correctly describe the test-particle quadrupole (TPQ) limit. However, Kozai's equations were later applied incorrectly in other studies. Kozai's equations imply that $H_1 = $ const, but outside the test-particle limit in the quadrupole-order evolution we have seen that H_1 is no longer constant. Moreover, even when ϵ_M is small, the octupole-order effects can lead to qualitatively different orbits than predicted at quadrupole order.

6.1. *Elimination of the Nodes and the Problem in Previous Quadrupole-Level Treatments*

Since the total angular momentum is conserved, the ascending nodes relative to the invariable plane follow a simple relation, $h_1(t) = h_2(t) - \pi$ (see Appendix D). If one inserts this relation into the Hamiltonian, which only depends on $h_1 - h_2$, the resulting "simplified" Hamiltonian is independent of h_1 and h_2. One might be tempted to conclude that the conjugate momenta H_1 and H_2 are

[7] The extent of the Fabrycky & Tremaine (2007) phase space over which ϵ_M is not negligible requires further investigation.

[8] Mikkola & Tanikawa (1998) also found somewhat different set of parameter when producing a fit for data set with less weight for the data of 1983 due to large noise in the active phase of the system.

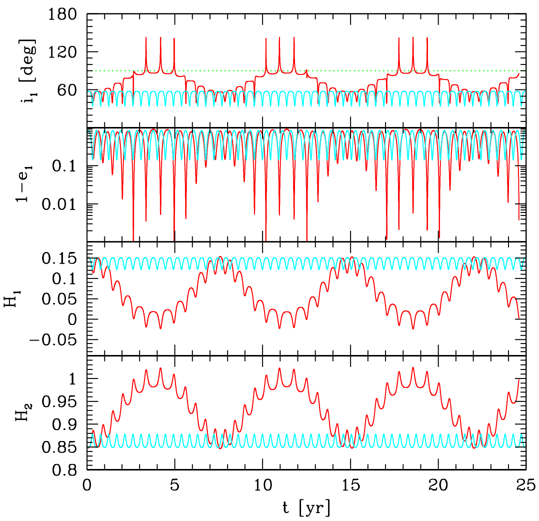

FIG. 10.— An example of dramatically different evolution between the quadruple and octupole approximations for a triple star system representing the best-fit parameters from the Mikkola & Tanikawa (1998) analysis of CH Cygni. The system has $m_1 = 3.51M_\odot$, $m_2 = 0.5M_\odot$ and $m_3 = 0.909M_\odot$, with $a_1 = 0.05$ AU and $a_2 = 0.21$ AU. We initialize the system with $e_1 = 0.32$, $e_2 = 0.6$, $g_1 = 145°$, $g_2 = 0°$ and $i_{\rm tot} = 72°$. For these initial conditions $i_1 = 57.02°$ and $i_2 = 14.98°$. We show both the (non-TPQ) quadrupole-level evolution (light-blue lines) and the octupole-level evolution (red lines). H_1 and H_2, the z-components of the angular momenta of the orbits, are normalized to the total angular momentum. Note that the octupole-level evolution produces periodic transitions from prograde to retrograde inner orbits (relative to the total angular momentum), while at the quadrupole-level the inner orbit remains prograde. To avoid clatter in the figure we omitted the TPQ limit result, we note however, that the evolution of the inclination and eccentricity are similar to the general quadrupole-level approximation, but with constant $H_{1,2}$.

constants of the motion. However, that conclusion is false. This incorrect argument has been made by a number of authors[9].

We now show that H_1 and H_2 are constant only in the TPQ limit. Outside of that limit they are not constant, and taking them to be constant breaks the conservation of the total angular momentum of the system. The quadrupole-level Hamiltonian (Eq. 19) depends on the angles as follows: $\mathcal{H} = \mathcal{H}(g_1, h_1 - h_2)$; that is, it is independent of g_2 and only depends on the difference between h_1 and h_2. Using the incorrect elimination of the nodes discussed above, one would conclude that H_1 is constant. Also, because the Hamiltonian is independent of g_2, $G_2 =$ const. From the geometric relation in equation (7), $H_1 = (G_{\rm tot}^2 + G_1^2 - G_2^2)/(2G_{\rm tot})$, and the constantcy of the total angular momentum, $G_{\rm tot}$, we have that $G_1 =$ const, but this is inconsistent with the dependence of $\mathcal{H}$ on g_1. The error comes from assuming that $H_1 =$ const; in fact, at the quadrupole level, we have

$$\dot{H}_1 = \frac{G_1}{G_{\rm tot}}\dot{G}_1, \tag{37}$$

which is consistent with both the geometric relation in equation (7) and the dynamical equations (24) and (25). When $G_1/G_{\rm tot} \to 0$, or, equivalently $G_2 \gg G_1$, $\dot{H}_1 \to 0$, and the TPQ result is achieved[10]. But, this is precisely the TPQ limit, where the outer orbit dominates the angular momentum of the system and therefore lies in the invariable plane.

In general, using *dynamical* information about the system—in this case that angular momentum is conserved, implying that $\mathbf{G}_1 + \mathbf{G}_2 = \mathbf{G}_{\rm tot}$ at all times and therefore $h_1 - h_2 = \pi$—to simplify the Hamiltonian is not correct. The derivation of Hamilton's equations relies on the possibility of making *arbitrary* variations of the system's trajectory, and such simplifications restrict the allowed variations to those which respect the dynamical constraints. Once Hamilton's equations are employed to derive equations of motion for the system, however, dynamical information can be employed to simplify these equations. See Appendix E for further discussion of this point.

In our particular case, quations of motion for components of the system that do not involve partial derivatives with respect to h_1 or h_2 will not be affected by the node-elimination substitution. For this reason, it is correct to derive equations of motion for all components *except* for H_1 and H_2 from the node-eliminated Hamiltonian; expressions for $\dot{H}_1$ and $\dot{H}_2$ can then be derived from conservation of angular momentum. This approach has been employed in at least one computer code for octupole evolution, though the discussion in the corresponding paper incorrectly eliminates the nodes in the Hamiltonian (Ford et al. 2000b).

In some later studies, (Sidlichovsky 1983; Innanen et al. 1997; Kiseleva et al. 1998; Eggleton et al. 1998; Mikkola & Tanikawa 1998; Kinoshita & Nakai 1999; Eggleton & Kiseleva-Eggleton 2001; Wu & Murray 2003; Valtonen & Karttunen 2006; Fabrycky & Tremaine 2007; Wu et al. 2007; Zdziarski et al. 2007; Perets & Fabrycky 2009), the assumption that $H_1 =$ const was built into the calculations of quadrupole-level secular evolution for various astrophysical systems, even when the condition $G_2 \gg G_1$ was not satisfied. Moreover many previous studies (Sidlichovsky 1983; Innanen et al. 1997; Kiseleva et al. 1998; Eggleton et al. 1998; Mikkola & Tanikawa 1998; Kinoshita & Nakai 1999; Eggleton & Kiseleva-Eggleton 2001; Wu & Murray 2003; Valtonen & Karttunen 2006; Fabrycky & Tremaine 2007; Wu et al. 2007; Perets & Fabrycky 2009) simply set $i_2 = 0$. In fact, given the mutual inclination i, the inner and outer inclinations i_1 and i_2 are set by the conservation of total angular momentum [see equations (9) and (10)].

In Figure 5 we show the ratio between the inner orbit's vertical angular momentum in the TPQ limit ($H_1^{TPQ} = G_1 \cos i$) and the H_1 in the correct quadruple-level approximation from the derivation shown here as a function of the ratio G_1/G_2. From this figure, it is clear that the standard formalism is only valid in the TPQ limit, where $G_1/G_2 \lesssim 10^{-4}$ (depending slightly on the mutual

[9] For example, Kozai (1962, p. 592) incorrectly argues that "As the Hamiltonian F depends on h and h' as a combination $h - h'$, the variables h and h' can be eliminated from F by the relation (5). Therefore, H and H' are constant."

[10] This is true as long as $\dot{G}_1$ is not larger then $G_1/G_{\rm tot}$. However, $\dot{G}_1$ is proportional to C_2 which is in turn proportional to $1/G_2^3 \sim 1/G_{\rm tot}^3$. Therefore, $\dot{G}_1 G_1/G_{\rm tot} \to 0$ in the test-particle limit.

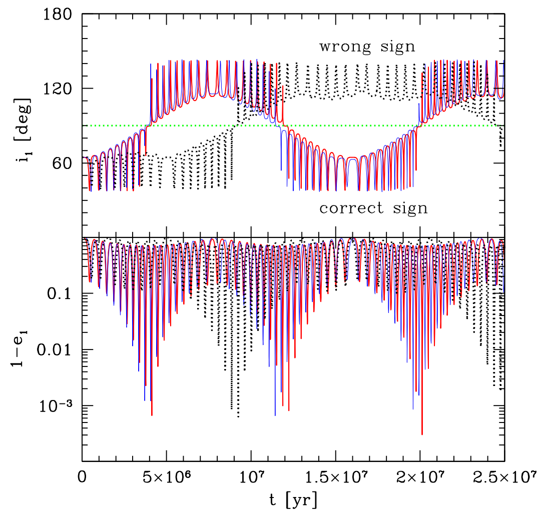

FIG. 11.— Comparison between direct integration and the octupole-level approximation. We consider both the wrong sign (i.e., $C_3 > 0$ for $m_1 > m_2$) and the correct sign (i.e., $C_3 < 0$ for $m_1 > m_2$) for the octupole term. The dotted curve represents the result from the integration with $C_3 > 0$, while the thick solid curve depicts the integration with the $C_3 < 0$. The thin curve, overlapping the thick curve shows the result of direct numerical integration of the three-body system using Burlish-Stoer integrator. We have used here the initial conditions in Figure 3.

inclination).

The quadrupole-level equations appearing commonly in the literature are correct only in the limit of quadrupole-level test particle (Kozai 1962; Lidov 1962). Nevertheless, these equations have been applied outside this limit, to massive bodies in the inner binary in many studies in the literature (Sidlichovsky 1983; Innanen et al. 1997; Kiseleva et al. 1998; Eggleton et al. 1998; Mikkola & Tanikawa 1998; Kinoshita & Nakai 1999; Eggleton & Kiseleva-Eggleton 2001; Wu & Murray 2003; Valtonen & Karttunen 2006; Fabrycky & Tremaine 2007; Wu et al. 2007; Zdziarski et al. 2007; Perets & Fabrycky 2009).

6.2. *Octupole-level Approximation and Truncation to Quadrupole*

The octupole-level Hamiltonian and equations of motion were previously derived by Harrington (1968, 1969); Sidlichovsky (1983); Marchal (1990); Krymolowski & Mazeh (1999); Ford et al. (2000b); Blaes et al. (2002) and Lee & Peale (2003b). Most of the equations of motion can be derived correctly when applying the elimination of the nodes—only the $\dot{H}_1$ and $\dot{H}_2$ equations are affected. These authors calculated the time evolution of the inclinations (i.e. H_1 and H_2 from the *total* (conserved) angular momentum, and thus avoided the problem that arises when eliminating the nodes from the Hamiltonian. In appendix C we show the complete set of equations of motion for the octupole-level approximation, derived from a correct Hamiltonian, including the nodal terms.

As previously noted by Blaes et al. (2002, equation 24), Ford et al. (2000b) introduced a sign error in the octupole coefficient, C_3, which was later corrected in Ford et al. (2004). The same sign error also exists in Marchal (1990), Sidlichovsky (1983, eq. 17), Krymolowski & Mazeh (1999, eq. 6b) and in Laskar & Boué (2010, their equation for $\mathcal{F}_3^{(0,0)}$). We note that Thompson (2010) and Lee & Peale (2003b,a) used the correct sign. To settle this point we use the direct N-body simulation shown in Figure 3 and compare it to the integration of the octupole-level approximation equations (see Appendix C) with and without the minus sign. We show our results in Figure 11 and find that C_3 as defined here [eq. (C1)] and in Blaes et al. (2002) is in agreement with the direct N-body, and thus indeed this is the correct sign. We note that this sign error can be easily resolved if $g_2 \rightarrow g_2 + \pi$ (which may imply to the source of this confusion).

As displayed here the octupole-level approximation gives rise to a qualitatively different evolutionary behavior for cases where ϵ_M [see eq. (35)] is not negligible. We note that many previous studies applied the quadrupole-level approximation, which may lead to significantly different results (e.g., Mazeh & Shaham 1979; Quinn et al. 1990; Bailey et al. 1992; Innanen et al. 1997; Eggleton et al. 1998; Mikkola & Tanikawa 1998; Eggleton & Kiseleva-Eggleton 2001; Valtonen & Karttunen 2006; Fabrycky & Tremaine 2007; Wu et al. 2007; Zdziarski et al. 2007; Perets & Fabrycky 2009). Neglecting the octupole-level approximation can cause changes in the dynamics varying from a few percent to completely different qualitative behavior.

Some other derivations of octupole-order equations of motion dealt with the secular dynamics in a general way, without using Hamiltonian perturbation theory or elimination of the nodes (Farago & Laskar 2010; Laskar & Boué 2010; Mardling 2010; Katz & Dong 2011). In these works there were no references to the discrepancy between these derivations and the previous studies. Also, note that the results of Holman et al. (1997) are based on a direct N-body integration, and thus are not subject to the errors mentioned above.

6.3. *Comparisons with Specific Papers*

Many previous studies applied the test particle quadrupole-level equations (TPQ) in various astrophysical settings, even in situations where those equations are not strictly applicable. We address some of these studies here in more detail.

As discussed above, Kozai (1962) studied the effect of Jupiter perturbations to an inclined asteroid. He specifically assumed that Jupiter's eccentricity is zero. However, as shown in Figure 8, taking into account Jupiters eccentricity (~ 0.05), produces a dramatically different evolutionary behavior, including retrograde orbits for the asteroid. Thomas & Morbidelli (1996) applied the same limit to the asteroid-Jupiter setting (see for example their Figure 2 for $a_1 = 3$ AU, where they explicitly show the (wrong) vertical angular momentum conservation). Kinoshita & Nakai (2007) developed an analytical solution for the TPQ limit (see also Kinoshita & Nakai 1991, 1999), however, they have applied it to the asteroid-Jupiter system, again assuming zero eccentricity for Jupiter. Note that all these works used the secular approximation for somewhat high α, with $a_1 = 3$ AU; for this value the secular approximation breaks when the asteroid's apo-center crosses Jupiter's orbit. In addition, Kinoshita & Nakai (2007) assumed $a_1 = 1.814$ AU, and as shown in the numerical integration in Figure 8, using the secular approximation for even this large value of α, may result in in-accurate evolutionary behavior.

The Kozai-Lidov mechanism has been applied to the study of the outer solar system. Kinoshita & Nakai (2007) also applied their analytical solution to Neptune's outer satellite Laomedeia. This system has $\epsilon \to 0$ and thus the TPQ limit there is justified. In addition, Perets & Naoz (2009) have studied the evolution of binary minor planets in the frame work of TPQ. In this problem $\epsilon \to 0$ and thus the results presented there and later applied in Naoz et al. (2010) and Grundy et al. (2011) justify the application of the TPQ limit.

Lidov & Ziglin (1976, sections 3–4) also solved analytically the quadrapole-level approximation but, unlike Kinoshita & Nakai (2007), they did not restrict themselves to the TPQ limit, and used the total angular momentum conservation law in order to calculate the inclinations. Later, Mazeh & Shaham (1979) also did not restrict their derivation to the TPQ limit (their eqs. A1-A8), and allowed for small eccentricities and inclinations of the outer body.

Kiseleva et al. (1998) and Eggleton & Kiseleva-Eggleton (2001) studied the Algol triple system (Lestrade et al. 1993) using the TPQ equations. The TPQ equations were also used in the paper that introduced the influential KCTF mechanism (Mazeh & Shaham 1979; Eggleton et al. 1998). Figure 12 compares the evolution computed in the (incorrect) "standard" quadrupole formalism, the correct quadrupole formalism, and the octupole-level formalism applied to the Algol system of Kiseleva et al. (1998). The correct quadrupole formalism decreases the minimum value of $1 - e_1$ by almost a factor of 2 relative to the previous "standard" formalism. The reduced pericenter distance would strongly increase the effects of tidal friction (not included here), which may lead to rapid circularization of the inner orbit. The octupole-level computation decreases the minimum pericenter distance by a further 40%.

Mikkola & Tanikawa (1998) analyzed the system CH Cygni, assuming that it was a triple system[11]. They used the TPQ limit to model the system and derive its orbital parameters. They found that the best fitted model has in fact comparable masses for the inner and outer orbits, and shorter periods for the outer and inner orbits then found in the literature using spectroscopy and interferometry (e.g., Hinkle et al. 2009; Mikołajewska et al. 2010). As we showed in Figure 10 using the octapule-level equations we found a dramatically different evolutionary path for their best-fit parameters then the quadrapole-level, and also the TPQ. It may even be that the triple model for this system cannot be excluded based on Mikkola & Tanikawa (1998) results, since the evolution is so different under the corrected formalism. We suggest that this work should be repeated.

Wu & Murray (2003), Wu et al. (2007), Fabrycky & Tremaine (2007) and Correia et al. (2011) studied the evolution of a Jupiter-mass planet in stellar binaries in the framework of KCTF. The case of HD 80606b (Wu & Murray (2003); Fabrycky & Tremaine (2007, their Fig. 1) and Correia et al. (2011, also Fig. 1)) was considered with an outer stellar companion at 1000 AU, and thus even if it that companion is assumed to be eccentric, ϵ_M is negligible, and the system is well described TPQ equations. However, the statistical distribution for closer stellar binaries in Wu et al. (2007) and Fabrycky & Tremaine (2007) is only valid in the approximation where the outer orbit's eccentricity is zero. In fact for the eccentric and packed systems considered in those studies, ϵ_M is not negligible, and the the octupole-level approximation results in dramatically different behavior (see §4.3). The same dramatic difference in behavior also exists in the analysis of triple stars (e.g., Fabrycky & Tremaine 2007; Perets & Fabrycky 2009), and thus we suggest that these studies should be repeated[12].

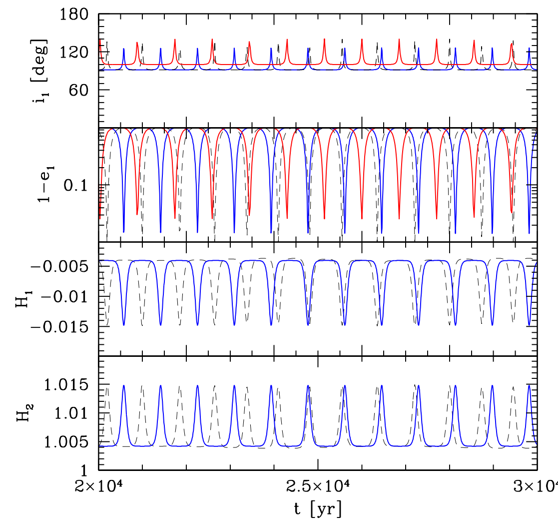

FIG. 12.— The time evolution of the system Algol (Eggleton et al. 1998), with $(m_1, m_2, m_3) = (2.5, 2, 1.7)$ $M_\odot$. The inner orbit has $a_1 = 0.095$ AU and the outer orbit has $a_2 = 2.777$ AU. The initial eccentricities are $e_1 = 0.01$ and $e_2 = 0.23$ and the initial relative inclination $i = 100°$. The z-components of the inner and outer orbital angular momentum, H_1 and H_2 are normalized to the total angular momentum. The initial mutual inclination of 100° corresponds to inner- and outer-orbit inclinations of 91.6° and 8.4°, respectively. We consider the (corrected) quadrupole-level evolution (blue lines), octupole-level evolution (dashed lines) and also the standard (incorrect) quadrupole-level evolution. In the latter we have assumed, as in previous papers, that $i_{\rm tot} = i_1$, which results in the discrepancy between the inclination values.

7. CONCLUSIONS

We have shown that the "standard" Kozai formalism had an error in the implementation of Hamiltonian mechanics (Kozai 1962; Lidov 1962). Correcting the formalism we find that the z-components of the *both* the inner and outer orbits' angular momenta in general change with time at both the quadrupole and octupole level. The conservation of the inner orbit's z-component

[11] It was recently claimed by Hinkle et al. (2009); Mikołajewska et al. (2010) that this system is in fact not a triple.

[12] We note that if ϵ_M is too big the system becomes unstable, which may suggests that the phase space for which the eccentric Kozai is significant may be somewhat limited.

of the angular momentum (the famous $\sqrt{1-e_1^2}\cos i =$ constant) only holds in the quadrupole-level *test particle* approximation. We have explained in details the source of the error in previous derivations (§6.1).

We have re-derived the secular evolution equations for triple systems using Hamiltonian perturbation theory to the octupole-level of approximation (Section 2 and Appendix A, 4 and Appendix C). We also showed that one can use the simplified Hamiltonian found in the literature (e.g., Ford et al. 2000b) as long as the equations of motion for the inclinations are calculated explicitly from the total angular momentum.

The correction shown here has important implications to the evolution of triple systems. We discuses a few interesting implications in Section 5. We showed that already at the quadrupole-level approximation the explicit assumption that the vertical angular momentum is constant can lead to erroneous results, see for example Figure 4. In this Figure we showed that far from the test particle limit in the quadrupole-level one can already find a significant difference in the evolutionary behavior. The corrected dynamics converges with the test particle in the limit where $G_2/G_1 < 10^{-4}$, (see Figure 5). During the evolution the inclination and eccentricity of both orbits oscillate. We show in Appendix B that at the quadrupole level of approximation, the inner eccentricity and the mutual inclination have a well defined maximum and minimum. At the test particle limit these values converge to the critical inclination ($39.2^\circ \geq i_0 \leq 140.8^\circ$) for large oscillatory amplitudes.

We have derived the complete set of equations for the octupole-level evolution, including the explicit equations of motion for the evolution of the inclinations and the z-component of the angular momentum of the inner and outer orbits.

The most notable outcome of the results presented here happens in the octupole-level of approximation, when the inner orbit flips from prograde to retrograde with respect to the total angular momentum (we call this flip the "eccentric Kozai mechanism"). We point out that, Krymolowski & Mazeh (1999); Ford et al. (2000b); Blaes et al. (2002); Lee & Peale (2003b) and Laskar & Boué (2010) had the correct equations of motion, and could, in principle, have observed this phenomena. However, it seems that the assumption of a constant vertical angular momentum was built into the community understanding of Kozai mechanism that the eccentric Kozai-Lidov effect was overlooked.

In Naoz et al. (2011) we suggested that this effect may play an important role in the formation mechanism of retrograde Hot Jupiters. There we showed the importance of this effect in a verity of planetary and stellar triple systems. For some examples see Figures 6–9, and Naoz et al. (2011) where we specifically discussed the evolution of two planet systems, triple stars and asteroids due to gravitational perturbations from Jupiter. We also compared our derivation with direct N-body integration and illustrated the same qualitative evolution. We also emphasized the importance of higher-orders approximations, where ϵ_M is significant.

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REFERENCES

Bailey, M. E., Chambers, J. E., & Hahn, G. 1992, A&A, 257, 315
Blaes, O., Lee, M. H., & Socrates, A. 2002, ApJ, 578, 775, arXiv:astro-ph/0203370
Brouwer, D. 1959, AJ, 64, 378
Carruba, V., Burns, J. A., Nicholson, P. D., & Gladman, B. J. 2002, Icarus, 158, 434
Chambers, J. E., & Migliorini, F. 1997, in Bulletin of the American Astronomical Society, Vol. 29, AAS/Division for Planetary Sciences Meeting Abstracts #29, 1024–+
Chatterjee, S., Ford, E. B., Matsumura, S., & Rasio, F. A. 2008, ApJ, 686, 580, arXiv:astro-ph/0703166
Correia, A. C. M., Laskar, J., Farago, F., & Boué, G. 2011, ArXiv e-prints, 1107.0736
Ćuk, M., & Burns, J. A. 2004, AJ, 128, 2518, arXiv:astro-ph/0408119
Donnison, J. R., & Mikulskis, D. F. 1995, MNRAS, 272, 1
Eggleton, P. P., Kiseleva, L. G., & Hut, P. 1998, ApJ, 499, 853, arXiv:astro-ph/9801246
Eggleton, P. P., & Kiseleva-Eggleton, L. 2001, ApJ, 562, 1012, arXiv:astro-ph/0104126
Eggleton, P. P., Kisseleva-Eggleton, L., & Dearborn, X. 2007, in IAU Symposium, Vol. 240, IAU Symposium, ed. W. I. Hartkopf, E. F. Guinan, & P. Harmanec, 347–355
Fabrycky, D., & Tremaine, S. 2007, ApJ, 669, 1298, 0705.4285
Farago, F., & Laskar, J. 2010, MNRAS, 401, 1189, 0909.2287
Ford, E. B., Joshi, K. J., Rasio, F. A., & Zbarsky, B. 2000a, ApJ, 528, 336, arXiv:astro-ph/9905347
Ford, E. B., Kozinsky, B., & Rasio, F. A. 2000b, ApJ, 535, 385
——. 2004, ApJ, 605, 966
Gaudi, B. S., & Winn, J. N. 2007, ApJ, 655, 550, arXiv:astro-ph/0608071
Geller, A. M., Hurley, J. R., & Mathieu, R. D. 2011, in Bulletin of the American Astronomical Society, Vol. 43, American Astronomical Society Meeting Abstracts #217, 327.02–+
Goldstein, H. 1950, Classical mechanics, ed. Goldstein, H.
Grundy, W. M. et al. 2011, Icarus, 213, 678, 1103.2751
Harrington, R. S. 1968, AJ, 73, 190
——. 1969, Celestial Mechanics, 1, 200
Hinkle, K. H., Fekel, F. C., Johnson, D. S., & Scharlach, W. W. G. 1993, AJ, 105, 1074
Hinkle, K. H., Fekel, F. C., & Joyce, R. R. 2009, ApJ, 692, 1360, 0811.0631
Holman, M., Touma, J., & Tremaine, S. 1997, Nature, 386, 254
Hut, P. 1980, A&A, 92, 167
Innanen, K. A., Zheng, J. Q., Mikkola, S., & Valtonen, M. J. 1997, AJ, 113, 1915
Ivanova, N., Chaichenets, S., Fregeau, J., Heinke, C. O., Lombardi, J. C., & Woods, T. E. 2010, ApJ, 717, 948, 1001.1767
Jefferys, W. H., & Moser, J. 1966, AJ, 71, 568
Katz, B., & Dong, S. 2011, ArXiv e-prints, 1105.3953
Katz, B., Dong, S., & Malhotra, R. 2011, ArXiv e-prints, 1106.3340
Kinoshita, H., & Nakai, H. 1991, Celestial Mechanics and Dynamical Astronomy, 52, 293
——. 1999, Celestial Mechanics and Dynamical Astronomy, 75, 125
——. 2007, Celestial Mechanics and Dynamical Astronomy, 98, 67
Kiseleva, L. G., Eggleton, P. P., & Mikkola, S. 1998, MNRAS, 300, 292
Kozai, Y. 1962, AJ, 67, 591
Kozai, Y. 1979, in IAU Symposium, Vol. 81, Dynamics of the Solar System, ed. R. L. Duncombe, 231–236

Krymolowski, Y., & Mazeh, T. 1999, MNRAS, 304, 720
Lai, D., Foucart, F., & Lin, D. N. C. 2010, ArXiv e-prints, 1008.3148
Laskar, J., & Boué, G. 2010, A&A, 522, A60+, 1008.2947
Lee, M. H., & Peale, S. J. 2003a, ApJ, 597, 644
——. 2003b, ApJ, 592, 1201, arXiv:astro-ph/0304454
Lestrade, J.-F., Phillips, R. B., Hodges, M. W., & Preston, R. A. 1993, ApJ, 410, 808
Lidov, M. L. 1962, planss, 9, 719
Lidov, M. L., & Ziglin, S. L. 1976, Celestial Mechanics, 13, 471
Lin, D. N. C., & Papaloizou, J. 1986, ApJ, 309, 846
Lithwick, Y., & Naoz, S. 2011, ArXiv e-prints, 1106.3329
Marchal, C. 1990, The three-body problem, ed. Marchal, C.
Mardling, R. A. 2010, MNRAS, 407, 1048, 1001.4079
Masset, F. S., & Papaloizou, J. C. B. 2003, ApJ, 588, 494, arXiv:astro-ph/0301171
Mazeh, T., & Shaham, J. 1979, AA, 77, 145
McKenna, J., & Lyne, A. G. 1988, Nature, 336, 226
Merritt, D., Gualandris, A., & Mikkola, S. 2009, ApJ, 693, L35, 0812.4517
Mikkola, S., & Tanikawa, K. 1998, AJ, 116, 444
Mikołajewska, J., Balega, Y., Hofmann, K.-H., & Weigelt, G. 2010, MNRAS, 403, L21, 0910.5176
Miller, M. C., & Hamilton, D. P. 2002, ApJ, 576, 894, arXiv:astro-ph/0202298
Murray, C. D., & Dermott, S. F. 2000, Solar System Dynamics, ed. Murray, C. D. & Dermott, S. F.
Nagasawa, M., Ida, S., & Bessho, T. 2008, ApJ, 678, 498, 0801.1368
Naoz, S., Farr, W. M., Lithwick, Y., Rasio, F. A., & Teyssandier, J. 2011, Nature, 473, 187, 1011.2501
Naoz, S., Perets, H. B., & Ragozzine, D. 2010, ApJ, 719, 1775, 1001.2558
Nesvorný, D., Alvarellos, J. L. A., Dones, L., & Levison, H. F. 2003, AJ, 126, 398
Perets, H. B., & Fabrycky, D. C. 2009, ApJ, 697, 1048, 0901.4328
Perets, H. B., & Naoz, S. 2009, ApJ, 699, L17, 0809.2095
Pribulla, T., & Rucinski, S. M. 2006, AJ, 131, 2986, arXiv:astro-ph/0601610
Quinn, T., Tremaine, S., & Duncan, M. 1990, ApJ, 355, 667
Sepinsky, J. F., Willems, B., Kalogera, V., & Rasio, F. A. 2010, ApJ, 724, 546, 1005.0625
Sidlichovsky, M. 1983, Celestial Mechanics, 29, 295
Skopal, A., Bode, M. F., Lloyd, H. M., & Drechsel, H. 1998, A&A, 331, 224
Takeda, G., Kita, R., & Rasio, F. A. 2008, ApJ, 683, 1063, 0802.4088
Thomas, F., & Morbidelli, A. 1996, Celestial Mechanics and Dynamical Astronomy, 64, 209
Thompson, T. A. 2010, ArXiv e-prints, 1011.4322
Tokovinin, A. A. 1997, Astronomy Letters, 23, 727
Triaud, A. H. M. J. et al. 2010, A&A, 524, A25+, 1008.2353
Valtonen, M., & Karttunen, H. 2006, The Three-Body Problem, ed. Valtonen, M. & Karttunen, H.
Vashkov'yak, M. A. 1999, Astronomy Letters, 25, 476
Wen, L. 2003, ApJ, 598, 419, arXiv:astro-ph/0211492
Wisdom, J., & Holman, M. 1991, AJ, 102, 1528
Wu, Y., & Murray, N. 2003, ApJ, 589, 605, arXiv:astro-ph/0303010
Wu, Y., Murray, N. W., & Ramsahai, J. M. 2007, ApJ, 670, 820, 0706.0732
Zdziarski, A. A., Wen, L., & Gierliński, M. 2007, MNRAS, 377, 1006, arXiv:astro-ph/0701470

APPENDIX

THE SECULAR AVERAGING AT THE QUADRUPOLE LEVEL

We develop the complete quadrupole-level secular approximation in this section. As mentioned, the main difference between the derivation shown here and those of previous studies lies in the "elimination of nodes" (e.g., Kozai 1962; Jefferys & Moser 1966), which relates to the transition the *invariable plane* (e.g., Murray & Dermott 2000) coordinate system, where the total angular momentum lies along the z-axis.

Transformation to the Invariable Plane

We choose to work in a coordinate system where the total initial angular momentum of the system lies along the z axis (see Figure 2),; the x-y plane in this coordinate system is known as the *invariable plane* (e.g., Murray & Dermott 2000), and therefore we call this coordinate system the *invariable coordinate system*. We begin by expressing the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ each in a coordinate system where the periapse of the orbit is aligned with the x-axis and the orbit lies in the x-y plane, called the "orbital coordinate system," and then rotating each vector to the invariable coordinate system. The rotation that takes the position vector in the orbital coordinate system to the position in the invariable coordinate system is given by (see Murray & Dermott 2000, chapter 2.8, and Figure 2.14 for more details)

$$\mathbf{r}_{1,\mathrm{inv}} = R_z(h_1)R_x(i_1)R_z(g_1)\mathbf{r}_{1,\mathrm{orb}} \ , \tag{A1}$$

where the subscript "inv" and "orb" refer to the invariable and orbital coordinate systems, respectively. The rotation matrices R_z and R_x as a function of rotation angle, θ, are

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A2}$$

and

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \ . \tag{A3}$$

Thus, the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by:

$$\cos\Phi = \hat{\mathbf{r}}^T_{2,\mathrm{orb}} R_z^{-1}(g_2)R_x^{-1}(i_2)R_z^{-1}(h_2)R_z(h_1)R_x(i_1)R_z(g_1)\hat{\mathbf{r}}_{1,\mathrm{orb}}, \tag{A4}$$

where $\hat{\mathbf{r}}_{1,2,\mathrm{orb}}$ are unit vectors that point along $\mathbf{r}_{1,2,\mathrm{orb}}$. In the orbital coordinate system, we have

$$\hat{\mathbf{r}}_{1,2,\mathrm{orb}} = \begin{pmatrix} \cos(l_{1,2}) \\ \sin(l_{1,2}) \\ 0 \end{pmatrix} \ . \tag{A5}$$

Note that $R_z^{-1}(h_2)R_z(h_1) = R_z(h_1 - h_2) \equiv R_z(\Delta h)$, eso the Hamiltonian will depend on the difference in the longitudes of the ascending nodes; in a similar manner, the Hamiltonian depends on l_1 and l_2 only through expressions of the form $l_1 + g_1$ and $l_2 + g_2$. Replacing $\cos\Phi$ in the Hamiltonian, eq. (15), with equation (A4) we find a general form of the Hamiltonian in terms of the orbital elements.

Transformation to Eliminate Mean Motions

Because we are interested in the long-term dynamics of the triple system, we now describe the transformation that eliminates the short-period terms in the Hamiltonian that depend of l_1 and l_2. The technique we will use is known as the Von Zeipel transformation (for more detalis, see Brouwer 1959).

Write the triple-system Hamiltonian in eq. (15) as

$$\mathcal{H} = \mathcal{H}_1^K + \mathcal{H}_2^K + \mathcal{H}_2, \tag{A6}$$

where $\mathcal{H}_1^K$ and $\mathcal{H}_2^K$ are the Kepler Hamiltonians that describe the inner and outer elliptical orbits in the triple system and $\mathcal{H}_2$ describes the quadrupole interaction between the orbits. Note that $\mathcal{H}_2$ is $\mathcal{O}\left(\alpha^2\right)$, and is the only term in $\mathcal{H}$ that depends on l_1 or l_2. Accordingly, we seek a canonical transformation that can eliminate the l_1 and l_2 terms from $\mathcal{H}_3$. Such a transformation must be close to the identity, since $\mathcal{H}_3 \ll \mathcal{H}$; let the generating function be

$$S(L_j^*, G_j^*, H_j^*, l_j, g_j, h_j) = \sum_{j=1}^{2}\left[L_j^* l_j + G_j^* g_j + H_j^* h_j\right] + \alpha^2 S_2(L_j^*, G_j^*, H_j^*, l_j, g_j, h_j), \tag{A7}$$

where we indicate the new momenta with a superscript asterix, and S_2 is the non-identity piece of the transformation that we will use to eliminate $\mathcal{H}_2$. The relationship between the new and old canonical variables is

$$p_i = \frac{\partial S}{\partial q_i} = p_i^* + \alpha^2 \frac{\partial S_2}{\partial q_i} \tag{A8}$$

and

$$q_i^* = \frac{\partial S}{\partial p_i^*} = q_i + \alpha^2 \frac{\partial S_2}{\partial p_i^*}, \tag{A9}$$

where the momenta $p_i \in \{L_i, G_i, H_i\}$, and the coordinates $q_i \in \{l_i, g_i, h_i\}$. Because our generating function is time-independent, the new and old Hamiltonians agree when evaluated at the corresponding points in phase space:

$$\mathcal{H}(q_i, p_i) = \mathcal{H}^*(q_i^*, p_i^*) \tag{A10}$$

when the phase space coordinates satisfy equations (A8) and (A9). Inserting these relations into the un-transformed Hamiltonian, and expanding to lowest order in α^2, we have

$$\mathcal{H}(q_i^*, p_i^*) + \alpha^2 \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial S_2}{\partial q_i} - \alpha^2 \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial S_2}{\partial p_i^*} = \mathcal{H}^*\left(q_i^*, p_i^*\right). \tag{A11}$$

Equating terms order-by-order in α gives

$$\mathcal{H}_1^K(q_i^*, p_i^*) = \mathcal{H}_1^{*K}(q_i^*, p_i^*), \tag{A12}$$

$$\mathcal{H}_2^K(q_i^*, p_i^*) = \mathcal{H}_2^{*K}(q_i^*, p_i^*), \tag{A13}$$

and

$$\mathcal{H}_2\left(q_i^*, p_i^*\right) + \alpha^2 \sum_{i=1}^{2} \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial S_2}{\partial q_i} - \alpha^2 \sum_{i=1}^{2} \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial S_2}{\partial p_i^*} = \mathcal{H}_2^*\left(q_i^*, p_i^*\right). \tag{A14}$$

Since the last two terms on the left-hand side of this latter equation are already $\mathcal{O}\left(\alpha^2\right)$, only the $\mathcal{H}_1^K$ and $\mathcal{H}_2^K$ parts of $\mathcal{H}$ contribute. These Kepler Hamiltonians only depend on L_1 and L_2, so there are only two non-zero partials of $\mathcal{H}$ at order α^2:

$$\mathcal{H}_2\left(q_i^*, p_i^*\right) + \alpha^2 \frac{\partial \mathcal{H}_1^K}{\partial L_1}\frac{\partial S_2}{\partial l_1} + \alpha^2 \frac{\partial \mathcal{H}_2^K}{\partial L_2}\frac{\partial S_2}{\partial l_2} = \mathcal{H}_2^*\left(q_i^*, p_i^*\right). \tag{A15}$$

We must use the terms that depend on S_2 to cancel any terms in H_2 that depend on l_1^* and l_2^*. Note that $\mathcal{H}_2$ is periodic in l_1^* and l_2^* with period 2π (see equations (A4) and (A5)), so we can write

$$\mathcal{H}_2\left(q_i^*, p_i^*\right) = \alpha^2 h_0 + \alpha^2 \sum_{k_1, k_2 = 1}^{\infty} h_{k_1 k_2} e^{-ik_1 l_1^* - ik_2 l_2^*}, \tag{A16}$$

with

$$h_{k_1 k_2} = \frac{1}{4\pi^2 \alpha^2} \int_0^{2\pi} dl_1^* dl_2^*\, \mathcal{H}_2\left(q_i^*, p_i^*\right) e^{ik_1 l_1^* + ik_2 l_2^*}. \tag{A17}$$

Now let $\partial\mathcal{H}_1^K/\partial L_1 \equiv \omega_1(L_1)$, and $\partial\mathcal{H}_2^K/\partial L_2 \equiv \omega_2(L_2)$. Suppose that S_2 is periodic in l_1 and l_2 (which are equivalent, at lowest order, to l_1^* and l_2^*). Then

$$\alpha^2 h_0 + \alpha^2 \sum_{k_1,k_2=1}^{\infty} h_{k_1k_2} e^{-ik_1l_1^* - ik_2l_2^*} + \alpha^2\omega_1 \sum_{k_1,k_2=1}^{\infty} -ik_1 s_{k_1k_2} e^{-ik_1l_1 - ik_2l_2} + \alpha^2\omega_2 \sum_{k_1,k_2=1}^{\infty} -ik_2 s_{k_1k_2} e^{-ik_1l_1-ik_2l_2} = \mathcal{H}_2^*\left(q_i^*, p_i^*\right), \quad \text{(A18)}$$

where

$$S_2 = s_0 + \sum_{k_1,k_2=1}^{\infty} s_{k_1k_2} e^{-ik_1l_1-ik_2l_2}. \quad \text{(A19)}$$

The terms dependent on l_1 will be eliminated from $\mathcal{H}_2^*$ if

$$s_{k_1k_2} = -i\frac{h_{k_1k_2}}{\omega_1k_1 + \omega_2k_2}. \quad \text{(A20)}$$

Assuming than the system is far from resonance (that is, that $\omega_1k_1 + \omega_2k_2 \neq 0$ for all k_1 and k_2), this gives us the necessary S_2 to eliminate all terms in $\mathcal{H}_2$ that depend on l_1 or l_2, leaving

$$\mathcal{H}_2^*\left(q_i^*, p_i^*\right) = \alpha^2 h_0 = \frac{1}{4\pi^2}\int_0^{2\pi} dl_1^* dl_2^*\, \mathcal{H}_2\left(q_i^*, p_i^*\right) e^{ik_1l_1^* + ik_2l_2^*}. \quad \text{(A21)}$$

That is, our canonical transformation to eliminate the rapidly-oscillating parts of $\mathcal{H}$ has left us with a Hamiltonian that is the average over the oscillation period of the original Hamiltonian[13].

The value of the Hamiltonian in equation (15) averaged over the mean motions is

$$\begin{aligned}\mathcal{H}_2^* = & \frac{C_2}{8}\{[1+3\cos(2i_2)]\big([2+3e_1^2][1+3\cos(2i_1)] \\ & + 30e_1^2\cos(2g_1)\sin^2(i_1)\big) + 3\cos(2\Delta h)[10e_1^2\cos(2g_1) \\ & \times(3+\cos(2i_1)) + 4(2+3e_1^2)\sin(i_1)^2]\sin^2(i_2) \\ & + 12(2+3e_1^2 - 5e_1^2\cos(2g_1))\cos(\Delta h)\sin(2i_1)\sin(2i_2) \\ & + 120e_1^2\sin(i_1)\sin(2i_2)\sin(2g_1)\sin(\Delta h) \\ & - 120e_1^2\cos(i_1)\sin^2(i_2)\sin(2g_1)\sin(2\Delta h)\}\ ,\end{aligned} \quad \text{(A22)}$$

where C_2 was defied in equation (20).

MAXIMUM ECCENTRICITY AND "KOZAI" ANGLES IN THE QUADRUPOLE APPROXIMATION

First note that setting $\dot{e_1} = 0$ also means that $\dot{G}_1 = 0$. The values of the argument of periapsis that satisfy these relations are: $g_1 = 0 + \pi n/2$, where $n = 0, 1, 2...$. Also, setting $\dot{G}_1(e_{1,\max,\min}) = 0$ means that $\dot{H}_1(e_{1,\max,\min}) = 0$ and $\dot{i}_1 = 0$, i.e., an extremum of the eccentricity is also an extremum of both the inner and outer inclinations.

The conservation of the total angular momentum, i.e., $\mathbf{G}_1 + \mathbf{G}_2 = \mathbf{G}_{\rm tot}$ sets the relation between the total inclination and inner orbit eccentricity. We re-write equation (6) as

$$L_1^2(1-e_1^2) + 2L_1L_2\sqrt{1-e_1^2}\sqrt{1-e_2^2}\cos i_{\rm tot} = G_{\rm tot}^2 - G_2^2\ , \quad \text{(B1)}$$

where in the quadrupole-level approximation e_2 and G_2 are constant. The right hand side of the above equation is set by the initial conditions. In addition, L_1, and L_2 [see eqs. (3) and (4)] are also set by the initial conditions. Using the conservation of energy we can write, for the minimum eccentricity case (i.e., setting $g_1 = 0$)

$$\frac{E}{2C_2} = 3\cos^2 i_{\rm tot}(1-e_1^2) - 1 + 6e_1^2\ , \quad \text{(B2)}$$

where we also used the relation $\Delta h = \pi$. We find a similar equation if we set $g_1 = \pi/2$:

$$\frac{E}{2C_2} = 3\cos^2 i_{\rm tot}(1+4e_1^2) - 1 - 9e_1^2\ . \quad \text{(B3)}$$

[13] Note that the canonical variables are also transformed. They differ from the original variables at $\mathcal{O}\left(\alpha^2\right)$. However, this difference is irrelevant when evaluating the interaction between the orbits described by $\mathcal{H}_2$, as this interaction is already $\mathcal{O}\left(\alpha^2\right)$, and so the differences between the original and transformed variables contribute at sub-leading order.

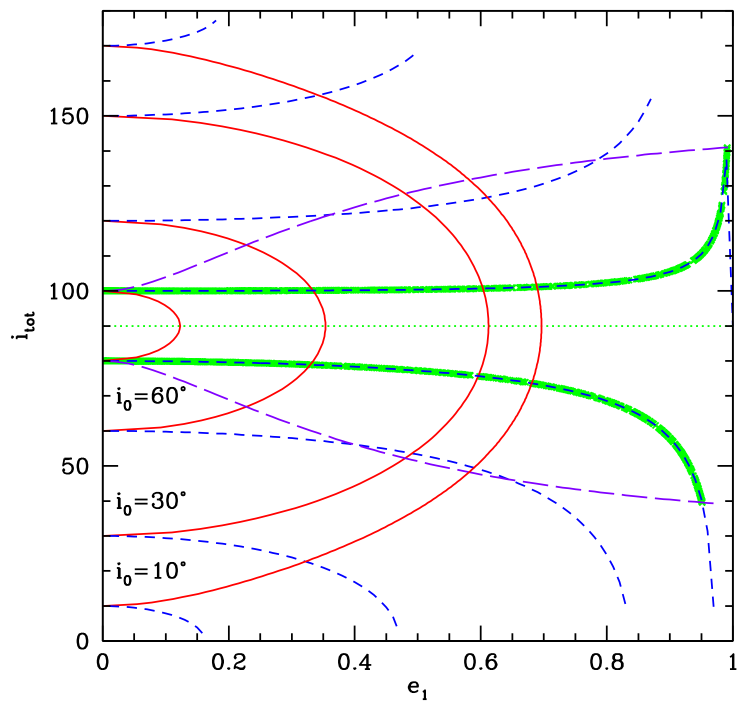

FIG. 13.— The total inclination and eccentricity relation. We show constant energy curves (solid curves, the "half" ellipses are for eq. (B5) and two examples for $i_0 = 80°$ and $i_0 = 100°$ of eq. (B6)), and constant total angular momentum curves (eq. (B4) dashed curves). The initial conditions considered here are $e_1^0 = 0$, g_1^0 and $e_2^0 = 0$ and $L_1/L_2 = 0.07$, appropriate for the Algol system (see §5.4). We consider four different initial inclinations and their symmetric 90° counterparts, from bottom to top $10, 30, 60$ and 80 degrees. We also show an example (highlighted curve) for the Algol system which is a result of integration of the quadrupole-level approximation equations.

Equations (B1), (B2) and (B3) give a simple relation between the total inclination and the inner eccentricity. The remainder of the parameters in the equations are defined by the initial conditions. Thus, using equations (B2) and (B1) we can find the minimum eccentricity reached during the oscillation and using equations (B3) and (B1) we can find also the maximum and the minimum inclinations. The following example illustrates the relation defined by these equations between the inclination and the eccentricity.

For simplicity we set initially $e_1^0 = 0$, g_1^0 and $e_2^0 = 0$ (the superscript 0 stand for initial values). In this appendix we consider only the quadrupole-level approximation, and thus e_2 doesn't change. Using these initial conditions (and for some initial mutual inclination i_0) we can write equation (B1) as

$$\sqrt{1-e_1^2}\cos i_{\rm tot} = \cos i_0 + \frac{L_1}{2L_2}e_1^2 \ . \tag{B4}$$

We show these curves for different i_0 in Figure 13 (short dashed curves) for a hypothetical system with the parameters of Algol (but with $e_2 = 0$, see §5.4). Note that there is a slight asymmetry between the prograde and retrograde orbits due to the L_1/L_2 factor (which is not the case for the test particle case, see Lithwick & Naoz 2011; Katz et al. 2011). We also write equations (B2) and (B3) using the initial conditions. Equation (B2) can be simplified to

$$(1-e_1^2)\cos^2 i_{\rm tot} = \cos^2 i_0 + 2e_1^2 \ , \tag{B5}$$

depicted in Figure 13 (solid curves, for different i_0). As can be seen from the Figure, this equation gives the minimum eccentricity, which is the crossing point with equation (B4). For these choice of initial conditions the minimum eccentricity is $e_1^0 = 0$. Equation (B3) becomes

$$(4+e_1^2)\cos^2 i_{\rm tot} = \cos^2 i_0 - 3e_1^2 \ , \tag{B6}$$

which is depicted in Figure 13 (long dashed curves, for $i_0 = 80°$ and $100°$). We now use this equation and equation (B4) to find the maximum eccentricity. After some algebra we find:

$$\left(\frac{L_1}{L_2}\right)^2 e_1^4 + \left(3 + 4\frac{L_1}{L_2}\cos i_0 + \left(\frac{L_1}{2L_2}\right)^2\right)e_1^2 + \left(\frac{L_1}{2L_2}\right)^2 - 3 + 5\cos^2 i_0 = 0 \ . \tag{B7}$$

As we approach the TPQ limit, $L_2 \gg L_1$, and this equation becomes

$$e_1^2 = 1 - \frac{5}{3}\cos^2 i_0 \ , \tag{B8}$$

which gives the maximum eccentricity as a function of mutual initial inclination with zero initial inner eccentricity. In Figure 13 we show that this approximation still hold fairly well even for the Algol system, where $L_1/L_2 \sim 0.07$. Equation (B8) has been found previously (e.g. Innanen et al. 1997; Kinoshita & Nakai 1999; Valtonen & Karttunen 2006) in the TPQ approximation, but in these works it is assumed to valid outside that limit. A solution exists only if the right hand side of this equations is positive, thus we find the critical angles for large Kozai oscillation in the TPQ limit:

$$39.2^\circ \geq i_0 \leq 140.8^\circ \ . \tag{B9}$$

For larger L_1/L_2 and/or for initial $e_1 > 0$ this limit and $e_{\rm max}$ are different and the full solution of equations (B1),B2) and B3) is required. In fact for each initial set of $e_1 > 0$ and $i_{\rm tot}$, there is a specific L_1/L_2 that will produce an angular momentum curve that crosses 90°. Thus, for initial $g_1 > 90^\circ$ the mutual inclination can oscillate from value below 90° to above. This happens because the inclination of the outer orbit i_2 changes considerably, while the inner orbit remain prograde (if started prograde). We emphasize that the "ocatpole-Kozai" behavior (§4) is of course present; and can only be neglected when $\epsilon_M \ll 1$.

THE FULL OCTUPOLE-ORDER EQUATIONS OF MOTION

We define:

$$C_3 = -\frac{15}{16}\frac{k^4}{4}\frac{(m_1+m_2)^9}{(m_1+m_2+m_3)^4}\frac{m_3^9(m_1-m_2)}{(m_1m_2)^5}\frac{L_1^6}{L_2^3G_2^5} \ . \tag{C1}$$

Note that this definition is with a different sign from Ford et al. (2000b), and consistent with Blaes et al. (2002); Ford et al. (2004). For equal mass m_1 and m_2 this factor is zero. We also define:

$$A = 4 + 3e_1^2 - \frac{5}{2}B\sin i_{\rm tot}^2 \ , \tag{C2}$$

where

$$B = 2 + 5e_1^2 - 7e_1^2\cos(2g_1) \ , \tag{C3}$$

and

$$\cos\phi = -\cos g_1\cos g_2 - \cos i_{\rm tot}\sin g_1\sin g_2 \ . \tag{C4}$$

The time evolution of the argument of periapse for the inner and outer orbits are given by:

$$\begin{aligned}
\dot{g}_1 = & 6C_2\Big\{\frac{1}{G_1}[4\cos^2 i_{\rm tot} + (5\cos(2g_1)-1) \\
& \times(1-e_1^2-\cos^2 i_{\rm tot})] + \frac{\cos i_{\rm tot}}{G_2}[2+e_1^2(3-5\cos(2g_1))]\Big\} \\
& - C_3e_2\Big\{e_1\left(\frac{1}{G_2}+\frac{\cos i_{\rm tot}}{G_1}\right) \\
& \times[\sin g_1\sin g_2(10(3\cos i_{\rm tot}^2-1)(1-e_1^2)+A) \\
& -5\cos i_{\rm tot}\cos\phi] - \frac{1-e_1^2}{e_1G_1}\times[\sin g_1\sin g_2 \\
& \times 10\cos i_{\rm tot}\sin i_{\rm tot}^2(1-3e_1^2) \\
& +\cos\phi(3A-10\cos i_{\rm tot}^2+2)]\Big\} \ ,
\end{aligned} \tag{C5}$$

and

$$
\begin{aligned}
\dot{g}_2 = 3C_2 \Bigg\{ & \frac{2\cos i_{\rm tot}}{G_1}[2 + e_1^2(3 - 5\cos(2g_1))] \\
& + \frac{1}{G_2}[4 + 6e_1^2 + (5\cos^2 i_{\rm tot} - 3)(2 + e_1^2[3 - 5\cos(2g_1)])\Bigg\} \\
& + C_3 e_1 \Bigg\{ \sin g_1 \sin g_2 \Bigg(\frac{4e_2^2+1}{e_2 G_2} 10\cos i_{\rm tot}\sin^2 i_{\rm tot}(1-e_1^2) \\
& - e_2 \left(\frac{1}{G_1} + \frac{\cos i_{\rm tot}}{G_2}\right)[A + 10(3\cos^2 i_{\rm tot} - 1)(1-e_1^2)]\Bigg) \\
& + \cos\phi \left[5B\cos i_{\rm tot} e_2 \left(\frac{1}{G_1} + \frac{\cos i_{\rm tot}}{G_2}\right) + \frac{4e_2^2+1}{e_2 G_2} A \right] \Bigg\}
\end{aligned}
\tag{C6}
$$

The time evolution of the longitude of ascending nodes is given by:

$$
\begin{aligned}
\dot{h}_1 = & -\frac{3C_2}{G_1 \sin i_1}\left(2 + 3e_1^2 - 5e_1^2\cos(2g_1)\right)\sin(2i_{\rm tot}) \\
& - C_3 e_1 e_2 [5B\cos i_{\rm tot}\cos\phi - A\sin g_1 \sin g_2 + 10(1 + 3\cos^2 i_{\rm tot}) \\
& \times (1-e_1^2)\sin g_1 \sin g_2] \frac{\sin i_{\rm tot}}{G_1 \sin i_1} \ ,
\end{aligned}
\tag{C7}
$$

where in the last part we have used again the law of sines for which $\sin i_1 = G_2 \sin i_{\rm tot}/G_{\rm tot}$. The evolution of the longitude of ascending nodes for the outer orbit can be easily obtained using:

$$
\dot{h}_2 = \dot{h}_1 \ . \tag{C8}
$$

The evolution of the eccentricities is:

$$
\begin{aligned}
\dot{e}_1 = & C_2 \frac{1-e_1^2}{G_1}[30e_1 \sin^2 i_{\rm tot}\sin(2g_1)] \\
& + C_3 e_2 \frac{1-e_1^2}{G_1}[35\cos\phi\sin^2 i_{\rm tot} e_1^2 \sin(2g_1) \\
& - 10\cos i_{\rm tot}\sin^2 i_{\rm tot}\cos g_1 \sin g_2 (1-e_1^2) \\
& - A(\sin g_1 \cos g_2 - \cos i_{\rm tot}\cos g_1 \sin g_2)] \ ,
\end{aligned}
\tag{C9}
$$

and

$$
\begin{aligned}
\dot{e}_2 = & -C_3 e_1 \frac{1-e_2^2}{G_2}[10\cos i_{\rm tot}\sin^2 i_{\rm tot}(1-e_1^2)\sin g_1 \cos g_2 \\
& - A(\sin g_1 \cos g_2 - \cos i_{\rm tot}\cos g_1 \sin g_2)] \ .
\end{aligned}
\tag{C10}
$$

We also write the angular momenta derivatives as a function of time; for the inner orbit

$$
\begin{aligned}
\dot{G}_1 = & -C_2 30 e_1^2 \sin(2g_1)\sin^2(i_{\rm tot}) + C_3 e_1 e_2 (\\
& - 35e_1^2 \sin^2(i_{\rm tot})\sin(2g_1)\cos\phi + A[\sin(g_1)\cos(g_2) \\
& - \cos(i_{\rm tot})\cos(g_1)\sin(g_2)] + 10\cos(i_{\rm tot})\sin(i_{\rm tot})[1-e_1^2]\cos(g_1)\sin(g_2)) \ ,
\end{aligned}
\tag{C11}
$$

and for the outer orbit (where the quadrupole term is zero)

$$
\begin{aligned}
\dot{G}_2 = & C_3 e_1 e_2 [A\{\cos(g_1)\sin(g_2) - \cos(i_{\rm tot})\sin(g_1)\cos(g_2)\} \\
& + 10\cos(i_{\rm tot})\sin^2(i_{\rm tot})[1-e_1^2]\sin(g_1)\cos(g_2)] \ .
\end{aligned}
\tag{C12}
$$

Also,

$$
\dot{H}_1 = \frac{G_1}{G_{\rm tot}}\dot{G}_1 - \frac{G_2}{G_{\rm tot}}\dot{G}_2 \ , \tag{C13}
$$

where using the law of sines we write:

$$
\dot{H}_1 = \frac{\sin i_2}{\sin i_{\rm tot}}\dot{G}_1 - \frac{\sin i_1}{\sin i_{\rm tot}}\dot{G}_2 \ . \tag{C14}
$$

The inclinations evolve according to

$$
(\dot{\cos i_1}) = \frac{\dot{H}_1}{G_1} - \frac{\dot{G}_1}{G_1}\cos i_1 \ , \tag{C15}
$$

and

$$(\dot{\cos i_2}) = \frac{\dot{H}_2}{G_2} - \frac{\dot{G}_2}{G_2}\cos i_2 \ . \tag{C16}$$

Our equations are equivalent to those of Ford et al. (2000b), but we give the evolution equations for H_1 and H_2 (and i_1 and i_2).

ELIMINATION OF THE NODES EQUATIONS DERIVATION

Here we show a derivation of equations (D7) which lead to the error in previous treatments. The angular momentum vectors of the two orbits are given by

$$\mathbf{G}_{1,2} = G_{1,2}\left(\sin i_{1,2}\sin h_{1,2}, -\sin i_{1,2}\cos h_{1,2}, \cos i_{1,2}\right) \ . \tag{D1}$$

Thus the total angular momentum vector is then:

$$\begin{aligned}\mathbf{G}_{\rm tot} = &(G_1 \sin i_1 \sin h_1 + G_2 \sin i_2 \sin h_2, \\ &- G_1 \sin i_1 \cos h_1 - G_2 \sin i_2 \cos h_2, G_1 \cos i_1 + G_2 \cos i_2) \ .\end{aligned} \tag{D2}$$

Recall that the z-component of the angular momentum is $H_j = G_j \cos i_j$.

In the elimination of the nodes we set $\mathbf{G}_{\rm tot}||\hat{z}$ thus,

$$G_1 \sin i_1 \sin h_1 = -G_2 \sin i_2 \sin h_2 \ , \tag{D3}$$

$$G_1 \sin i_1 \cos h_1 = -G_2 \sin i_2 \cos h_2 \ , \tag{D4}$$

$$H_1 + H_2 = G_{\rm tot} \ . \tag{D5}$$

Dividing Eqs (D3) and (D4), we obtain

$$\tan h_1 = \tan h_2, \tag{D6}$$

implying that

$$h_1 - h_2 = \pi \ . \tag{D7}$$

Because the dynamics of the system conserves total angular momentum, this result will always hold. This is, however, a *dynamical* restriction, and does not imply any restriction on the partial derivatives that produce the equations of motion from the Hamiltonian. In other words, we cannot substitute

$$h_1 = h_2 - \pi \ . \tag{D8}$$

into the Hamiltonian before computing equations of motion that may involve terms with

$$\frac{\partial \mathcal{H}}{\partial h_i}. \tag{D9}$$

EXAMPLE

As a simple example to illustrate the source of the mistake, let us consider a 1-D system with two equal masses connected by a spring. The Hamiltonian for such a problem is

$$\mathcal{H} = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2}k_s(x_1 - x_2)^2 \ , \tag{E1}$$

where k_s is the spring constant. Qualitatively equivalent to the elimination of the nodes would be here to transform to the center of mass of the system, so that $x_1 = -x_2$. If we now substitute this relationship between the coordinates into the Hamiltonian, we get

$$\mathcal{H} = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2}k_s(2x_1)^2 \ . \tag{E2}$$

But this is incorrect! This Hamiltonian implies, for example, that $P_2 = \text{const}$.

Note that the error that leads to the incorrect secular three-body Hamiltonian is analagous: conservation of momentum gives a relation between two coordinates ($h_1 = h_2 - \pi$), and substitution of this relation into the Hamiltonian gives the incorrect relation $\sqrt{1 - e_1}\cos i_1 = \text{const}$.