THE ASTROPHYSICAL JOURNAL, 336:L63–L66, 1989 January 15

ON THE EXISTENCE OF STABLE RELATIVISTIC STAR CLUSTERS WITH ARBITRARILY LARGE CENTRAL REDSHIFTS

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Received 1988 September 12; accepted 1988 October 26

ABSTRACT

We have developed a new numerical method for determining the dynamical evolution of a collisionless system in full general relativity. Using this new method we demonstrate the existence of stable relativistic star clusters with arbitrarily large central redshifts. Such clusters may result from the secular evolution of an initially Newtonian, isotropic cluster toward a relativistic state.

Subject headings: clusters: dynamics — galaxies: redshifts — gravitation — quasars — relativity

I. INTRODUCTION

It is now possible to solve Einstein's equations for the nonlinear dynamical evolution of collisionless systems in general relativity, at least in spherical symmetry (Shapiro and Teukolsky 1985*a*, *b*, 1986, hereafter collectively ST; and Rasio, Shapiro and Teukolsky 1988, hereafter RST). In this *Letter*, we apply these new techniques to address several long-standing questions concerning the stability of relativistic star clusters. In particular, we discuss the existence of stable relativistic clusters with arbitrarily large central redshifts.

This subject became active in the period that followed Hoyle and Fowler's (1967) proposal that quasars might be lying at the centers of massive relativistic star clusters and derive their large redshifts from the strong gravitational fields of those clusters. The main theoretical question was whether stable relativistic structures with central redshifts as large as ~ 2 could exist. At the time, the stability of relativistic clusters against radial perturbations could only be assessed by approximate semianalytical calculations. The linearized perturbation equations for an equilibrium cluster were recast in the form of a variational principle (Ipser and Thorne 1968; Ipser 1969*a*). Using suitable trial functions, one can locate the point along an equilibrium sequence where the radial oscillation frequency ω^2 becomes negative. This gives a sufficient condition for instability. In all the equilibrium sequences that were studied at that time, the clusters became unstable when their central redshifts satisfied $Z_c \gtrsim 0.5$ (Ipser 1969*a*, *b*; Fackerell 1970). It has generally been believed since then that *all* relativistic star clusters with central redshifts $Z_c \gtrsim 0.5$ should be dynamically unstable.

This belief persists in spite of speculations that some self-similar clusters with infinite central densities and redshifts, constructed by Bisnovatyi-Kogan and Zel'dovich (1969), might be stable (see Bisnovatyi-Kogan and Thorne 1970). The unrealistic nature of these clusters made them appear of little interest. Moreover, all techniques for testing stability that were applied to them yielded inconclusive results. Only certain corresponding *fluid* structures were proved to be stable.

By analogy with equilibrium sequences of fluid stars, one can also examine the fractional binding energy (i.e., the binding energy per unit rest mass) of clusters as a diagnostic for stability. It it well known (see, e.g., Shapiro and Teukolsky 1983) that the onset of instability along equilibrium sequences of relativistic *fluid* configurations is indicated by a maximum of the fractional binding energy along the sequence. An important question is whether the same should be true for equilibrium sequences of *collisionless* systems, such as star clusters. In most cases the first maximum of the fractional binding energy coincides, to within numerical accuracy, with the point where ω^2 becomes negative. In no cases does ω^2 become negative before the first maximum. This has led to the conjecture that the onset of instability for relativistic star clusters also coincides with a maximum of the fractional binding energy. Interest in this subject has weakened, however, after the cosmological origin of quasar redshifts gained general acceptance, and the conjectures were left without final proofs.

An important recent result is Ipser's (1980) proof that the above mentioned conjecture indeed gives at least a *sufficient* condition for stability. Specifically, Ipser proved that all clusters along an appropriately constructed one-parameter equilibrium sequence are stable at least up to the first maximum of the fractional binding energy. Recently, powerful new numerical techniques have been developed, leading to the fully relativistic particle simulations of ST, and the relativistic phase-space calculations of RST. These calculations completely confirm that the conjecture is indeed true: in all cases a dynamical instability, leading to the collapse of the cluster to a black hole, is observed as soon as the first maximum of the fractional binding energy along the sequence has been passed.

We now raise the following question: if one could construct an equilibrium sequence where the fractional binding energy increases monotonically with redshift, would all clusters remain stable even when $Z_c \gg 0.5$? Below we answer yes to this question by providing a concrete example. The clusters we study do have a fractional binding energy increasing monotonically with central redshift and do not collapse when dynamically evolved. Moreover, these clusters have perfectly reasonable physical properties and may even be of astrophysical relevance: they could represent the relativistic final states of initially Newtonian clusters undergoing the gravothermal catastrophe.

II. BASIC EQUATIONS AND METHOD

We study the stability properties of equilibrium clusters by using them as initial data in our fully nonlinear dynamical calculations. These calculations are restricted to spherical symmetry, but the gravitational fields can become arbitrarily

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strong and the matter velocities can approach the speed of light. Only spherically symmetric unstable modes of oscillation in the clusters are allowed to grow. However, this restriction is of little practical importance here since we will be dealing with distribution functions which are monotonically decreasing functions of the energy only. It is known theoretically (Ipser 1975) that such distribution functions are dynamically stable to *nonradial* modes of oscillations.

We adopt the notations of Misner, Thorne, and Wheeler (1973) and set $c = G = 1$ throughout. The metric is written in the ADM form (Arnowitt, Deser, and Misner 1962), with isotropic radial coordinate,

$$ds^2 = -(\alpha^2 - A^2\beta^2)dt^2 + 2A^2\beta\, dr\, dt + A^2(dr^2 + r^2\, d\theta^2 + r^2 \sin^2 \theta\, d\phi^2)\,. \qquad (1)$$

Here α and β are the lapse and shift functions of ADM; see also Smarr and York (1978*a*, *b*).

Our method follows the evolution of the (smooth) distribution function of matter in phase space. The components of the stress-energy tensor are computed directly by quadratures of the distribution function over velocity space. Here the time evolution of the distribution function f is determined by the collisionless Boltzmann (Vlasov) equation in general relativity, which in spherical symmetry can be written

$$\frac{\partial f}{\partial t} + \left(\frac{dr}{dt}\right)\frac{\partial f}{\partial r} + \left(\frac{du_r}{dt}\right)\frac{\partial f}{\partial u_r} = 0\,, \qquad (2)$$

where $f = f(r, u_r, j, t), j \equiv (u_\theta{}^2 + u_\phi{}^2/\sin^2\theta)^{1/2}$ is the conserved "angular momentum at infinity" per unit mass of a star, and u_α is the 4-velocity. The geodesic equation in the metric (1) can be used to obtain explicit expressions for the coefficients dr/dt and du_r/dt appearing in equation (2).

To obtain f from equation (2), we make direct use of Liouville's theorem, which states that for a collisionless system, values of f are conserved along dynamical trajectories in phase space (i.e., geodesics). Numerically, we evaluate f at some point (r_1, u_{r1}, j_1) in phase space and time $t = t_1$, by *constructing* the geodesic that leads to this point from the initial time $t = 0$, and thereby determine the initial position (r_0, u_{r0}, j_0) from which the geodesic originates (note that $j_0 = j_1$ since j is conserved). Then we set $f(r_1, u_{r1}, j_1, t_1) = f(r_0, u_{r0}, j_0, t = 0)$, where the right-hand side can be evaluated directly from the initial data.

This method is very accurate and robust, but requires large amounts of computing time (for example, following the collapse of an unstable, moderately uniform cluster, all the way to the formation of an event horizon can take ~1 CPU hr on an IBM 3090-600 supercomputer). A detailed description of the method together with a series of test-bed calculations is given in RST.

III. THE STABILITY OF RELATIVISTIC STAR CLUSTERS

We have considered truncated isothermal, polytropic, and power-law equilibrium sequences. These sequences had already been studied in the past (see Zel'dovich and Podurets 1965; Ipser 1969*b*; Fackerel 1970; ST), and our results completely confirmed those of previous investigations: the fractional binding energy along these sequences reaches its first maximum at a central redshift $Z_c \sim 0.5$ and clusters beyond this point are unstable. All unstable clusters collapsed to black holes in a few dynamical times (see RST for details).

A very important distribution function, which was never considered before from the point of view of stability, is the one that could result from the *secular* evolution of an initially Newtonian cluster via the gravothermal catastrophe to the point where its core becomes relativistic. This distribution function may be relevant to the formation process of supermassive black holes at the center of dense galactic nuclei (Shapiro and Teukolsky 1985*c*; Kochanek, Shapiro, and Teukolsky 1987). Specifically, such a black hole might result from the *dynamical instability* of a relativistic, extreme core-halo cluster of compact stars that forms in a galactic nucleus via the gravothermal catastrophe.

Numerical integrations of the *Newtonian* Fokker-Planck equation for the secular evolution of an isotropic, one-component cluster during the gravothermal catastrophe yield a unique, self-similar form of the distribution function at late times (Cohn 1980). This distribution function is plotted in Figure 1 in terms of the nondimensional energy variable of Cohn,

$$x \equiv 3\left(\frac{E_{\rm NR} - \phi(0)}{\bar{v}^2(0)}\right). \qquad (3)$$

Here $\bar{v}(0)$ is the central velocity dispersion, $\phi(0)$ is the central Newtonian potential, and $E_{\rm NR}$ is the nonrelativistic energy per unit mass of a star in the cluster. At late times we have

$$\bar{v}^2(0) \approx 0.226\,|\phi(0)|\,. \qquad (4)$$

Fokker-Planck evolution of star clusters in the relativistic domain has never been studied. However, a natural relativistic generalization to the Newtonian distribution function described above can be obtained by the following procedure, originally given by Kochanek, Shapiro, and Teukolsky (1987). First note that in the Newtonian limit, $|\phi(0)| \approx Z_c$, where Z_c is the central redshift of the cluster, and $E_{\rm NR} \approx E/m - 1$, where E is the relativistic energy and m the rest mass of a star. Then

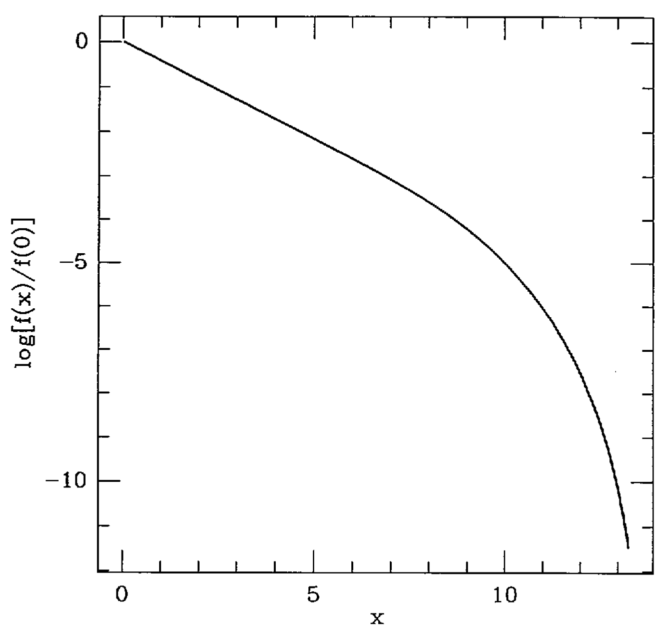

FIG. 1.—Gravothermal catastrophe distribution function plotted as a function of the dimensionless energy variable x. See text for discussion.

redefine the energy variable x accordingly, as

$$x \equiv 13.25\left(\frac{E/m - 1 + Z_c}{Z_c}\right), \qquad (5)$$

and adopt the same form for the distribution function $f(x)$ as plotted in Figure 1. This distribution function is an equilibrium solution to the relativistic Vlasov equation (2), and reproduces the essential features of a highly relaxed stellar system. In particular, it exhibits a homogeneous isothermal core surrounded by an extended halo with density $\rho \sim r^{-2.2}$ (see Fig. 2).

The equilibrium sequence of relativistic equilibrium clusters generated by equation (5) has a most unusual feature. When we plot the fractional binding energy E_b/M_0 as a function of central redshift Z_c (Fig. 3), we find that the curve has no turning point: *the fractional binding energy increases monotonically with redshift*, at least up to $Z_c \sim 30$. Moreover we know that (1) as $Z_c \to 0$, the clusters become Newtonian and are therefore known to be stable, since $df/dE < 0$ (cf. Doremus, Feix, and Baumann 1971; Sygnet *et al.* 1984; Kandrup and Sygnet 1985); and (2) the value f_{cut} of the distribution function at the boundary of the system in phase space ($E = E_{\text{cut}}$) is constant along the sequence (here $f_{\text{cut}} = 0$). Ipser (1980) has proven a theorem which indicates that under these circumstances, *all clusters along the sequence should remain dynamically stable, even when their central redshifts become arbitrarily large!*

We studied the stability properties of these clusters using our fully relativistic dynamical code. Six clusters along the equilibrium sequence, ranging in central redshifts from $Z_c = 0.123$ to $Z_c = 3.75$, were used as initial conditions (they are identified by triangles in Fig. 3). All of them were evolved for more than

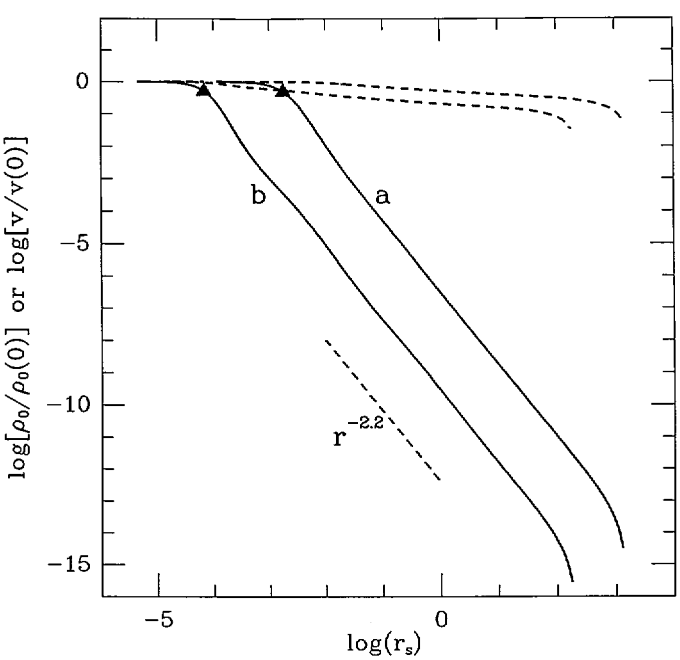

FIG. 2.—Rest-mass density profiles (*solid lines*) and velocity dispersion profiles (*dashed lines*) for the relativistic gravothermal catastrophe distribution function; (*a*) is for $Z_c = 0.123$, (*b*) for $Z_c = 3.75$. Here r_S is the radius in Schwarzschild coordinates. All clusters have a nearly isothermal core, surrounded by an extended halo with density $\rho_0 \sim r^{-2.2}$. The triangles indicate the "core radius," where the density falls to one-half its central value.

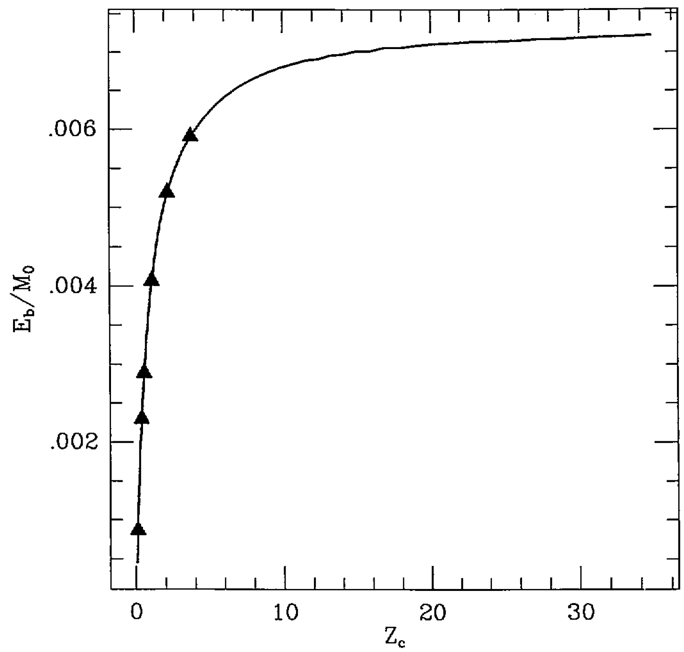

FIG. 3.—Fractional binding energy E_b/M_0 vs. central redshift Z_c for the relativistic gravothermal catastrophe distribution function. Note the absence of turning point in the curve, indicating stability. The triangles indicate clusters that have been studied numerically. All have been found to be stable against radial collapse.

15 dynamical times. To within our numerical accuracy, they showed no sign of evolution whatsoever. For example, the density profiles of Figure 2 were maintained to better than 3% in all cases. Since our code never failed to accurately locate the onset of instability in test-bed runs involving unstable clusters (cf. RST), we conclude that the clusters corresponding to equation (5) are indeed dynamically stable, at least up to a central redshift $Z_c = 3.75$. At even higher redshifts, the dynamical range in the system increases rapidly and makes the computational time prohibitively large (at $Z_c = 3.75$, the ratio of central to mean stellar densities is $\approx 10^{14}$). However, it appears quite certain that the stability will continue as $Z_c \to \infty$.

IV. CONCLUSIONS

Our recent numerical calculations completely confirm the long-standing conjecture that, as in the case of fluid stars, *the onset of instability in relativistic collisionless star clusters coincides with the first maximum of the fractional binding energy along an equilibrium sequence.*

We have constructed a possible relativistic generalization to the distribution function representing the endpoint of the Newtonian gravothermal catastrophe. It generates an equilibrium sequence of relativistic star clusters in which the fractional binding energy increases monotonically with central redshift. We have verified numerically that, when allowed to evolve dynamically, these clusters do not collapse. Instead, they maintain their structure to high accuracy, regardless of how large the central redshift is. We conclude that *these clusters provide the first examples of finite, asymptotically flat equilibrium systems which can become arbitrarily relativistic at the center and still remain stable against gravitational collapse.*

We thank J. Ipser for a useful conversation and a critical reading of the manuscript. We are grateful to D. Chernoff, C. Kochanek, and M. Weinberg for stimulating discussions, and G. Quinlan for providing us with the Newtonian gravothermal catastrophe distribution function in tabular form. This work has been supported by National Science Foundation grants AST 87-14475 and PHY 86-03284 to Cornell University. Computations were performed on the Cornell National Supercomputer Facility, which is supported in part by the National Science Foundation, IBM Corporation, New York State, and the Cornell Research Institute.

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