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THERMAL AND DYNAMICAL EQUILIBRIUM IN TWO-COMPONENT STAR CLUSTERS

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ABSTRACT

We present the results of Monte Carlo simulations for the dynamical evolution of star clusters containing two stellar populations with individual masses m_1 and $m_2 > m_1$, and total masses M_1 and $M_2 < M_1$. We use both King and Plummer model initial conditions, and we perform simulations for a wide range of individual and total mass ratios, m_2/m_1 and M_2/M_1. We ignore the effects of binaries, stellar evolution, and the galactic tidal field. The simulations use $N = 10^5$ stars and follow the evolution of the clusters until core collapse. We find that the departure from energy equipartition in the core follows approximately the theoretical predictions of Spitzer and Lightman & Fall, and we suggest a more exact condition that is based on our results. We find good agreement with previous results obtained by other methods regarding several important features of the evolution, including the precollapse distribution of heavier stars, the timescale on which equipartition is approached, and the extent to which core collapse is accelerated by a small subpopulation of heavier stars. We briefly discuss the possible implications of our results for the dynamical evolution of primordial black holes and neutron stars in globular clusters.

Subject headings: celestial mechanics, stellar dynamics — globular clusters: general — instabilities — methods: *n*-body simulations

1. INTRODUCTION

Remarkable advances have been made over the last three decades in our understanding of globular cluster dynamics (see, e.g., Meylan & Heggie 1997 for a recent review). The simple case of a two-component cluster is traditionally regarded as the second level of sophistication and therefore a logical challenge for new methods that have tackled the single-component case. Two-component clusters were originally examined because they better resemble real clusters, which contain a continuous spectrum of masses. While somewhat more realistic in this regard, clusters with only two mass components still represent a simplification with respect to real clusters. It has been suggested recently, however, that for a range of configurations in mass types and the relative size of the two populations, two-component clusters can resemble real clusters that are mostly composed of compact objects and main-sequence stars (Kim, Lee, & Goodman 1998). Similarly, clusters containing both single main-sequence stars and primordial binaries can be modeled, in first approximation, as two-component systems, although dynamical interactions involving binaries are expected to play an important role for these systems (Gao et al. 1991). Perhaps the best reason to examine a simplified model of any stellar system, however, is to obtain a more profound understanding of individual physical processes.

Much of the discourse regarding two-component systems has focused on the following questions. First, for what configurations of the cluster is dynamical equilibrium precluded (i.e., the system is not stable on dynamical timescales)? Second, for what configurations of the cluster is thermal equilibrium precluded (i.e., equipartition of kinetic energies between each component is not allowed)? Both questions originate from an analysis by Spitzer (1969), in which he noticed that simultaneous thermal and dynamical equilibrium is impossible for some clusters. In particular, the heavier stars sink into the center as they lose kinetic energy to the lighter stars during the approach to equipartition. If equipartition is not attained, then the heavier stars will continue sinking until their self-gravity dominates the cluster's potential in the core. Shortly thereafter, the heavier component will undergo a gravothermal collapse, forming a small dense core composed mainly of the heavier stars (Spitzer 1969). Refinements to this analysis have obtained similar constraints on the configurations of two-component clusters in dynamical and thermal equilibrium (Lightman & Fall 1978).

Several methods have been used to address questions about dynamical and thermal equilibrium in two-component systems. These include the construction and study of one-parameter families of models in dynamical equilibrium (Kondrat'ev & Ozernoy 1982; Katz & Taff 1983), Monte Carlo approaches to the numerical integration of the Fokker-Planck equation (Spitzer & Hart 1971), direct integration of the Fokker-Planck equation in phase space (Inagaki & Wiyanto 1984; Kim, Lee, & Goodman 1998), and direct *N*-body simulations (Portegies Zwart & McMillan 2000). The majority of work using any one of these methods has been undertaken at least partly in order to confirm Spitzer's conclusion (Yoshizawa et al. 1978) or refute it (Merritt 1981).

Dynamical equilibrium is attained and maintained on timescales that are very short compared to the amount of time needed for relaxation or equipartition. A so-called equilibrium model (i.e., whose phase-space distribution function satisfies the equation for hydrostatic equilibrium) therefore resembles a possible stage or snapshot in the evolution of a dynamically stable cluster. It is interesting to construct a parameterized family of equilibrium models for

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which either equipartition is assumed or the temperature ratio is allowed to vary, in order to determine under what conditions the dynamical equilibrium becomes impossible. In the majority of previous work, the distribution functions of such families take the form of lowered Maxwellians or spatially truncated isothermal spheres.

Yoshizawa et al. (1978) examined a linear series of equilibrium models of two-component isothermal spheres with reflecting walls and found that turning points of the total energy (stability limits) were positive for some clusters. In such cases, a cluster was not self-bounded and would dissociate if the walls were removed. These configurations, which were interpreted to preclude dynamical equilibrium, occurred under conditions corresponding closely to the ones Spitzer proposed. Katz & Taff (1983) examined the limits of stability for a linear series of equilibrium models of self-bounded two-component clusters with a lowered Maxwellian velocity distribution. They found that the number of possible configurations for clusters in dynamical and thermal equilibrium were diminished dramatically in the regime that Spitzer proposed. Kondrat'ev & Ozernoy (1982) constructed a family of equilibrium models, based on a generalization of the single-component King models, for which equipartition of kinetic energies was generally impossible. By contrast, Merritt (1981) described a one-parameter family of equilibrium models for which equipartition in the core is possible for all cluster configurations within Spitzer's unstable regime. It seems clear, however, that these models violate an important assumption of Spitzer's analysis and that they are highly unrealistic (Merritt 1981).

Few studies that examine the *evolution* of two-component systems have been undertaken for a wide range of cluster configurations and using the other methods mentioned above. Among the more notable efforts are the Monte Carlo calculations of Spitzer & Hart (1971; later extended to three-component systems by Spitzer & Shull 1975), which were interpreted to partially confirm Spitzer's original analysis, and the direct Fokker-Planck integrations of Inagaki & Wiyanto (1984; see also Inagaki 1985), which also partially support Spitzer's claim, although by examining a limited range of clusters that do not totally satisfy his assumptions. The former study finds that a single model, which belongs to the unstable regime in which equipartition is precluded, develops a collapsing subsystem of heavier stars. Both studies find that central equipartition of kinetic energies is never attained throughout the evolution of some clusters.

In this paper we present the results of calculations used to model the evolution of two-component clusters with a wide range of configurations. We examine several features of the evolution, including the tendency toward equipartition, the evolution of mass densities in the core, and the tendency for core collapse to be accelerated by the presence of a small subpopulation of heavier stars. Our aim is partly to assess the accuracy of Spitzer's analysis and the refinement supplied by Lightman & Fall (1978). We have used a new Monte Carlo code for modeling the evolution as a sequence of equilibrium models whose velocities are perturbed according to the average effect of long-range stellar encounters (Joshi, Rasio, & Portegies Zwart 2000b). Our initial two-component systems are isolated King and Plummer models with a subpopulation of heavier stars.

Our paper is organized as follows. In § 2 we review the theoretical arguments that suggest conditions under which simultaneous dynamical and thermal equilibrium are not allowed. In § 3 we discuss our numerical method and the quantities we calculate, and in § 4 we present our main results. In § 5 we discuss some additional comparisons between our findings and those of other studies as well as some astrophysical implications of our results for subpopulations of compact objects in globular clusters.

2. EQUIPARTITION OF KINETIC ENERGIES IN TWO-COMPONENT CLUSTERS

According to Spitzer (1969), simultaneous dynamical and thermal equilibrium is impossible for two-component star clusters within a certain range of configurations. Let us consider a cluster with stars of two masses, m_1 and m_2, where $m_2 > m_1$. Moreover, let M_2 and M_1 be the total mass in each component. Spitzer assumed that $M_2 \ll M_1$, which is normally the case for real clusters. He concluded that for certain values of the ratios M_2/M_1 and m_2/m_1, equipartition will not be attained as the heavier and lighter stars exchange kinetic energy, and hence the heavier stars will sink very far into the center. Moreover, the heat exchange with lighter stars promotes them to higher orbits, so that eventually insufficient numbers will remain in the center to conduct heat rapidly away from the heavier stars. If the mass stratification proceeds far enough, then the self-gravity of the heavier stars will dominate the potential in the core, and this subsystem will undergo gravothermal collapse. The result is a very dense core composed exclusively of heavier stars.

In his analysis, Spitzer begins by assuming global equipartition. Global equipartition is not realistic, however, because the relaxation and equipartition times vary greatly throughout the cluster, becoming longer than the age of the universe in the outer halo. In fact, we expect equipartition only in the inner region, where relaxation times are shortest. His discussion is mostly unchanged by this, so long as we confine its relevance to processes in this inner region. We shall reproduce here only the main strategy of his argument and its conclusions. Let v_{m1}^2 and v_{m2}^2 represent the mean square velocities of stars in each component. As mentioned, the assumption of equipartition implies that the temperature ratio ξ is equal to unity:

$$\xi \equiv \frac{(1/2)m_2 v_{m2}^2}{(1/2)m_1 v_{m1}^2} = 1 \ . \tag{1}$$

Spitzer then applies a component-wise virial theorem, which for the heavier component gives

$$v_{m2}^2 = \frac{(2/5)GM_2}{r_2} + \frac{G}{M_2}\int_0^\infty \frac{\rho_2 M_1(r) dV}{r} \ , \tag{2}$$

where the first term represents the gravitational self-binding energy of the heavier stars, r_2 is the virial radius of the heavier stars, ρ_2 is the density of the heavier stars at a distance r from the center, $M_1(r)$ is the total mass of the heavier stars contained within the radius r, and dV is the volume element. Spitzer assumes that, since $M_2 \ll M_1$, the second term in the corresponding equation for v_{m1}^2 can be ignored and also that, since $m_2 \gg m_1$, the heavier stars will be concentrated in the center of the system and $M_1(r)$ may be approximated by $4\pi\rho_1(0)r^3/3$ in equation (2). Spitzer & Hart (1971) noticed that the second assumption often does not hold strictly. In particular, even far into the evolution many heavier stars can still reside well outside the core.

Merritt (1981) constructed equilibrium models that violate this assumption by great amounts and discovered that equipartition is possible for some (admittedly unrealistic) configurations that can be realized for all values of m_2/m_1 and M_2/M_1.

Under Spitzer's assumptions, after a series of manipulations, one obtains the following expression for the quantity S as a direct consequence of equation (1):

$$S \equiv \left(\frac{M_2}{M_1}\right)\left(\frac{m_2}{m_1}\right)^{3/2} = \frac{(\rho_{h1}/\rho_{h2})^{1/2}}{(1+\alpha\rho_{h1}/\rho_{h2})^{3/2}}\,, \qquad (3)$$

where ρ_{h1} and ρ_{h2} are the densities within the half-mass radii for each component and where α depends on the distribution of mass within the cluster. In particular, if we denote by r_{m2} and r_{h2} the rms and half-mass radius for the heavier component, respectively, then α is given by

$$\alpha = \frac{5\rho_1(0)}{4\rho_{h1}}\left(\frac{r_{m2}}{r_{h2}}\right)^2 . \qquad (4)$$

Spitzer estimates a value of 5.6 for α. Merritt (1981) contests the value 3.5 assigned by Spitzer to $\rho_1(0)/\rho_{h1}$ for polytropes with n between 3 and 5 but finds that it corresponds approximately to its minimum value, which, as we shall see, does not change Spitzer's conclusion. In particular, since the right-hand side of equation (3) has a maximum value $0.38\alpha^{-1/2}$ with respect to variation in ρ_{h1}/ρ_{h2}, it follows that equipartition is possible only if S does not exceed a critical value $\beta \equiv 0.38\alpha^{-1/2}$,

$$S < \beta\,, \quad \text{where } \beta = 0.16 \text{ for } \alpha = 5.6\,. \qquad (5)$$

Spitzer suggests that for smaller values of the individual mass ratio m_2/m_1, the inequality (5) remains valid, except that $\beta \to 1$ as $m_2/m_1 \to 1$.

While it is useful to assess under what conditions simultaneous dynamical and thermal equilibrium are not expected, it is also interesting for our purposes to estimate the extent of departure from thermal equilibrium where dynamical equilibrium holds. To that end, let us maintain the condition stated in equation (2), while not insisting that $\xi = 1$ in equation (1). By following the value ξ through Spitzer's analysis, we find that the temperature ratio has a lower bound for a given value of S,

$$\xi > \left(\frac{S}{\beta}\right)^{2/3} . \qquad (6)$$

Inagaki & Wiyanto (1984) performed an analysis similar to Spitzer's, except by casting equations (1) and (2) in terms of core values for the component-wise total mass, mean square velocity, and density. They obtained a minimum difference between the core temperatures of each component. Letting M_{c2} and M_{c1} denote the total mass contained within the core for the heavier and lighter components, respectively, and letting r_{c1} represent the core radius of the lighter stars, this difference is given by

$$\frac{1}{2}\,m_2(v_{m1}^2)_{\min} - \frac{1}{2}\,m_1 v_{m1}^2 = \left[\frac{1}{2}\left(\frac{M_{c2}}{M_{c1}}\right)^{2/3} - 0.2m_1\right]\frac{GM_{c1}}{r_{c1}}\,. \qquad (7)$$

For now, we note that the minimum ratio (given by the inequality [6]) and difference (given by eq. [7]) of component-wise temperatures increases with increasing values of the total mass ratio and that this increase is steeper for increasing m_2/m_1 (in eq. [6], recall that $\beta \to 1$ as $m_2/m_1 \to 1$).

Lightman & Fall (1978) developed an approximate theory for the core collapse of two-component clusters that resembles that of Spitzer. They examined two constant-density isothermal spheres representing the cores of the heavier and lighter components, where the radius of the former is smallest. By applying a component-wise virial theorem and several simplifying assumptions, they find a set of four ordinary differential equations for the virial radii and total masses in each component. They obtain the following condition for equipartition of energies between the two components, where we let $\tilde{m} \equiv m_2/m_1$ and $\tilde{M} \equiv M_2/M_1$:

$$\Gamma(\tilde{m}, \tilde{M}) \equiv \tilde{m}^3\tilde{M}^2\left[\frac{27}{4}\left(1+\frac{3\tilde{M}}{2\tilde{m}}\right)\left(1+\frac{5}{2}\,\tilde{M}\right)^{-3}\right] \le 1\,. \qquad (8)$$

Solutions to the differential equations exhibit a minimum temperature ratio $\xi_{\min}$, which they suggest bears the following approximate relation to Γ,

$$\xi_{\min} \simeq \Gamma^{1/3}\,. \qquad (9)$$

In this case also, the minimum temperature ratio increases with increasing values of the individual and total mass ratios.

The majority of attempts to evaluate the theoretical predictions of Spitzer (1969) and Lightman & Fall (1978) have met with moderate success. For example, Yoshizawa et al. (1978) obtained 0.25 for the value of β in the case of spatially truncated two-component isothermal spheres. Nevertheless, few investigations have examined models with a comprehensive range of individual and total mass ratios (outside studies based on turning points along a sequence of equilibrium models). In the next two sections we discuss the methods we used and results we obtained for a relatively broad survey of the parameter space determined by M_2/M_1 and m_2/m_1.

3. NUMERICAL METHODS AND DEFINITIONS

We used a Monte Carlo method for modeling the dynamical evolution of clusters as a sequence of equilibrium models subject to regular velocity perturbations. The velocity perturbations represent the average effect of long-range stellar encounters (Hénon 1971). Our Monte Carlo code has been described in detail by Joshi et al. (2000b) and Joshi, Nave, & Rasio (2000a). The code allows us to perform dynamical simulations for realistic clusters containing up to $N \sim 10^5$–10^6 stars on a parallel supercomputer. In the present study, we ignore the effects of binaries, stellar evolution, and the galactic tidal field. Our units are defined in Joshi et al. (2000b, § 2.8). The unit of length is close to the virial radius of the cluster, the mass is measured in units of the total initial cluster mass, and the unit of time is of order the initial half-mass relaxation time $t_{\rm rh}$. In this paper, when reporting times in units of $t_{\rm rh}$, we have adopted the same definition used in previous studies of two-component clusters since Spitzer & Hart (1971),

$$t_{\rm rh} = \frac{0.06M^{1/2}r_h^{3/2}}{G^{1/2}m\log_{10}(0.4N)}\,, \qquad (10)$$

where $M = M_1 + M_2$ is the total cluster mass, r_h is the initial cluster half-mass radius, and $m = M/N$ is the average stellar mass.

We undertook two sets of calculations, hereafter called sets A and B. All calculations are performed for a cluster containing $N = 10^5$ single stars. The initial model used in each calculation of set A was a two-component King model (King 1966). In particular, the velocities and positions for all stars with a mass m_1 were chosen according to the King model distribution function with dimensionless central potential $W_0 = 6$. Although the initial King model is truncated at its finite tidal radius, we do not enforce that tidal boundary during the evolution, allowing the cluster to expand indefinitely. In each case, some fraction of the stars was then changed to a mass $m_2 > m_1$ according to a chosen value of the total mass ratio M_2/M_1. The initial ratio of mean temperatures in the heavier and lighter components was therefore m_2/m_1. The initial models for calculations in set B were generated in a similar way, except using the Plummer distribution function (polytrope with $n = 5$; see, e.g., Binney & Tremaine 1989). Both sets of calculations are listed in Table 1. Set A includes calculations undertaken for a range of total mass ratios [$M_2/M_1 = (3 \times 10^{-3})$–0.6] and a range of individual mass ratios ($m_2/m_1 = 1.5$–6). Set B comprises only nine systems, including four studied by Inagaki & Wiyanto (1984), all with $m_2/m_1 = 2$. Every calculation is terminated at core collapse, measured at the instant that a number density of 10^8 (in our units) is attained in the core. Our results are not sensitive to the exact value of the core density used to terminate the calculation. However, the

TABLE 1

Models and Results

S	m_2/m_1	M_2/M_1	N_2	ξ_{min}	t_ρ/t_{rh}	t_{cc}/t_{rh}	Model
0.05	1.50	0.0272	1782	1.010	12.8	12.8	King
0.05	1.75	0.0216	1219	1.020	12.0	12.0	King
0.05	2.00	0.0176	876	1.028	11.2	11.2	King
0.05	3.00	0.00962	320	1.027	9.2	9.2	King
0.05	4.00	0.00625	156	1.016	8.7	9.0	King
0.05	6.00	0.00340	57	1.018	8.2	8.2	King
0.10	1.50	0.0544	3502	1.023	12.3	12.3	King
0.10	1.75	.0432	2409	1.035	10.9	11.1	King
0.10	2.00	0.0354	1737	1.043	9.2	9.6	King
0.10	3.00	0.0192	637	1.039	5.9	6.4	King
0.10	4.00	0.0125	311	1.053	4.9	5.5	King
0.10	6.00	0.00680	113	1.061	3.7	4.3	King
0.15	1.50	0.0816	5162	1.027	11.5	11.7	King
0.15	1.75	0.0648	3570	1.043	9.7	10.1	King
0.15	2.00	0.0530	2583	1.044	7.5	8.5	King
0.15	3.00	0.0289	953	1.082	4.4	5.3	King
0.15	4.00	0.0188	467	1.094	3.3	3.9	King
0.15	6.00	0.0102	170	1.128	2.4	2.6	King
0.20	1.50	0.109	6767	1.021	11.0	11.5	King
0.20	1.75	0.0864	4704	1.049	8.4	9.5	King
0.20	2.00	0.0707	3415	1.052	6.8	7.8	King
0.20	3.00	0.0385	1267	1.090	3.8	4.6	King
0.20	4.00	0.0250	621	1.124	2.6	3.2	King
0.20	6.00	0.0136	226	1.138	1.6	1.9	King
0.25	1.50	0.136	8318	1.035	10.3	11.0	King
0.25	1.75	0.108	5812	1.049	8.1	9.2	King
0.25	2.00	0.0883	4232	1.072	6.2	7.6	King
0.25	3.00	0.0481	1578	1.104	3.2	4.2	King
0.25	4.00	0.0313	775	1.154	2.0	2.6	King
0.25	6.00	0.0170	283	1.175	1.2	1.5	King
0.50	1.50	0.272	15358	1.043	6.9	10.2	King
0.50	1.75	0.216	10986	1.057	5.0	8.4	King
0.50	2.00	0.176	8121	1.079	3.8	6.6	King
0.50	3.00	0.0962	3108	1.173	2.0	3.2	King
0.50	4.00	0.0625	1539	1.312	1.4	2.0	King
0.50	6.00	0.0340	564	1.355	1.0	1.2	King
1.10	1.50	0.599	28529	1.046	1.64	10.2	King
1.24	5.0	0.111	2172	1.60	0.7	1.2	King
0.0042......	2.00	0.0015	74	1.008	14.5	14.5	Plummer
0.0057......	2.00	0.0020	101	1.021	14.3	14.3	Plummer
0.0085......	2.00	0.0030	150	1.021	14.0	14.0	Plummer
0.0101......	2.00	0.0036	178	1.008	13.9	13.9	Plummer
0.0141......	2.00	0.0050	249	1.021	13.7	13.7	Plummer
0.0286......	2.00	0.0101	503	1.005	12.8	12.8	Plummer
0.113	2.00	0.040	1960	1.049	9.0	9.3	Plummer
0.314	2.00	0.111	5262	1.066	5.6	7.5	Plummer
2.824	2.00	1.000	33333	1.127	0.014	7.5	Plummer

value of the core collapse time t_{cc} determined numerically can have a large statistical uncertainty, particularly when the core is dominated by a small number of heavier stars near the end of the evolution (in those cases we estimate that the statistical uncertainty on the values of t_{cc}/t_{rh} reported in Table 1 can be as large as $\sim 5\%$). The majority of our calculations require 10–20 central processing unit hours on an SGI/Cray Origin2000 supercomputer to reach core collapse.

The range of values we consider for m_2/m_1 and M_2/M_1 includes a number of astrophysically relevant cases. In particular, a subpopulation of neutron stars in a dense globular cluster might have $m_2/m_1 \simeq 2$ (e.g., corresponding to $m_2 = 1.4\ M_\odot$ for an average background stellar mass $m_1 = 0.7\ M_\odot$) and $M_2/M_1 \sim 10^{-3}$–10^{-2} depending on the neutron star retention fraction. A subpopulation of black holes might have $m_2/m_1 \simeq 5$–10 and $M_2/M_1 \sim 10^{-3}$–10^{-2}. Massive blue stragglers or primordial binaries containing main-sequence stars near the turn-off mass would have $m_2/m_1 \simeq 2$–3 and $M_2/M_1 \sim 10^{-3}$–10^{-1}.

For each calculation we record several quantities at each program time step (the time steps are proportional to a fraction of the core relaxation time; see Joshi et al. 2000b). These continuous measurements include the total cluster core radius and several component-wise Lagrange radii. Within each of these radii we also count the number of stars and calculate the mean temperature and mean mass density for each component. Of particular interest are the quantities calculated inside the core radius, where relaxation times are shortest and where thermal equilibrium is the most likely to occur. We use the customary definition for the total cluster core radius r_c (Spitzer 1987),

$$r_c = \left[\frac{3v_m(0)^2}{4\pi G\rho(0)}\right]^{1/2}, \quad (11)$$

where $v_m(0)^2$ is the mean square velocity and $\rho(0)$ is the mean density of stars at the center. We calculate the ratio of core densities in each component (ρ_{c2}/ρ_{c1}) and the ratio of core temperatures. We also calculate the minimum core temperature ratio ξ_{min} that is reached after approximately 90% of the precollapse evolution in all cases. Specifically, ξ_{min} is calculated as the average temperature ratio from 90% to 95% of the core collapse time. The core collapse time t_{cc} and the time t_ρ at which the core mass densities of each component become equal (i.e., $\rho_{c2}/\rho_{c1} = 1$) are also measured. We report our main results in Table 1, and we discuss these in the following section.

4. RESULTS

The evolution of the core temperature ratio ξ is shown in Figure 1 for three calculations in set A (two-component King models), namely, for $S = 0.05$ (*top*), $S = 0.5$ (*middle*), and $S = 1.24$ (*bottom*). Figure 2 shows the core temperatures of the lighter stars (*top*) and the heavier stars (*bottom*) for the case $S = 1.24$. Several features that we expect and that have been mentioned already in § 2 are easily recognizable. The temperature ratio begins with a value m_2/m_1 and then decreases gradually as equipartition is approached. It is clear that ξ reaches a minimum value that is greater than 1 for the case $S = 1.24$, so that equipartition is clearly never attained. Equipartition is nearly attained for $S = 0.5$, and $\xi_{min} = 1$ to within 5% for $S = 0.05$. It is clear from Figure 2 that the heavier component cools initially, then maintains a constant mean kinetic energy, and then begins to heat prior to core collapse. At the same time, the lighter component steadily becomes hotter as it receives energy from the heavier component. The temperature ratio in the last time steps becomes very noisy because the temperatures are computed using the relatively few stars that remain in the core.

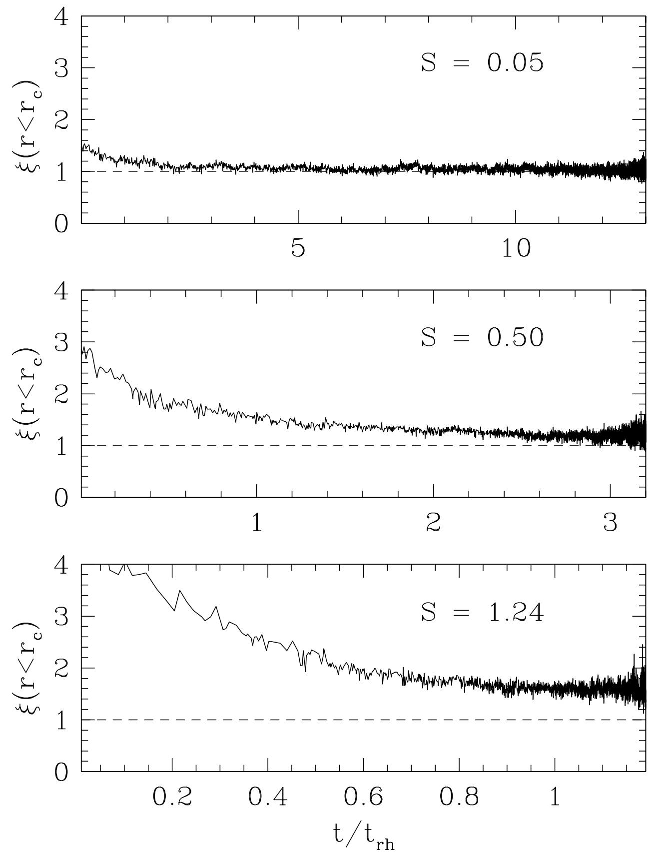

FIG. 1.—Evolution of the temperature ratio in the core for $S = 0.05$ and $m_2/m_1 = 1.5$ (*top*), $S = 0.5$ and $m_2/m_1 = 3$ (*middle*), and $S = 1.24$ and $m_2/m_1 = 5.0$ (*bottom*). The minimum temperature ratio attained in each calculation increases with S. Time is displayed in units of the initial half-mass relaxation time (t_{rh}). In each case the evolution is shown until shortly before core collapse. Equipartition is clearly not reached prior to core collapse for large S. Notice also that core collapse occurs sooner with increasing S. The initial condition in each case was a two-component King model with $W_0 = 6$.

The temperature ratio ξ reaches a minimum value at different times with respect to core collapse for each of the models shown in Figure 1. In cases where the minimum value is greater than 1, it sometimes appears that the gravothermal catastrophe has beaten the approach to equipartition. In such cases, one may ask whether an initial model with a less concentrated spatial distribution and a different initial relaxation time would yield a different minimum temperature ratio. In fact, we find that ξ_{min} is robust with respect to changes in the initial value of the dimensionless potential W_0. The evolution of the temperature ratio for three calculations, where $W_0 = 1$, 5, and 10 for $S = 1$ and $m_2/m_1 = 5$, are shown in Figure 3. In all three cases the minimum temperature ratio is approximately 1.55.

The evolution of the core mass density ratio ρ_{c2}/ρ_{c1} is shown in Figure 4 for the same three clusters and in the same order. The core mass densities of each component are

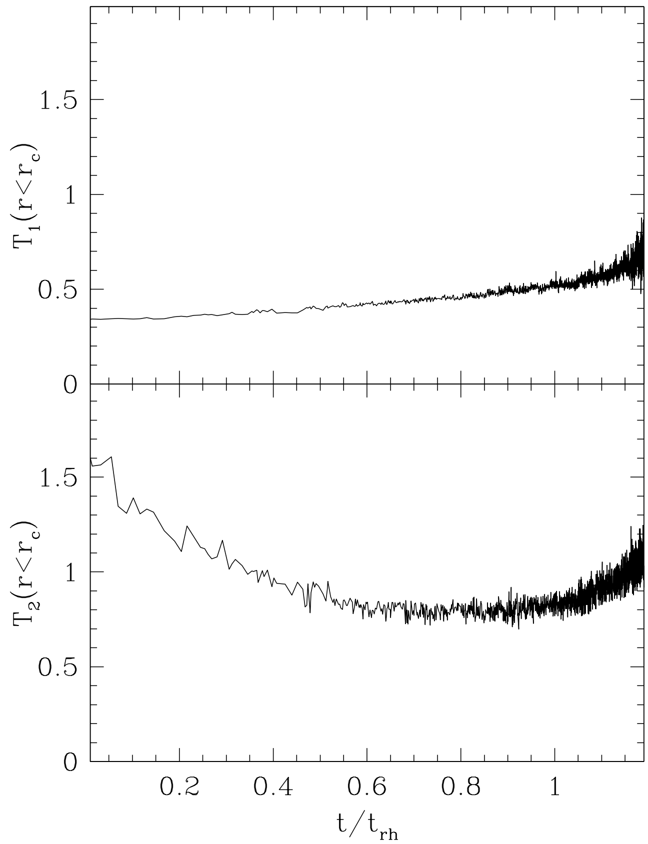

FIG. 2.—Evolution of the core temperature for the lighter stars [$T_1(r < r_c)$; *top*] and the heavier stars [$T_2(r < r_c)$, *bottom*] for the case $S = 1.24$ and $m_2/m_1 = 5.0$ (the ratio of these is shown in Fig. 1, *bottom*). Time is displayed in units of the initial half-mass relaxation time (t_{rh}).

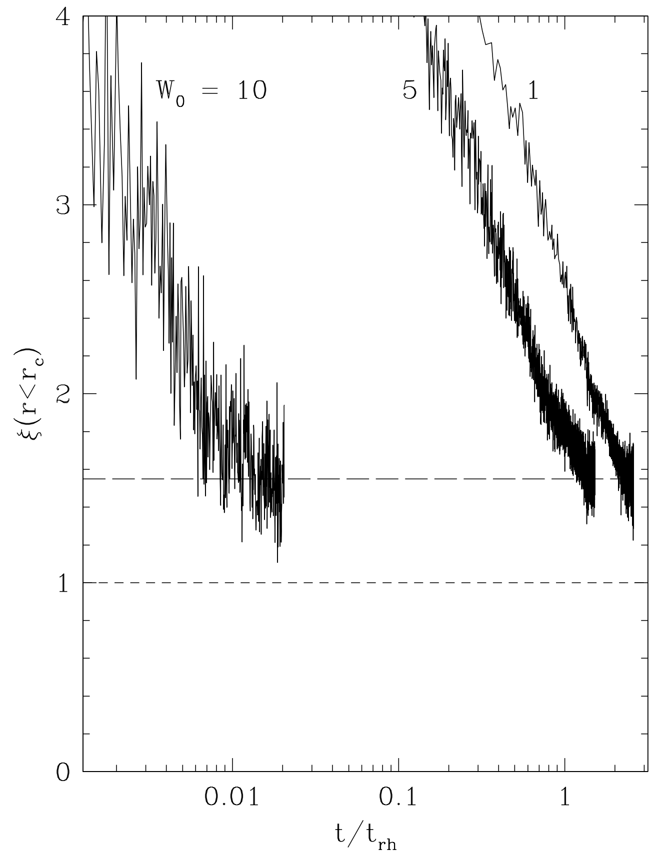

FIG. 3.—Evolution of the core temperature ratio for three calculations with different initial values of the dimensionless King model potential W_0, but all with $S = 1$ and $m_2/m_1 = 5$ ($M_2/M_1 \simeq 0.09$). *From left to right*: $W_0 = 10$, $W_0 = 5$, and $W_0 = 1$. While the relaxation and core collapse times for these calculations span a wide range, in each case the temperature ratio reaches the same minimum value of approximately 1.55. The evolution is shown until shortly before core collapse in each case. Note that the logarithmic timescale has compressed the shapes of these curves, so that the leveling in the temperature ratio prior to core collapse is not as clearly apparent as in Fig. 1.

shown in Figure 5 for the model with $S = 1.24$. One can see clearly that, as S increases, the core mass densities in each component become equal sooner with respect to core collapse. That is, for larger S the self-gravity of the heavier stars dominates the potential in the core for a longer period prior to the onset of the gravothermal catastrophe. Moreover, we can see that, although the core density is initially dominated by the lighter stars, the heavier stars overtake and exceed the density of lighter stars by more than an order of magnitude prior to core collapse for $S = 0.5$ and $S = 1.24$.

Approximate values of the minimum core temperature ratio $\xi_{\min}$ are plotted using three symbols in the parameter space determined by M_2/M_1 and m_2/m_1 in Figure 6 for 37 calculations in set A. Also drawn are the Spitzer and Lightman-Fall "stability boundaries," above which simultaneous dynamical and thermal equilibrium are supposedly prohibited ($S = 0.16$ and $\Gamma = 1$, respectively; cf. eqs. [5] and [8]). Our simulations allow us to determine $\xi_{\min}$ with a numerical accuracy of about 5%. Specifically, $\xi_{\min}$ is calculated as the average core temperature ratio from 90% to 95% of the precollapse evolution, and this average has a standard deviation of approximately 0.05 in our calculations for $N = 10^5$ stars. Therefore, those calculations marked with an open square in Figure 6 have been determined to reach equipartition within our numerical accuracy. One can see that the Spitzer boundary $S = 0.16$ is approximately respected for $m_2/m_1 \geq 2$. By comparison, the Lightman-Fall boundary falls well inside the range of clusters that have clearly not attained equipartition. In spite of this, the Lightman-Fall boundary better reproduces the shape of boundaries between regions of constant $\xi_{\min}$. A more properly drawn Spitzer boundary has a similar shape, recalling that $\beta \to 1$ as $m_2/m_1 \to 1$. Based on the results shown in Figure 6, we propose our own condition for equipartition,

$$\Lambda \equiv \left(\frac{M_2}{M_1}\right)\left(\frac{m_2}{m_1}\right)^{2.4} \geq 0.32 \, . \qquad (12)$$

The boundary determined by equation (12) is strictly valid for $1.75 < m_2/m_1 < 7$ and is also drawn in Figure 6. For $m_2/m_1 < 1.75$, equipartition is achieved for all clusters considered. For $m_2/m_1 > 7$, extrapolation of equation (12) seems reasonable.

The dependence of $\xi_{\min}$ on S for several values of m_2/m_1 is shown in Figure 7. Also drawn is the Spitzer stability boundary. These curves are broadly consistent with trends anticipated by the inequality (6). In particular, $\xi_{\min}$ increases with S, and the initial slope of this increase becomes larger with increasing m_2/m_1 (again recalling that $\beta \to 1$ as $m_2/m_1 \to 1$). The dependence of $\xi_{\min}$ on Γ for several values of m_2/m_1 is shown in Figure 8. Also drawn is

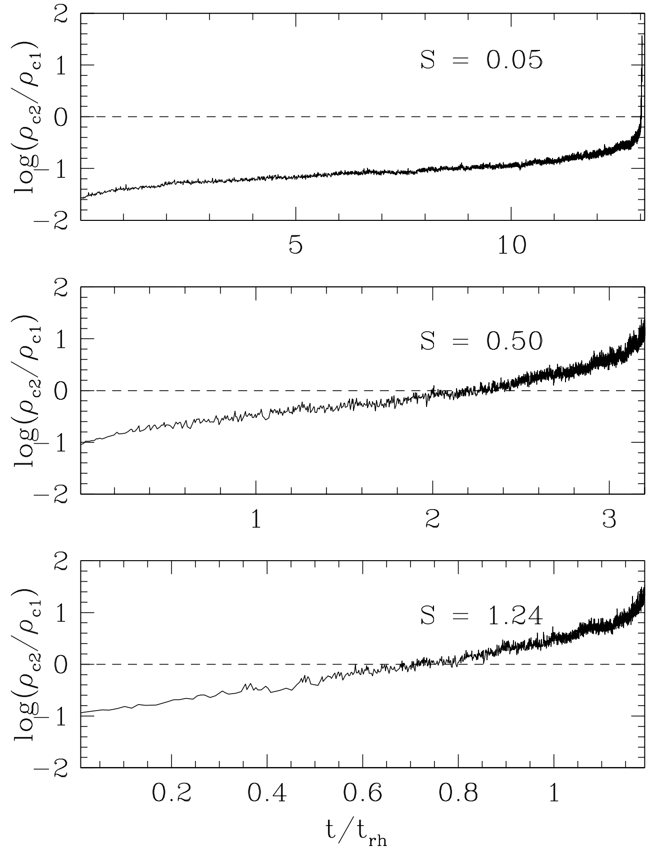

FIG. 4.—Evolution of the mass density ratio in the core (ρ_{c2}/ρ_{c1}) for the same three cases as in Fig. 1: $S = 0.05$ and $m_2/m_1 = 1.5$ (*top*), $S = 0.5$ and $m_2/m_1 = 3$ (*middle*), and $S = 1.24$ and $m_2/m_1 = 5.0$ (*bottom*). The evolution is shown until shortly before core collapse in two cases (*middle*, *bottom*) and until core collapse in one case (*top*). As S increases, the time at which equal mass densities are attained in the core (t_ρ) occurs sooner with respect to core collapse ($t_\rho \simeq t_{cc}$ for the case $S = 0.05$).

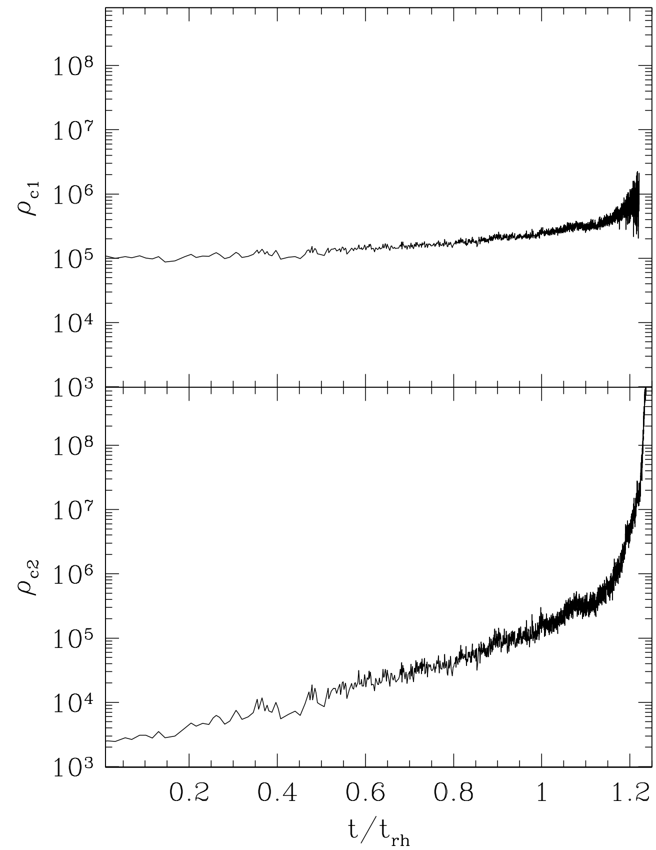

FIG. 5.—Evolution of the core mass density for the lighter stars (*top*) and the heavier stars (*bottom*) for the same case as in Fig. 2: $S = 1.24$ and $m_2/m_1 = 5.0$ (the ratio of these is shown in Fig. 4, *bottom*). The evolution of ρ_{c1} is shown until no lighter stars remain in the core, whereas the evolution of ρ_{c2} is shown until core collapse.

the value of ξ_{min} anticipated by equation (9), and the Lightman-Fall stability boundary. One can see that, while the numerical results are displaced from the predicted curve, they have the same curvature and display a similar trend. The minimum temperature *difference* for calculations in set B (two-component Plummer models) are shown in Figure 9, where they are compared with the results of Inagaki & Wiyanto (1984). Here agreement is excellent except in the limit of small M_2/M_1, where the temperature of the heavier component is calculated using very few stars, so that the difference is characterized by a large amount of noise. As a last comparison, we note that, using a Monte Carlo scheme different from ours, Spitzer & Hart (1971) found that $\xi_{min} = 1.34$ for a Plummer model with $S = 1.24$ and $m_2/m_1 = 5$, whereas for the same system we obtain $\xi_{min} = 1.60$.

Recall that t_ρ is the time at which the mass densities of each component become equal in the core. Approximate values of the time t_ρ as a fraction of the core collapse time t_{cc} are plotted using three symbols in the parameter space determined by M_2/M_1 and m_2/m_1 in Figure 10 for 37 calculations in set A. This plot confirms the trend anticipated by the previous examination of three individual clusters (Fig. 4), namely, the amount of time—as a fraction of the core collapse time—during which the heavier stars dominate the potential in the core increases with S. The trends appear not

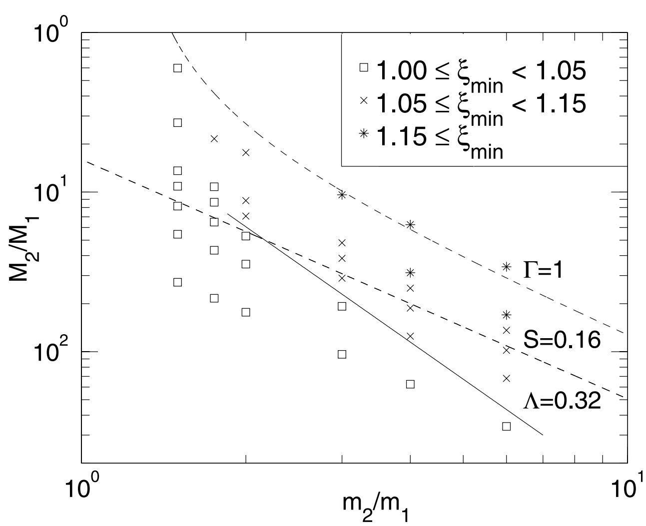

FIG. 6.—Minimum temperature ratio in the core for the 37 calculations in set A, represented here using three symbols at points in the parameter space determined by M_2/M_1 and m_2/m_1. Also drawn are the Spitzer and Lightman-Fall stability boundaries ($S = 0.16$ and $\Gamma = 1$, respectively) and the boundary $\Lambda = 0.32$ suggested by these results (eq. [12]). Calculations marked with an open square are determined to have reached equipartition within our numerical accuracy.

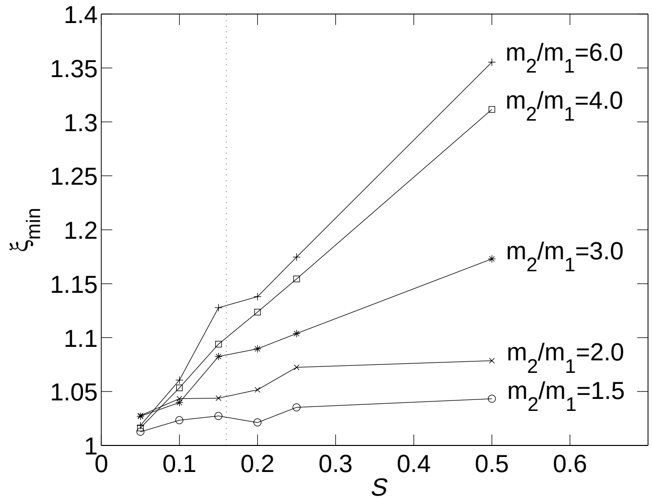

FIG. 7.—Minimum temperature ratio in the core vs. S for several values of m_2/m_1. Also drawn is the Spitzer stability boundary ($S = 0.16$).

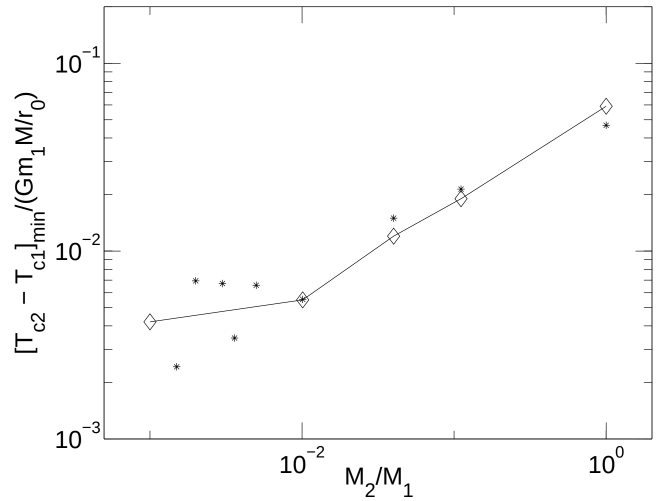

FIG. 9.—Minimum temperature difference in the core vs. the total mass ratio as found in the present study (*asterisks*) and by Inagaki & Wiyanto (1984) (*diamonds*). Each of these calculations was begun with a two-component Plummer model (set B) where $m_2/m_1 = 2$. Here r_0 is the Plummer scale length and M is the total cluster mass. Agreement is very good except in the vicinity of small M_2/M_1. In this domain, the temperature of the heavier component is calculated using very few stars, so that the difference is characterized by a large amount of noise.

to respect any of the previous stability boundaries very well, but our condition given by equation (12) fares best. They may nevertheless shed light on the related question of whether a dense subsystem of heavy stars collapses *independently*, since the self-gravity of the heavier stars must dominate the potential in the core in order for this to happen. Where equipartition is attained, the mass segregation is retarded or stopped, so that equal mass densities may not occur until core collapse (i.e., $t_\rho \simeq t_{cc}$).

All of the calculations were terminated at core collapse, at which time the radius containing 1% of the mass in the heavier component diminishes sharply. The time t_{cc} is measured at the instant when a number density of 10^8 in our units is attained within the core (see § 3). The variation of the core collapse time t_{cc} with S for several values of m_2/m_1 is shown in Figure 11. The trends confirm that the onset of core collapse is accelerated by the presence of a small and heavier subpopulation, in agreement with the findings of others (Inagaki & Wiyanto 1984; Inagaki 1985; Quinlan 1996).

5. DISCUSSION

In this section we discuss several features of the evolution in more detail, and we mention some possible astrophysical applications of our results to the evolution of compact stellar remnants in globular clusters.

The temperature ratio ξ in Figure 1 initially has the value m_2/m_1. While this is an artifact of the way our initial models were constructed, m_2/m_1 happens to be also the most realistic value of ξ for equilibrium models with a relatively shallow potential. In families of equilibrium models it is

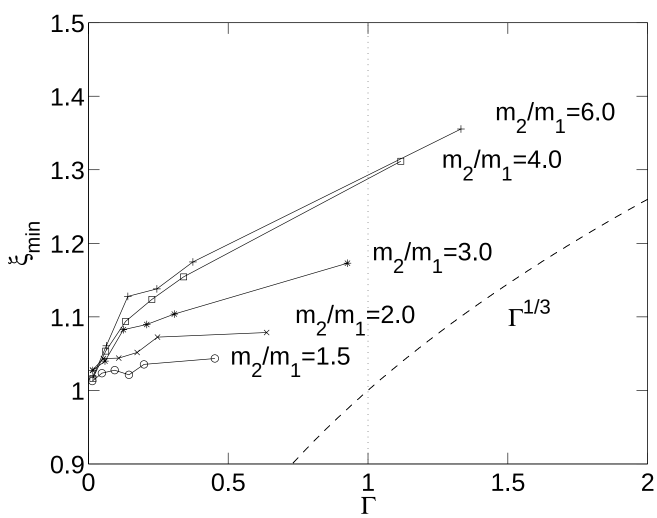

FIG. 8.—Minimum temperature ratio in the core vs. Γ for several values of m_2/m_1. Also drawn is the Lightman-Fall stability boundary ($\Gamma = 1$) and a theoretical estimate of the minimum core temperature ratio, $\Gamma^{1/3}$ (Lightman & Fall 1978).

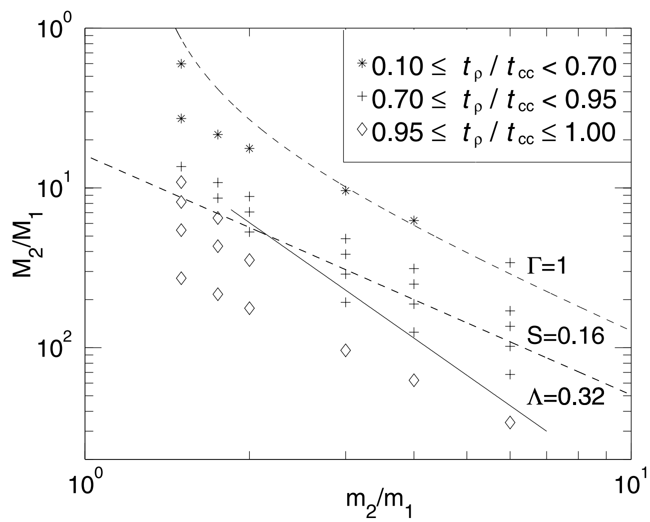

FIG. 10.—Fraction of the core collapse time when equal mass density is attained in the core (t_ρ/t_{cc}) for the 37 calculations in set A, represented here using three symbols at points in the parameter space determined by M_2/M_1 and m_2/m_1. Also drawn are the Spitzer and Lightman-Fall stability boundaries ($S = 0.16$ and $\Gamma = 1$, respectively) and the boundary suggested by the results shown in Fig. 6 ($\Lambda = 0.32$). Note that our boundary appears to apply approximately throughout the range $1 < m_2/m_1 < 7$ (in contrast to Fig. 6, where it does not apply below $m_2/m_1 \simeq 1.75$).

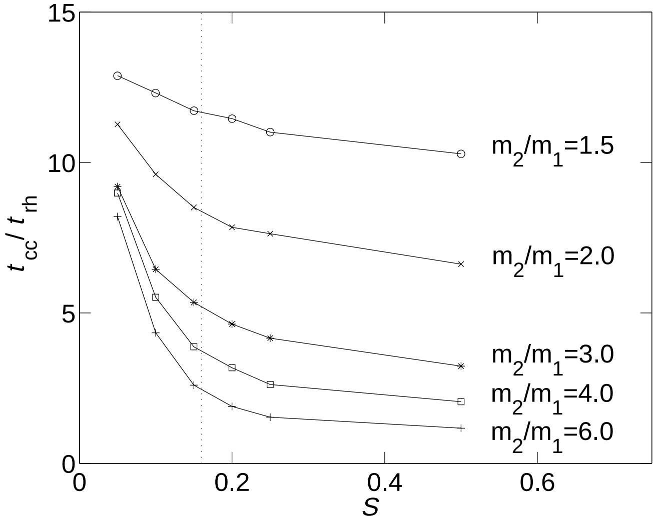

FIG. 11.—Core collapse times vs. S for several values of m_2/m_1, for 30 calculations in set A. The initial condition in each case was a two-component King model with $W_0 = 6$. The times are displayed in units of the initial half-mass relaxation time ($t_{\rm rh}$).

typical to find that $\xi \rightarrow m_2/m_1$ as $W_0 \rightarrow 0$ (Kondrat'ev & Ozernoy 1982; Katz & Taff 1983). In trials for which initial models were modified so that ξ had some value other than m_2/m_1, a brief period of rapid relaxation ensued that restored the value m_2/m_1. This effect has been observed in simpler models of the evolution calculated using other methods (Lightman & Fall 1978).

In the core, the initial behavior of the temperature ratio is mostly determined by the temperature of the heavier component, while the mean kinetic energy of the lighter stars, which are more abundant at first, increases gradually (Fig. 2). Spitzer (1969) suggested that the approach to equipartition is characterized by the exponential decay of kinetic energy in the heavier component, with a time constant equal to *twice* the so-called equipartition time,

$$t_{\rm eq} = t_{r1} \frac{3\pi^{1/2}}{16} \frac{m_1}{m_2} \left(1 + \frac{v_{m2}}{v_{m1}}\right)^{3/2} , \quad (13)$$

where t_{r1} is a relaxation time for the stars of mass m_1. In the case where the mean square velocities of each component are initially equal, the initial equipartition time is $t_{\rm eq} \simeq t_{r1}(m_1/m_2)$. (It should be noted that $t_{\rm eq}$ decreases as equipartition is approached.) This characterization of the decline in kinetic energy of heavier stars agrees very well with our results for stars contained within the half-mass radius, where we assume $t_{r1} \simeq t_{\rm rh}$, the initial half-mass relaxation time for the cluster as a whole. In particular, the kinetic energy of the heavier component diminishes to a fraction $1/e$ of its initial value (after subtracting its minimum value for the entire evolution) within $0.39 t_{\rm rh}$ for $S = 1$, $m_2/m_1 = 5$, and within $1.3 t_{\rm rh}$ for $S = 0.5$, $m_2/m_1 = 1.5$, in good agreement with the theory [which predicts a time $2t_{\rm eq} \simeq 2(m_1/m_2)t_{\rm rh}$]. However, we find that equipartition is approached on a similar timescale in the core, where the theory predicts that $t_{\rm eq}$ should be shorter by approximately $\frac{1}{5}$, and hence agreement is poor (the ratio of initial core and half-mass relaxation times for King models with $W_0 = 6$ is approximately $\frac{1}{5}$; see Quinlan 1996).

A leveling in the temperature ratio at a minimum value greater than 1 has been observed in calculations using other methods as well (Inagaki & Wiyanto 1984; Lightman & Fall 1978). We find that this leveling is approximately coincident with the heavier stars reaching their maximum numbers within the core. Inagaki & Wiyanto (1984) found that an increase in the core temperature ratio prior to or during collapse is coincident with t_ρ, the time at which equal mass densities are attained in the core. While we are not able to resolve adequately the late-collapse behavior of ξ (because our calculation loses accuracy in this regime), it is clear from Figure 2 that the heavier component begins to heat prior to collapse. Indeed, we find that the temperature of the heavier component does not begin to rise until $t > t_\rho$.

Katz & Taff (1983) examined the turning points in a linear series of equilibrium models. In particular, they studied a one-parameter family of self-bounded isolated equilibrium models with a lowered Maxwellian velocity distribution. Turning points representing the limits of stability for dynamical equilibrium were obtained for several values of m_2/m_1 and M_2/M_1 in terms of maximum possible values of the dimensionless potential k. Katz & Taff also calculated the core temperature ratio ξ that corresponds to each maximum value of k. Since for their models ξ was found to approach 1 for large values of k, the core temperature ratios calculated for each turning point represent the minimum allowed value of ξ for given values of m_2/m_1 and M_2/M_1. These are plotted with respect to S for several values of m_2/m_1 in Figure 12 and should be compared with our results, shown in Figure 7. While these numbers exhibit a similar trend with S for given m_2/m_1, the trend in m_2/m_1 evidently disagrees with our results and therefore also the prediction of the inequality (6). It is important to bear in mind, however, that these ratios are obtained for only the members of this particular family, which do not necessarily represent states that can be obtained by real dynamical processes. Our results contradict those of Katz & Taff (1983) insofar as we find clusters with smaller values of $\xi_{\rm min}$ for identical values of m_2/m_1 and M_2/M_1 that appear to be stable on dynamical timescales. Katz & Taff found that the number of possible configurations for their models in dynamical equilibrium diminish sharply for $S > 0.16$. It is interesting to note also that the values they obtain for $\xi_{\rm min}$ appear not to diminish below 1.10 for small values of M_2/M_1.

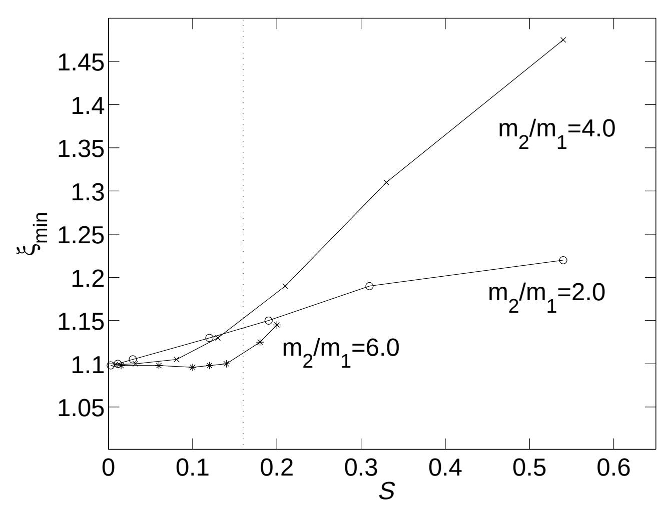

FIG. 12.—Minimum core temperature ratio for turning points in a linear series of equilibrium models (Katz & Taff 1983). Also drawn is the Spitzer stability boundary ($S = 0.16$).

We concur with the findings of Spitzer & Hart (1971) that many heavier stars remain outside the core throughout the evolution. This is clear from plots of Lagrange radii as a function of time for the heavier component (Fig. 13). This casts doubt on the assumption, committed in Spitzer's original analysis, that all of the heavier stars quickly become concentrated in the core (see § 2). In particular, we find that for $S = 1.24$, $M_2/M_1 = 0.111$, and $m_2/m_1 = 5$, the radius containing 75% of the mass in the heavier component diminishes to only 50% of its initial value (and hence remains larger than the core radius) throughout the evolution. Nevertheless, by the onset of core collapse, we frequently observe for calculations with large m_2/m_1 that no lighter stars remain in the core.

Our results may have important implications for the dynamical evolution of various subpopulations of interesting objects in globular clusters. In particular, a subcomponent of primordial black holes with $m_2/m_1 \simeq 10$ is expected to remain far from energy equipartition with the rest of the cluster and to evolve very quickly to core collapse on its own relaxation timescale. For a typical cluster initial mass function, and assuming that all black holes formed initially by the stellar population are retained in the cluster, we expect $M_2/M_1 \simeq 10^{-3}$–10^{-2}, well above our stability boundary in Figure 6. Recent N-body simulations for clusters containing primordial black holes indicate that, after reaching core collapse, the dense subcluster of black holes evaporates quickly in the background cluster potential. Three-body processes occurring in the postcollapse phase produce a significant number of tight black hole binaries that will coalesce in a few billion years, making these binaries important sources of gravitational waves for current ground-based laser interferometers (Portegies Zwart & McMillan 2000; see also Kulkarni, Hut, & McMillan 1993 and Sigurdsson & Hernquist 1993). For neutron stars, with $m_2/m_1 \simeq 2$, our results suggest that equipartition can be reached if the fraction of the total cluster mass in neutron stars is $\lesssim 5\%$. This fraction is higher than would be predicted for a standard cluster initial mass function and neutron star progenitor masses (even if, in contrast to what is suggested by many observations, neutron stars were born without the kicks that might eject a large fraction from the cluster). However, many multimass King models and dynamically evolving Fokker-Planck models of globular clusters based on fits to both photometric and kinematic data suggest that much higher fractions of neutron stars may be present in many clusters. For example, the recent Fokker-Planck models of Murphy et al. (1998) for 47 Tuc suggest that 4.6% of the total cluster mass is in the form of dark stellar remnants of mass 1.4 $M_\odot$. With such a high mass fraction, it is possible that the neutron stars in 47 Tuc may remain out of energy equipartition with the rest of the cluster. More sophisticated dynamical simulations taking into account the full mass spectrum of the cluster, stellar evolution, and binaries will be necessary to resolve the issue.

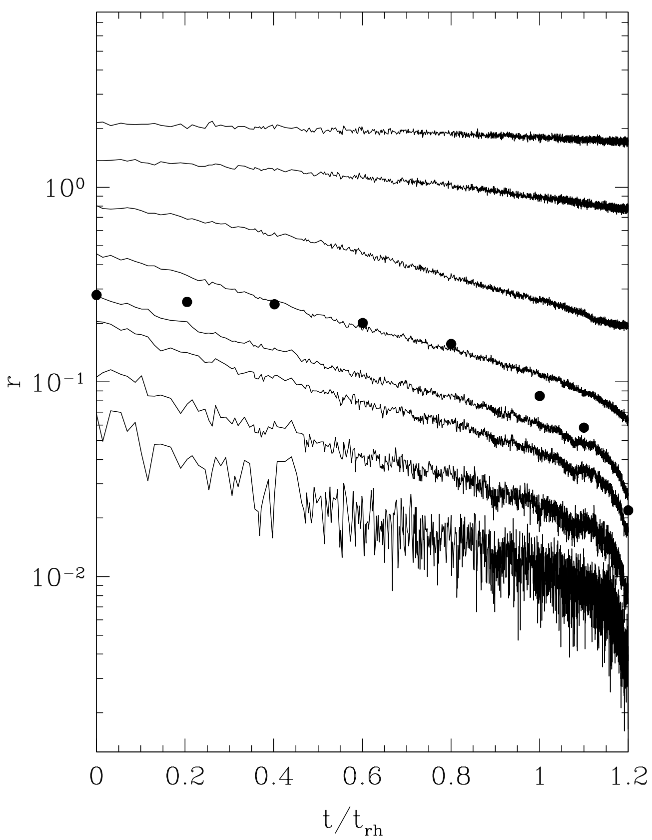

FIG. 13.—Eight Lagrange radii for the heavier component in the two-component King model with $S = 1.24$, $m_2/m_1 = 5$ (same model as in Fig. 2). *From top to bottom*: The radii containing 90%, 75%, 50%, 25%, 10%, 5%, 1%, and 0.1% of the total mass in the heavier component. Also drawn are several points in the evolution of the cluster core radius (*filled circles*). Note that many stars in the heavier component remain well outside of the core throughout the precollapse evolution.

6. SUMMARY AND CONCLUSIONS

Although we have omitted the effects of binaries (which have been shown to retard the onset of core collapse; see, e.g., Gao et al. 1991) and stellar evolution (which can have important implications for the early evolution; see Joshi et al. 2000a), the following observations and conclusions seem justified.

1. For some two-component clusters the core temperature ratio becomes constant over some fraction of the evolution at a minimum value greater than 1, in agreement with previous results obtained using other methods.

2. The departure from equipartition calculated for a range of individual and total mass ratios approximately respects the theoretical predictions of Spitzer (1969) and Lightman & Fall (1978). The agreement with Spitzer is reasonable for $m_2/m_1 \geq 2$, and the Lightman-Fall stability boundary ($\Gamma = 1$) appears to reflect the shape of regions of constant ξ_{min} in the parameter space determined by M_2/M_1 and m_2/m_1, although it lies well inside the region occupied by clusters for which equipartition is clearly not attained. A more accurate boundary suggested by our results is given by equation (12). The trend in the minimum values predicted for the temperature ratio by Lightman & Fall (1978) is similar to what we observe.

3. Stars in the heavier component do not immediately fall into the cluster core, as assumed by Spitzer in his analysis, and instead many remain well outside the core throughout the evolution.

4. The approach to equipartition within the half-mass radius appears to occur on the timescale $2t_{eq}$, as suggested by Spitzer (1969; see eq. [13]).

5. A core temperature ratio of m_2/m_1 appears to be a robust quantity for equilibrium models with a relatively shallow potential, in agreement with previous results obtained using other methods.

6. For clusters with $M_2/M_1 \ll 1$ and $m_2/m_1 > 1$, core collapse times decrease with increasing M_2/M_1 and m_2/m_1, in agreement with previous results.

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