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LETTERS TO NATURE

An observational test for the existence of a planetary system orbiting PSR1257+12

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FOLLOWING the first report[1] of an object of planetary mass orbiting a pulsar, Wolszczan and Frail[2] have now reported the even more surprising discovery of two planet-size companions in orbit around the nearby millisecond pulsar PSR1257+12. The orbital periods of the two planets are about 98 days and 67 days, very close to a 3:2 ratio. Here we point out that, because of this near commensurability, the mutual gravitational perturbations of the two planets should produce not only small secular changes, but also larger periodic changes in their orbital elements. In particular, we find that changes in the eccentricities and orbital periods should become measurable within a few years. Such a measurement would help determine the three masses in the system and the inclinations of the orbits. More importantly, a detection of these changes, if they accord with the theoretical predictions presented here, would provide irrefutable confirmation that the periodic residuals observed by Wolszczan and Frail are indeed caused by orbiting planets, rather than some other effect. For the single planet-size object previously reported[1] around the pulsar PSR1829−10, there is no dynamical test analogous to the one proposed here to confirm the planetary interpretation.

To each planet orbiting the pulsar there corresponds a mass function[3] $f_j = (m_j \sin i)^3/(M+m_j)^2$. Here M is the mass of the central neutron star, m_j is the mass of the planet, with $j=1$ for the inner planet and $j=2$ for the outer planet, and i is the angle between the orbital plane and the plane of the sky (we assume that the orbital plane is the same for both planets; see below). From the observed parameters of the system, we find $f_1 = 5.45\times10^{-16}M_{\odot}$ and $f_2 = 3.12\times10^{-16}M_{\odot}$. We can safely neglect terms of order $m_j/M \sim 10^{-5}$ in the above expression for the mass function. We deduce that $m_1 = 3.4M_{\oplus}M_{1.4}^{2/3}(\sin i)^{-1}$ and $m_2 = 2.8M_{\oplus}M_{1.4}^{2/3}(\sin i)^{-1}$, where $M_{1.4}$ is the neutron star mass in units of the canonical value of $1.4M_{\odot}$, and $M_{\oplus}$ is the mass of the Earth. The corresponding semimajor axes in AU are $a_1 = 0.36M_{1.4}^{1/3}$ and $a_2 = 0.47M_{1.4}^{1/3}$. The other orbital elements are directly measurable, independent of M and sin i. The values currently observed[2] are $e_1 = 0.022$ and $e_2 = 0.020$ for the eccentricities, and $\omega_1 = 252°$ and $\omega_2 = 107°$ for the longitudes of pericentre (measured from the ascending node).

Using these orbital elements, and the times of pericentre passage given by Wolszczan and Frail[2], one can construct a complete set of initial data for the system[4]. The dynamical evolution of the system is then determined by solving Newton's equations for three point masses interacting gravitationally. We have done a high-accuracy numerical integration of these equations, using the Bulirsch–Stoer algorithm[5]. We make the problem dimensionless by setting $G = a_1 = m_1 = 1$. The numerical solution then depends on only one parameter, the ratio M/m_1. The results corresponding to $M/m_1 = 1.4\times10^5$ are illustrated in Fig. 1. These results are little changed if other, more distant planets are present in the system. In particular, we have verified that the presence of a third planet with $m_3 \approx 1M_{\oplus}$ and $a_3 \approx 1$ AU (for which there is preliminary evidence[2]) has practically no influence on the results shown in Fig. 1. We find that, on a timescale of ~10 yr, any orbital element w closely follows an evolution of the type

$$w(t) = A + Bt + C\sin(Dt+E) \qquad (1)$$

The linear term is only apparent in the evolution of ω_j, for which $B \approx 0.04°\ \mathrm{yr}^{-1}$. The evolution of all orbital elements is dominated by the periodic term. The largest fractional change is that of the eccentricity of the outer planet, for which $C \approx 4\times10^{-4}$, corresponding to $\Delta e_2/e_2 \approx 2\times10^{-2}$. The period is $2\pi/D \approx 5.5$ yr for all elements. The constants A and E are determined by the initial conditions.

We can gain a better physical understanding of these results by examining Lagrange's equations for the system (see ref. 6 for a general discussion and refs 7 and 8 for a clear summary and useful Solar System examples). The disturbing function for each planet, due to its gravitational perturbation by the other, can be separated into two parts. The secular part is made of

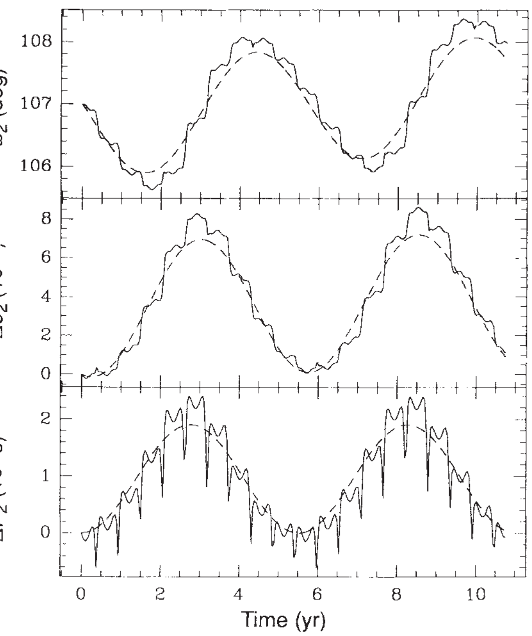

FIG. 1 Orbital evolution of the PSR1257+12 system. The solid lines are from a direct numerical integration of Newton's equations for three point masses. The dashed lines are from perturbation theory (equations (2) and (6)). The instantaneous (osculating) values of the longitude of pericentre, and of the changes in eccentricity and orbital period of the outer planet, are shown as a function of time. The orbital elements of the inner planet follow a similar evolution, but with somewhat smaller amplitudes.

TABLE 1 Secular evolution parameters

	$i=1$ eigenmode	$i=2$ eigenmode
g_i (deg yr^{-1})	4.39×10^{-2}	2.93×10^{-3}
$2\pi/g_i$ (yr)	8,210	123,000
E_{1i}	-1.94×10^{-2}	-6.52×10^{-3}
E_{2i}	$+2.06\times10^{-2}$	-6.50×10^{-3}
β_i (deg)	88.7	13.2

terms that do not depend on the mean longitudes of either planet. In addition, the existence of a near 3:2 commensurability in the mean motions of the two planets implies that those terms in the disturbing function that depend on the particular combination of mean longitudes $2\lambda_1-3\lambda_2$ will contribute large, periodic changes. We treat each contribution in turn and obtain simple analytical expressions for the coefficients of equation (1).

Following Brouwer and Clemence[6], we write the solution of Lagrange's equations corresponding to the secular part of the disturbing function as

$$h_j \equiv e_j \sin \omega_j = \sum_{i=1,2} E_{ji} \sin(g_i t+\beta_i)$$
$$k_j \equiv e_j \cos \omega_j = \sum_{i=1,2} E_{ji} \cos(g_i t+\beta_i) \qquad (2)$$

There is no secular variation of the semimajor axes or orbital periods. The frequencies g_i and the amplitudes E_{ji} can be calculated as the eigenvalues and eigenvector components of a 2×2 matrix with coefficients given explicitly in terms of the orbital elements[7]. The norms of the eigenvectors and the phases β_i can be calculated from a set of initial conditions. The results for the PSR1257+12 system, assuming $M=1.4M_\odot$ and $\sin i=1$, are given in Table 1. We have numbered the two eigenstates so that $g_1>g_2$. We find that the current state of the system is close to a pure g_1 eigenstate with $E_{11}\approx -E_{21}$. This corresponds to $e_j\approx|E_{j1}|=$ constant, $\omega_1\approx g_1t+\beta_1+\pi$, and $\omega_2\approx\omega_1-\pi$. A similar eigenmode structure is observed for the two outer satellites of Uranus, Titania and Oberon, which are also in a near 3:2 resonance[7,8]. By comparing equations (2) and (1) for ω_j, we see that $B\approx g_1$. This can be calculated explicitly as

$$g_1=\frac{1}{8}n_1\frac{m_1}{M}\alpha b^{(1)}_{3/2}(\alpha)\{q\alpha+\nu+[(q\alpha-\nu)^2+4q\alpha\nu\beta^2]^{1/2}\} \qquad (3)$$

In this expression, n_j is the mean orbital motion, the $b^{(n)}_{3/2}$ are Laplace coefficients[6], and we have defined the non-dimensional ratios $\alpha=a_1/a_2$, $q=m_2/m_1$, $\nu=n_2/n_1$, and $\beta=b^{(2)}_{3/2}(\alpha)/b^{(1)}_{3/2}(\alpha)$.

We now turn to the near-resonant terms in the disturbing function. For the 3:2 case, there are two such terms, which depend on the mean longitudes as $\cos\phi_j$ with $\phi_j=2\lambda_1-3\lambda_2+\omega_j$. The corresponding amplitudes are given explicitly in terms of the orbital elements[6]. To lowest order in e, we find that Lagrange's equations reduce to

$$\dot\omega_j=F_j(\alpha)\frac{n_j}{e_j}\frac{m_k}{M}\cos\phi_j, \qquad \dot e_j=F_j(\alpha)n_j\frac{m_k}{M}\sin\phi_j \qquad (4)$$

and a similar expression for $\dot a_j$, with $k=2(1)$ if $j=1(2)$, and

$$F_1(\alpha)=-\frac{1}{2}\left(6\alpha+\alpha^2\frac{d}{d\alpha}\right)b^{(3)}_{1/2}(\alpha)$$
$$F_2(\alpha)=\frac{1}{2}\left(5+\alpha\frac{d}{d\alpha}\right)b^{(2)}_{1/2}(\alpha) \qquad (5)$$

To solve equations (4) analytically, we use the fact that $|\dot\omega_j|\ll|2n_1-3n_2|$ and we keep the orbital elements constant at their present values when evaluating the right-hand sides. (These approximations are justified for $t\ll2\pi/|\dot\omega_j|$ and as long as the perturbations are small.) We can then write $\phi_j=(2n_1-3n_2)t+\phi_j^{(0)}$, where $\phi_j^{(0)}$ is the initial value of ϕ_j, and find

$$\omega_j(t)=\frac{F_j(\alpha)}{e_j}\frac{m_k}{M}\frac{n_j}{(2n_1-3n_2)}\{\sin[(2n_1-3n_2)t+\phi_j^{(0)}]-\sin\phi_j^{(0)}\} \qquad (6)$$

and similar expressions for $e_j(t)$ and the osculating orbital period $P_j(t)$. These analytical solutions are shown in Fig. 1 for the outer planet (dashed lines). By comparing them with equation (1), we obtain simple expressions for the coefficients C and D. In particular we have $D=2n_1-3n_2$ for all orbital elements.

The rapid fluctuations apparent in the numerical solution (Fig. 1) are caused by all the nonresonant and nonsecular terms in the disturbing function, which have been neglected in our analytical treatment. The largest such fluctuations are those corresponding to close encounters between the two planets, which occur roughly every $3\times(2\pi/n_1)\approx200$ days. In particular, these close encounters produce changes in the instantaneous (osculating) orbital periods of amplitude $\Delta P/P\approx10^{-4}$, occurring over a timescale of ~1 month. The current error bars on the orbital periods[2] (which are measured by fitting data over a period of time $\gg$ 1 month) are $\sim10^{-4}$, indicating that these effects may be marginally detectable already.

The amplitudes of both secular and periodic changes in the orbital elements are all proportional to the ratio m_j/M, which scales as $M^{-1/3}(\sin i)^{-1}$. Therefore, a measurement of any of these changes will provide a constraint on the combination $M^{1/3}\sin i$, but will not allow M and $\sin i$ to be determined independently. If the inclinations of the two orbits are slightly different, a measurement of any orbital change for both planets will provide independent estimates of $M^{1/3}\sin i_1$ and $M^{1/3}\sin i_2$. Our analysis remains valid as long as the relative inclination $\delta i=i_1-i_2$ is small. This is because near-resonant periodic terms involving the inclinations and node longitudes do not arise to lowest order in δi for the 3:2 case[8]. Secular variations in these quantities do arise, but, in the limit of small eccentricities and small relative inclination, they are completely decoupled from those involving eccentricities and longitudes of pericentre[7]. Should δi turn out to be large, a more general analysis would be necessary.

A strict upper limit on the masses of the two planets can also be obtained by considering the long-term dynamical stability of the system. Stability requires that the minimum separation between the two planets always remain larger than the radius of their 'Hill's spheres'. This condition can be written approximately[9] as $\Delta a/a_j>2.1(m_j/M)^{1/3}$. As $m_1\approx m_2$, this implies for both planets $M/m_j>7\times10^2$, or $m_j/M_\oplus<6\times10^2M_{1.4}$. By numerically integrating the equations of motion of the system over a timescale of $\sim10^4$ yr for many different values of the ratio M/m_1, we find a stability condition $M/m_1>8\times10^2$, in good agreement with the simple analytical estimate.

Quite apart from the determination of the masses and orbital inclinations, our results provide a critical test for the basic interpretation of the radio observations. The measured orbital elements of the two planets must all be evolving exactly as described above. A confirmation of this evolution would provide irrefutable evidence that a planetary system outside our own Solar System has indeed been found. □

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