Dispensability of the randomization device in "Belief-free Equilibria in Repeated Games", by Ely, Hörner and Olszewski (2004)

For B a subset of a Euclidean space, let $\mathring{B}$ denote its interior. Recall that V denotes the limit of the belief-free equilibrium payoff set, as δ tends to 1, given that a public randomization device is available. We show that:

Lemma 1 *In the positive case, if $v \in \mathring{V}$, then there exists $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, v is a payoff vector achieved by a belief-free equilibrium without any public randomization device.*

Proof. Suppose that $v^0 \in \mathring{V}$ is a belief-free equilibrium payoff. Then, by Proposition 5, there exists $p = (p_1, \ldots, p_J)$, $p \geq 0$, $\sum_j p_j = 1$, such that:

$$pM_i > v_i^0 > pm_i, \ i = 1, 2$$

where the inequalities can be taken to be strict as $v^0 \in \mathring{V}$. Therefore, there exists $\bar{\delta} < 1$ and a finite sequence $\{j_k\}_{k=1}^K$, $j \in \mathcal{J}$, such that, for $\delta > \bar{\delta}$,

$$q^t M_i > v_i^0 > q^t m_i, \ i = 1, 2, \text{ all } t.$$

where:

$$q_j^t = (1 - \delta) \sum_{\tau = t}^{\infty} \delta^{\tau - t} I_{\{\tau = k \ (\mathrm{mod}\ K)\}}, \ j = j_k.$$

and $I_{\{x=y\}} = 1$ if $x = y$, and 0 otherwise.

We consider the sequence of regimes A^{j_1}, $A^{j_2}, \ldots, A^{j_K}$, $A^{j_1}, A^{j_2}, \ldots, A^{j_K}, \ldots$ That is, we consider a cyclic sequence of regimes with cycle A^{j_1}, $A^{j_2}, \ldots, A^{j_K}$. Thus, $q^t = (q_1^t, \ldots, q_J^t)$ defines a probability measure that can be interpreted as the occupation measure, from period t on, given discount factor δ, over the different regimes, for the sequence under consideration.

By definition of M_i and m_i, there exists, for each player $i = 1, 2$, and each regime $\mathcal{A}^j$, a pair $\left\{ \bar{\alpha}_{-i}^{\mathcal{A}^j}, \underline{\alpha}_{-i}^{\mathcal{A}^j} \right\}$ of mixed actions in $\triangle \mathcal{A}_{-i}^j$ and a pair $\left\{ \bar{x}_i^{\mathcal{A}^j}, \underline{x}_i^{\mathcal{A}^j} \right\}$ of functions satisfying Eq.(4) and (5). Define a Markovian strategy for each player $i = 1, 2$ as follows:

(i) the state space in period t is $[q^t m_{-i}, q^t M_{-i}]$.

(ii) the initial state is v_{-i}^0, the equilibrium value to be achieved. Let $\mathcal{A}^j$ be the regime in period t, given the sequence described above.

(iii) at any state $v_{-i} \in [q^t m_{-i}, q^t M_{-i}]$ such that $v_{-i} = \lambda q^t m_{-i} + (1 - \lambda) q^t M_{-i}$, player i plays $\underline{\alpha}_i^{\mathcal{A}^j}$ with probability λ and $\underline{\alpha}_i^{\mathcal{A}^j}$ with complementary probability.

(iv) (a) if the mixed action used by player i in period t is $\bar{\alpha}_i^{\mathcal{A}^j}$, the realized action is $a_i \in A_i$ and the signal observed is $\sigma_i \in \Sigma_i$, the next period state, v'_{-i}, is given by:

$$v'_{-i} = \frac{1 - \delta}{\delta} \left(\bar{x}_{-i}^{\mathcal{A}^j} (a_i, \sigma_i) - M_{-i}^{\mathcal{A}^j} \right) + \frac{q^t M_{-i}}{\delta},$$

which is an element of $\left[q^{t+1} m_{-i}, q^{t+1} M_{-i} \right]$ by construction.

(b) if the mixed action used by player i in period t is $\underline{\alpha}_i^{\mathcal{A}^j}$, the realized action is $a_i \in A_i$ and the signal observed is $\sigma_i \in \Sigma_i$, the next period state, v'_{-i}, is given by:

$$v'_{-i} = \frac{1 - \delta}{\delta} \left(\underline{x}_{-i}^{\mathcal{A}^j} (a_i, \sigma_i) - m_{-i}^{\mathcal{A}^j} \right) + \frac{q^t m_{-i}}{\delta},$$

which is an element of $\left[q^{t+1} m_{-i}, q^{t+1} M_{-i} \right]$ by construction.

This pair of Markovian strategies is a sequential equilibrium and achieves the payoff vector v^0. A similar argument applies to the abnormal case, and the proof in the negative case is standard. ■