Large Contests without Single Crossing

Wojciech Olszewski and Ron Siegel*

August 2015

Abstract

We show that the equilibria of asymmetric contests with many players and prizes are approximated by certain single-agent mechanisms. This complements the work of Olszewski and Siegel (2015), who considered a more restricted environment in which players' utility function satisfies a single crossing condition. Relaxing single crossing enlarges the set of approximating mechanisms, and weakens the notion of approximation. When the approximating mechanism is unique, the stronger notion of approximation is recovered.

1 Introduction

We study contests with many, possibly heterogeneous, players and prizes that include many existing contest models as special cases. We show that the equilibria of such contests are approximated by certain incentive-compatible and individually-rational mechanisms in an environment with a single-agent that has a continuum of possible types. This makes it possible to approximate the equilibria of contests whose exact equilibrium characterization is complicated, as well as the equilibria of contests for which there is no existing equilibrium characterization. This facilitates contest design, welfare analysis, and comparative statics.

*Department of Economics, Northwestern University, Evanston, IL 60208 (e-mail: wo@northwestern.edu) and The Pennsylvania State University, University Park, IL 16802 (e-mail: rus41l@psu.edu). We thank Ivan Canay and Joel Horowitz for very helpful suggestions.

Our analysis complements that of Olszewski and Siegel (2015), henceforth: OS, who considered a similar environment, with the added assumption that players' utility function satisfies a single crossing condition. Relaxing this assumption, as we do here, is valuable because the assumption does not hold in some applications. One example is the contest studied in Barut and Kovenock (1998), henceforth: BK, which we discuss in Section 3; perhaps other applications in which single crossing does not hold have not yet been studied because their analysis has proven difficult.

The cost of relaxing single crossing is that the set of approximating mechanisms may not be a singleton, even for contests that have a unique equilibrium, and the notion of approximation is weaker than in OS. Nevertheless, multiple approximating mechanisms may share common features, as is the case in the setting of BK. And when the approximating mechanism is unique, the approximation is as strong as in OS.

The rest of the paper is organized as follows. Section 2 introduces the basic terminology and notation. Section 3 contains our main result, illustrates the result in the setting of BK, and compares it to the results of OS. The Appendix contains the proof of the result.

2 Terminology and notation

2.1 Agents and prizes

An agent is characterized by a type $x \in X = [0,1]$. We will use the terms "player" for discrete contests and "agent" for the limit case. A prize is characterized by a number $y \in Y = [0,1]$. Prize 0 is "no prize."

Agents' utilities are given by a continuous function $U(x,y,t)$, where x is the agent type, y is the single prize he obtains, and $t \geq 0$ is his bid. The utility of obtaining no prize by bidding 0 is normalized to 0, i.e., $U(x,0,0) = 0$ for all x. Higher prizes are better and higher bids are more costly, so $U(x,y,t)$ strictly increases in y for every $x > 0$ and $t \geq 0$, and strictly decreases in t for every $x \geq 0$ and $y \geq 0$. Sufficiently high bids are prohibitively costly, so $U(x,1,b_{\max}) < 0$ for some $b_{\max}$ and all x. We therefore restrict the range of bids that agents can make to $B = [0, b_{\max}]$. Unlike OS, we do not assume that the utility satisfies strict single crossing.[1]

2.2 Contests

For every n, we define "the n-th contest," in which n players compete for n known prizes $y_1^n \leq y_2^n \leq ... \leq y_n^n$ (some of which may be no prize). Player i's privately-known type x_i^n is distributed according to a CDF F_i^n, and these distributions are commonly known and independent across players.[2] In the special case of complete information, each CDF corresponds to a Dirac measure. Each player, after learning her type, chooses a bid in B, the player with the highest bid obtains the highest prize, the player with the second-highest bid obtains the second-highest prize, and so on. Ties are resolved by a fair lottery. The utility of player i from bidding t and obtaining prize y_j^n is $U\left(x_i^n, y_j^n, t\right)$. A slight adaptation of the proof of Corollary 1 in Siegel (2009) shows that when each player's set of possible types is finite the contest has at least one mixed-strategy Bayesian Nash equilibrium. For general distributions F_i^n, equilibrium existence follows from Corollary 5.2 in Reny (1999).

We let $F^n = \left(\sum_{i=1}^n F_i^n\right)/n$, so $F^n(x)$ is the expected percentile ranking of type x given the vector of players' types. We denote by G^n the empirical distribution of prizes, which assigns a mass of $1/n$ to each y_j^n (recall that each prize y_j^n is known). We assume that F^n converges in weak*-topology to a continuous and strictly increasing distribution F, and G^n converges to some distribution G.[3] See OS for a detailed discussion of this assumption.

Given an equilibrium of the n-th contest, we denote by D_i^n the distribution on $X \times Y \times B$ that describes player i's type, the prize she obtains, and her bids. We refer to the distribution $D^n = \left(\sum_{i=1}^n D_i^n\right)/n$ as the *equilibrium outcome*, and later relate the sequence of distributions $D^1, D^2, \ldots$ to probability distributions D that describe the outcomes of some mechanisms.

[1] Strict single crossing means that for any $x_1 < x_2$, $t_1 < t_2$, and $y_1 < y_2$ we have that $U(x_1, y_2, t_2) \geq U(x_1, y_1, t_1)$ implies $U(x_2, y_2, t_2) > U(x_2, y_1, t_1)$.

[2] All probability measures are defined on the σ-algebra of Borel sets.

[3] Convergence of probability measures in weak*-topology is equivalent to pointwise convergence at points of continuity of the limit distribution. See Billingsley (1995) for the definition of this topology and its properties.

2.3 Limit mechanism-design setting

A *consistent allocation* is a probability distribution H on $X \times Y$ whose marginal on X coincides with F and whose marginal on Y coincides with G. With a continuum of agents and prizes distributed according to F and G, this condition says that all the prizes are allocated to agents, and each agent obtains exactly one prize (which can be no prize). The conditional distribution H_x is interpreted as the lottery over prizes faced by an agent of type x.

A (direct) *mechanism* prescribes for each reported type x a joint probability distribution $Q_x(y,t)$ over prizes and bids. A mechanism is *incentive compatible* (IC) if the expected utility of each agent is maximized by reporting truthfully, i.e.,

$$\int_{y\in Y}\int_{t\in B} U(x,y,t)dQ_z(y,t)$$

is maximized at $z = x$. A mechanism is *individually rational* (IR) if the expected utility of each agent from reporting truthfully is at least as high as the utility from bidding 0 and obtaining the "lowest" available prize, i.e.,

$$\int_{y\in Y}\int_{t\in B} U(x,y,t)dQ_x(y,t) \geq U\left(x, y_{\text{inf}}, 0\right),$$

with an equality for at least one type x, where $y_{\text{inf}} = \inf\left\{y : G\left(y\right) > 0\right\}$.

An *inverse tariff* is a non-decreasing, upper semi-continuous function that maps bids to prizes. Given an inverse tariff, a *tariff mechanism* is an IR mechanism that prescribes for each type x a distribution $Q_x(y,t)$ that assigns probability 1 to the set of prize-bid pairs that maximize $U\left(x,y,t\right)$ among the prize-bid pairs in the graph of the inverse tariff. A tariff mechanism is clearly IC.

A mechanism *implements* an allocation H if the marginal of Q_x on Y coincides with H_x for almost every type x. Distributions H and $\{Q_x : x \in X\}$ may not determine a probability distribution on $X \times Y \times B$.[4] When such a distribution D exists, which will be the case in our result, we say that the mechanism that implements H is *regular*, and refer to D as the *outcome* of the mechanism.

[4]For example, take some non-measurable function $f : X \to [0,\infty)$, have H distributed uniformly on, and assigning probability 1 to, the diagonal $\{(x,x) : x \in X\}$, and have Q_x assigning probability 1 to the pair $(x, f(x))$. That is, type x is prescribed prize x and bid $f(x)$.

3 Result

We can now formulate our result, whose proof is in the Appendix.

Theorem 1 *For any $\varepsilon > 0$ and any metrization of the weak*-topology, there is an N such that for all $n \geq N$, for any equilibrium of the n-th contest there is a regular tariff mechanism that implements a consistent allocation whose outcome is ε-close (in the metrization) to the outcome of the equilibrium.*

Unlike the results in OS, Theorem 1 applies to settings in which the utility function may not satisfy strict single crossing. As an example, consider $U(x, y, t) = y - t$, let $x_i^n = i/n$ (so F_i^n is a Dirac measure), and let $y_j^n = j/n$. The limit distributions F and G are uniform. The n-th contest is an all-pay auction with n symmetric players and n heterogeneous prizes, and the value of prize j to all players is j/n. Players are symmetric because their type does not enter the utility function (in particular, single crossing fails). Such contests were studied by BK, who considered grading, promotions, procurement settings, and political competitions. BK showed that the n-th contest has a unique equilibrium, in which all players randomize uniformly across all bids $t \in [0, 1]$. For the approximation, consider the uniform allocation, whose density is $h(x, y) = 1$ for all values of x and y. The unique IC-IR mechanism that implements this allocation has $Q_x(y, t)$ distributed uniformly on the diagonal $y = t$. This is a tariff mechanism with a continuous inverse tariff that maps every bid $t \in [0, 1]$ to prize t. This mechanism approximates the unique equilibrium in the sense of Theorem 1.

The approximation in Theorem 1 is in general weaker than in OS, because it only provides an approximation of the aggregate behavior of players. In particular, it may be that even for large n the approximating mechanism does not approximate the behavior of any player individually. To illustrate this possibility, consider a version of the setting of BK, in which there are half as many prizes as players, and the prizes are identical. That is, the limit type distribution F is uniform on $[0, 1]$ and the limit prize distribution is $G(y) = 1/2$ for all $y \in [0, 1)$ and $G(1) = 1$. Let $U(x, 1, t) = 1 - t$ and $U(x, 0, t) = -t$. Let F_i^n be the Dirac measure on $F^{-1}(i/n)$, and let $y_j^n = G^{-1}(j/n)$ for $i, j = 1, ..., n$. For $n = 2k + 1$, the n-th contest has an equilibrium in which the k even players $2, 4, \ldots, 2k$ bid 0 and obtain no prize, and the $k + 1$ odd players $1, 3, \ldots, 2k + 1$ employ the same mixed strategy on $[0, 1]$.

As n increases, the mixing players bid close to 1 and obtain a prize with probability close to 1. Therefore, the distributions D^n converge in weak*-topology to the distribution D in which every type x bids 0 and obtains 0 with probability 1/2 and bids 1 and obtains 1 with probability 1/2. Consequently, the individual strategies of all players in every contest qualitatively differ from the conditionals of D.

If, however, for every type x there is a unique prize-bid pair $(y(x), t(x))$ that maximizes $U(x, y, t)$ among the prize-bid pairs from the graph of the inverse tariff, then convergence in weak*-topology implies convergence in a sense similar to that of Theorem 2 in OS. Namely, for any $\varepsilon > 0$ and sufficiently large n, for a fraction $1 - \varepsilon$ of the players, with probability $1 - \varepsilon$ the prize that player i obtains differs from $y(x_i^n)$ by at most ε and the bid of player i differs from $t(x_i^n)$ by at most ε.

Another limitation of Theorem 1 is that the set of tariff mechanisms that implement a consistent allocation may be quite large, even if every contest has a unique equilibrium. For example, in the setting of BK there is a continuum of consistent allocations and tariff mechanisms that implement them. All of them, however, are associated with the same inverse tariff that maps each bid $t \in [0, 1]$ to prize t, so much can be gleaned from the approximation about the structure of equilibria despite the indeterminacy in the approximating mechanism.

Finally, we conjecture that Theorem 1 is the strongest general convergence result that one can obtain, because some contests have many equilibria, and different sequences of equilibria may be approximated by different mechanisms. For example, consider again the version described above of the setting of BK with a mass 1/2 of identical prizes. For $n = 2k + 1$, the n-th contest has an equilibrium in which players $1, \ldots, k$ bid 0 and obtain no prize, and players $k + 1, \ldots, n$ employ the same mixed strategy on $[0, 1]$. As n increases, the mixing players bid close to 1 and obtain a prize with probability close to 1. Therefore, the distributions D^n converge in weak*-topology to the distribution D in which every type $x < 1/2$ bids 0 and obtains no prize and every type $x \geq 1/2$ bids 1 and obtains a prize. Similarly, there is a sequence of equilibria with a limit distribution in which every type $x \leq 1/2$ bids 1 and obtains a prize and every type $x > 1/2$ bids 0 and obtains no prize, as well as many other sequences of equilibria with different limit distributions.

A Proof of Theorem 1

We denote an equilibrium by $\sigma^n = (\sigma_1^n, \ldots, \sigma_n^n)$, where σ_i^n is player i's equilibrium strategy in the n-th contest. In order to prove Theorem 1, it suffices to show every subsequence σ^{n_k} of sequence σ^n contains a further subsequence such that the equilibrium outcomes of this further subsequence converge in weak*-topology to the outcome of a regular tariff mechanism that implements a consistent allocation. With no loss of generality, we can take the subsequence σ^{n_k} to be the entire sequence σ^n.

We denote by $R_i^n(t)$ the random variable that is the percentile location of player i in the ordinal ranking of the players in the n-th contest if she bids slightly above t and the other players employ their equilibrium strategies.[5] That is,

$$R_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} 1_{\{\sigma_k^n \in X \times [0,t]\}}\right),$$

where $1_{\{\sigma \in X \times [0,t]\}}$ is 1 if $\sigma \in X \times [0,t]$ and 0 otherwise. Let

$$A_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} \Pr\left(\sigma_k^n \in X \times [0,t]\right)\right)$$

be the expected percentile ranking of player i. Then, by Hoeffding's inequality, for all t in B we have

$$\Pr\left(\left|R_i^n(t) - A_i^n(t)\right| > \delta\right) < 2\exp\left\{-2\delta^2(n-1)\right\}. \tag{1}$$

Finally, let

$$A^n(t) = \frac{1}{n}\sum_{i=1}^{n} A_i^n(t)$$

be the average of the expected percentiles rankings of the players in the n-th contest if they bid t and the other players employ their equilibrium strategies.

Let T^n be the mapping from bids to prizes induced by A^n. That is, $T^n(t) = (G^n)^{-1}(A^n(t))$, where $(G^n)^{-1}(z) = \inf\{y : G^n(y) \geq z\}$ for $z > 0$, and $(G^n)^{-1}(0) = \inf\{y : G^n(y) > 0\}$. (In words, $(G^n)^{-1}(z)$ is the prize of an agent with percentile ranking z when prizes are distributed according to G^n.)

[5] This is the infimum of her ranking if she bids above t, which is equivalent to bidding t and winning any ties there. If ties happen with probability 0, then this is equivalent to bidding t.

For expositional simplicity, we prove the theorem assuming that G is strictly increasing, which implies that G^{-1} is continuous. This assumption can be relaxed by applying analogous (almost identical) arguments to those used in OS to obtain their Theorem 2 from Theorem 1.

Now, let T^n be the mapping from bids to prizes induced by A^n. That is, $T^n(t) = G^{-1}(A^n(t))$. OS show that the sequence T^n contains a subsequence that uniformly converges to a continuous T. We will assume again that this subsequence is the entire sequence T^n.

OS relates players' equilibrium behavior in large contests to the inverse tariff T in the following way. For every x, denote by BR_x type x's set of optimal bids given T, i.e., the bids t that maximize $U(x, T(t), t)$. Denote by $BR(\varepsilon)$ the ε-neighborhood of the graph of the correspondence that assigns to every $x \in [0,1]$ the set BR_x, i.e., $BR(\varepsilon)$ is the union over all types x and bids $t \in BR_x$ of the open balls of radius ε centered at (x,t). For every type x denote by $BR_x(\varepsilon)$ the set of bids t such that $(x,t) \in BR(\varepsilon)$.

Note that $BR(\varepsilon)$ is a 2-dimensional open set, while each $BR_x(\varepsilon)$ is a 1-dimensional "slice." Note also that $BR(\varepsilon)$ is in general larger than the union across x of the pairs (x,t) for which the distance between t and some bid in BR_x is less than ε. In particular, $BR_x(\varepsilon)$ may contain bids whose distance from every bid in BR_x is more than ε. Using the sets $BR_x(\varepsilon)$, players' equilibrium behavior is characterized by the following lemma.

Lemma 1 *(OS) For every $\varepsilon > 0$, there is an N such that for every $n \geq N$, the equilibrium bid of any player $i = 1, ..., n$ of type x_i^n in the n-th contest belongs to $BR_{x_i^n}(\varepsilon)$ with probability 1.*

This result appears as Lemma 3 in the Online Appendix of OS. OS prove this lemma, as well as uniform convergence of T^n to T and continuity of T, without assuming their single crossing condition.

From the sequence of equilibria that corresponds to the sequence T^n that converges uniformly to T, choose a subsequence such that D^n converges to some probability distribution D in weak*-topology. We will now show that D assigns probability 1 to the set $C = \{(x, y, t) : t \in BR_x \text{ and } y = T(t)\} \subset X \times Y \times B$.

By standard arguments the correspondence that assigns BR_x to type x is upper hemi-continuous. Therefore, the set $\{(x,t) : t \in BR_x\} \subset X \times B$ is closed, and by continuity of T, the set C is also closed. Suppose to the contrary that D assigns a positive probability to the complement of C. Then, for some $\varepsilon > 0$, D assigns a positive probability to the complement of the 2ε-neighborhood O of C, that is, to the set $X \times Y \times B - O$. Consider the ε-neighborhood V of C and its closure $\bar{V}$ (which is contained in O), and take a continuous function $f : X \times Y \times B \to [0,1]$ such that $f(\bar{V}) = 1$ and $f(X \times Y \times B - O) = 0$. Then,

$$\int f dD < 1.$$

But by Lemma 1, for sufficiently large n every player i with probability 1 bids t such that $(x_i^n, T(t), t)$ is close to C. By uniform convergence of T^n to T, also $(x_i^n, T^n(t), t)$ is close to C. Finally, by (1) and continuity of G^{-1}, for sufficiently large n player i obtains with high probability a prize y such that $(x_i^n, y, t) \in V$. Thus,

$$\int f dD^n \to 1, \tag{2}$$

a contradiction.

Thus, the limit mechanism determined by D prescribes for each of a measure 1 of types x bids $t \in BR_x$ and corresponding prizes $T(t)$ with probability 1. This implies that D determines a tariff mechanism.[6] It is regular, because each D^n is regular, and it implements a consistent allocation, because each D^n implements a consistent allocations. We show this last statement for the marginal with respect to x; the proof for the marginal with respect to y is analogous.

Consider a continuous function $f_\varepsilon : X \times Y \times B \to [0,1]$ whose value is 1 on the set of all (x,y,t) such that $x \leq x^*$ and 0 on the set of all (x,y,t) such that $x \geq x^* + \varepsilon$, for some $\varepsilon > 0$. Then, by the definition of weak*-convergence,

$$\int f_\varepsilon dD^n \to \int f_\varepsilon dD.$$

For large enough n the integrals on the left-hand side belong to $[F(x^*) - \varepsilon, F(x^* + \varepsilon) + \varepsilon]$. These integrals would belong to $[F(x^*), F(x^* + \varepsilon)]$ if the marginals of D^n with respect to x

[6] If needed, change D on a measure 0 set of types x so that it assigns to every type x bids $t \in BR_x$ and corresponding prizes $T(t)$.

were equal to F. But the marginal of D^n with respect to x is not equal to F; rather, it is a measure that assigns probability $1/n$ to every x_i^n. These measures, however, converge in weak*-topology to F, which implies that the integrals belong to $[F(x^*) - \varepsilon, F(x^* + \varepsilon) + \varepsilon]$. Therefore, $\int f_\varepsilon dD$ also belongs to $[F(x^*) - \varepsilon, F(x^* + \varepsilon) + \varepsilon]$. Taking the limit for $\varepsilon \to 0$, and applying right-continuity of CDFs, we obtain that F is the marginal of D with respect to x.

References

[1] **Billingsley, Patrick**. 1995. Probability and Measure, 3rd Edition, John Wiley and Sons, Inc.

[2] **Barut, Yasar, and Dan Kovenock.** 1998. "The Symmetric Multiple Prize All-Pay Auction with Complete Information." *European Journal of Political Economy*, 14(4): 627-644.

[3] **Olszewski, Wojciech, and Ron Siegel.** 2015. "Large Contests." *Econometrica*, forthcoming.

[4] **Reny, Philip J.** 1999. "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games." *Econometrica*, 67(5): 1029-1056.

[5] **Siegel, Ron.** 2009. "All-Pay Contests." *Econometrica*, 77(1): 71-92.