Two lectures on information aggregation in markets

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Abstract

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1 Introduction

These lectures examine situations in which dispersed information is aggregated by prices. Particular attention will be paid to situations in which such aggregation is sufficient to bring about the same outcomes that would arise if all the information were public.

- Many participants with private signals about value fundamentals.
- A simple such environment is the sale of a good whose value is common.
- **Question:** To what extent prices formed in natural trading modes aggregate the dispersed information?
- Fundamental question of economic theory.
- Part of the broader question of how well the price system performs the allocation of resources.
- The focus is on information aggregation by certain price formation modes that are interesting because they are "natural" or wide-spread.

- It differs from the mechanism design question of how to aggregate the information well, which in this environment should not be difficult.

- The information aggregation by price question was first investigated in the context of competitive markets (by the REE literature).

- A subsequent step is its investigation in environments with strategic agents. Milgrom (1979) & wilson (1977) addressed this question in the context of auctions. Most likely, they did not have in mind a structured auction but rather as a component of a market price formation.

- But there are of course other modes of price formation some of which might give rise to different insights.

- The lectures cover mainly Milgrom (1979) and parts of Lauermann & Wolinsky (2013a&b).

2 Example of REE and bidding equilibrium

Unit mass of agents.

An indivisible good.

$$\text{Agent i's utility} = \begin{cases} v_i + t & \text{if} \quad \text{owns unit + sum of money } t \\ t & \text{if} \quad \text{no unit + sum of money } t \end{cases}$$

$v_i = v(x_i, w)$

$w \in R =$ common component unknown to the agents.

$x_i \in R =$ private component and privately observed by i.

$F(x_i|w) =$ the distribution of x_i conditional on w.

v is increasing in both arguments; F is stochastically increasing in w.

There is a mass $\frac{1}{2}$ of goods. Initially $\frac{1}{2}$ the agents own a unit each and $\frac{1}{2}$ do not.

REE

Let $x_m(w)$ denote the median of $F(x_i|w)$

$P = v(x_m(w), w)$ is a REE

P reveals all relevant uncertainty.

If $x_i > x_m(w)$, Agent i demands a unit (or keeps it if she already owns it).

If $x_i < x_m(w)$, Agent i "supplies" a unit (if she owns one and does not demand if she does not own).

Since $x_m(w)$ is the median of x_i over the population, the market clears: demand=supply.

Thus, information is perfectly aggregated and allocation is efficient.

But how does information get into the price?

A BIDDING MODEL–double auction

Each agent submits a bid.
A potential buyer submits bid $b(x_i)$–the highest price they are willing to pay.
A potential seller submits bid $s(x_i)$–the lowest price at which they are willing to part with the good.

Given the realized bids, the allocation is determined as follows.
Let $D(p)$ be the measure of $\{x_i \mid b(x_i) \geq p\}$
Let $S(p)$ be the measure of $\{x_i \mid s(x_i) \leq p\}$

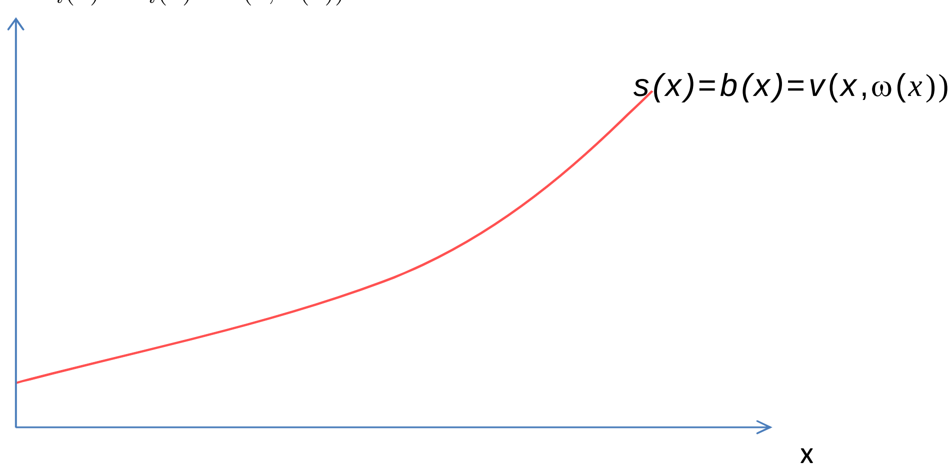
Let p^* be a price such that $D(p^*) = S(p^*)$ (if there are multiple employ some selection).
Allocate a unit to each agent whose bid exceeds p^*.
A buyer (seller) who gets (sells) a unit pays (receives) p^*.

An equilibrium is such that each agent's bid is optimal given the population's bidding behavior.

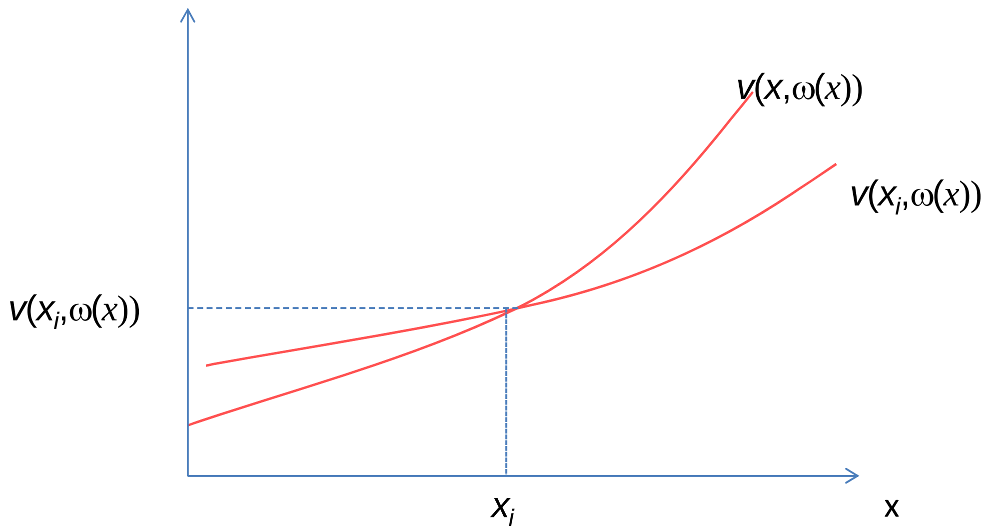
Assume that for any x there is w such that $x = x_m(w)$ (assume that the range of x and w is the whole of R).

Let $w(x)$ be the state in which the median is equal to x, i.e., $x_m(w(x)) = x$.

Following is an equilibrium
$b_i(x) = s_i(x) = v(x, w(x))$.



With these bids $p^* = v(x_m(w), w)$
The following figure proves that it is an equilibrium



If $x_i > x_m$, then $v(x_i, w(x_i)) > v(x_i, w) = v(x_i, w(x_m)) > v(x_m, w(x_m)) = p^*$. Agent i gets a unit at a price below her valuation (or keeps the unit rather than sell it at a price below her valuation).

If $x_i < x_m$, then $v(x_i, w(x_i)) < v(x_i, w) = v(x_i, w(x_m)) < v(x_m, w(x_m)) = p^*$. Agent i does not get a unit at a price above her valuation (or sells the unit rather than keep it at a price above her valuation).

This bidding equilibrium's outcome coincides with the REE. But here it is clear how the information was aggregated into the price.

More specific assumptions:
$v(x_i, w) = x_i + w$
$F(x_i|w)$ is uniform on $[0, w]$
$P = (3/2)w$ is a REE
$b_i(x) = s_i(x) = v(x_i, w(x_i)) = 3x_i$

3 Information aggregation in a CV auction

First price auction for a single object (in case of a tie the good is not awarded).

n bidders.

V = The unknown value of the object is in $\{v_1, v_2, ..., v_m\}$, $v_i < v_{i+1}$.

x_i = signal received by bidder i. $x_i \in R$

P_n = Probability distribution on $\{v_1, v_2, ..., v_m\} \times R^n$, i.e., value and signals.

Commonly known prior $P_n(V = v_i) > 0$ for all i.

Conditional independence: $P_n(x_i \in A|v_k, x_j) = P_n(x_i \in A|v_k)$ for all $i, j \leq n$, $A \subseteq R$ and $v_k \in \{v_1, v_2, ..., v_m\}$.

Symmetry and independence of n: $P_n(x_i \in A|B) = P_n(x_j \in A|B) = P(A|B)$ for all n, $i, j \leq n$, $A \subseteq R$ and $B \subseteq \{v_1, v_2, ..., v_m\}$

Therefore, from now on we will omit the subscript n from the distribution and write $P(A|B)$.

The conditional distribution of (any) x_i conditional on state v_k is absolutely continuous with continuous density function f_{v_k}.

$b_{ni}(x_i)$ bidder i's equilibrium strategy.

$W_n = \max_{i \leq n} b_{ni}(x_i)$

Let C, $D \subseteq \{v_1, v_2, ..., v_m\}$ and assume $P(C) > 0$.

Definition: C can be distinguished from D using signal x, if either $P(D) = 0$ or

$$\inf_A \frac{P(x \in A|D)}{P(x \in A|C)} = 0$$

Proposition: The following conditions are equivalent: (i) The event $\{V = v_k\}$ can be distinguished from the event $\{V < v_k\}$ using x_1; (ii) The sets $A^L = \{t| \max_{i<k} f_{v_i}(t)/f_{v_k}(t) < 1/L\}$ are non-empty for every $L > 0$; (iii) There is a sequence $\{t_\ell\}$ such that for every $i < k$, $\lim_{\ell \to \infty} f_{v_i}(t_\ell)/f_{v_k}(t_\ell) = 0$

Proof: Straightforward.

Proposition: $W_n \to V$ in probability[1] iff, for every $k > 1$, the event $\{V = v_k\}$ can be distinguished from the event $\{V < v_k\}$ using x_1.

Proof: (i) NECESSITY. Suppose $W_n \to V$ in probability. Fix $k > 1$ and choose $\alpha = (v_{k-1} + v_k)/2$. By conditional independence of the signals, for any i,

$$\begin{aligned} P(B_n &< \alpha|V = v_i) = \prod_{\ell=1}^{n} P(b_{n\ell}(x_\ell) < \alpha|V = v_i) = \prod_{\ell=1}^{n} [1 - P(b_{n\ell}(x_\ell) \geq \alpha|V = v_i)] \\ &\leq \exp\left(-\sum_{\ell=1}^{n} P(b_{n\ell}(x_\ell) \geq \alpha|V = v_i)\right) \end{aligned}$$

where the last inequality follows from $\prod\limits_{\ell=1}^{n} [1 - y_\ell] = \exp\left(\sum_{\ell=1}^{n} \ln(1 - y_\ell)\right)$ and from $\ln(1 - y) \leq -y$ for $y \in [0, 1)$.

Choose $i < k$. So, $v_i < \alpha$. Since $W_n \to V$ in probability, we have $P(W_n < \alpha|V = v_i) \to 1$. This and the above equation together imply that for $i < k$,

$$\sum_{\ell=1}^{n} P(b_{n\ell}(x_\ell) \geq \alpha|V = v_i) \to 0$$

Hence,

$$0 = \lim_{n\to\infty} \sum_{i=1}^{k-1} P(V = v_i) \sum_{\ell=1}^{n} P(b_{n\ell}(x_\ell) \geq \alpha|V = v_i) = \lim_{n\to\infty} \sum_{\ell=1}^{n} P(b_{n\ell}(x_\ell) \geq \alpha|V < v_k) \tag{@}$$

Since

$$1 - \sum_{\ell=1}^{n} P(b_{n\ell}(x_\ell) \geq \alpha|V = v_k) \leq P(W_n < \alpha|V = v_k) \to 0$$

where "$\to 0$" follows from convergence in probability and $\leq$ follows from $\Pr(\max b_{n\ell}(x_\ell) \geq \alpha|...) \leq \sum_{\ell=1}^{n} P(b_{n\ell}(x_\ell) \geq \alpha|...)$. It then follows that

$$\liminf \sum_{\ell=1}^{n} P(b_{n\ell}(x_\ell) \geq \alpha|V = v_k) \geq 1 \tag{@@}$$

Combining (@) and (@@) and replacing each x_ℓ by x_1 yields

$$\lim_{n\to\infty} \frac{\sum_{\ell=1}^{n} P(b_{n\ell}(x_1) \geq \alpha|V < v_k)}{\sum_{\ell=1}^{n} P(b_{n\ell}(x_1) \geq \alpha|V = v_k)} = 0 \tag{@@@}$$

Notice that there must be $j(n) \in \{1, 2, ..., n\}$ such that

$$\lim_{n\to\infty} \frac{P(b_{nj(n)}(x_1) \geq \alpha|V < v_k)}{P(b_{nj(n)}(x_1) \geq \alpha|V = v_k)} = 0 \tag{@@@@}$$

[1] $P(|W_n - V| > \varepsilon) \to 0$ for all $\varepsilon > 0$.

since o/w $\exists\varepsilon > 0$, such that for all n and all $j \leq n$, $\frac{P(b_{nj}(x_1)\geq\alpha|V<v_k)}{P(b_{nj}(x_1)\geq\alpha|V=v_k)} \geq \varepsilon$ hence $\frac{\sum_{\ell=1}^{n} P(b_{n\ell}(x_1)\geq\alpha|V<v_k)}{\sum_{\ell=1}^{n} P(b_{n\ell}(x_1)\geq\alpha|V=v_k)} \geq \varepsilon$ contradicting (@@@).

Let $A^n = b_{nj(n)}^{-1}[\alpha, \infty]$ and rewrite (@@@@) as

$$\lim_{n\to\infty} \frac{P(x_1 \in A^n|V < v_k)}{P(x_1 \in A^n|V = v_k)} = 0$$

This establishes necessity, i.e., $W_n \to V$ in probability implies that $\{V = v_k\}$ can be distinguished from the event $\{V < v_k\}$ using x_1.

NOTE THAT THE NECESSITY PROOF ABOVE DOES NOT USE THE EQUILIBRIUM.

(ii) SUFFICIENCY. Suppose that the event $\{V = v_k\}$ can be distinguished from the event $\{V < v_k\}$ using x_1. That is, for any k and any $\varepsilon > 0$, there exists a set $A(k, \varepsilon)$ such that

$$\frac{P(x_1 \in A(k,\varepsilon)|V < v_k)}{P(x_1 \in A(k,\varepsilon)|V = v_k)} < \varepsilon \tag{\%}$$

Suppose that W_n does not converge to V in probability. Then, there is $k, \delta > 0$ and $\beta > 0$ such that

$$\limsup_{n\to\infty} P(W_n < v_k - \delta|V = v_k) > \beta \tag{\%\%1}$$

or

$$\limsup_{n\to\infty} P(W_n > v_k + \delta|V = v_k) > \beta \tag{\%\%2}$$

Suppose that (%%1) holds and choose the maximal k for which it does. Let

$$\varepsilon = \frac{P(V = v_k)\beta\delta}{2P(V < v_k)(v_k - \delta)}$$

Define the bidding strategy

$$b(x) = \begin{cases} v_k - \delta & if \quad x \in A(k,\varepsilon) \\ 0 & o/w \end{cases}$$

For sufficiently large n, the payoff of a bidder who uses $b(x)$ when all else use their equilibrium strategies is at least

$$P(x_1 \in A(k,\varepsilon), V = v_k)\beta\delta + P(x_1 \in A(k,\varepsilon), V < v_k)(\delta - v_k)$$

where the first term is the δ that will be gained in the event that $V = v_k$ the W_n is below $v_k - \delta$ and the "distinguishing" signal occurs. The second term accounts for the maximal possible loss $\delta - v_k$ in case that signal occurs but $V < v_k$. Rewrite this payoff as

$$\begin{aligned}
P(x_1 &\in A(k,\varepsilon)|V = v_k)P(V = v_k)\beta\delta + P(x_1 \in A(k,\varepsilon)|V < v_k)P(V < v_k)(\delta - v_k) = \\
P(x_1 &\in A(k,\varepsilon)|V = v_k)\left[P(V = v_k)\beta\delta + \frac{P(x_1 \in A(k,\varepsilon)|V < v_k)}{P(x_1 \in A(k,\varepsilon)|V = v_k)}P(V < v_k)(\delta - v_k)\right] > \\
P(x_1 &\in A(k,\varepsilon)|V = v_k)\left[P(V = v_k)\beta\delta + \varepsilon P(V < v_k)(\delta - v_k)\right] \\
P(x_1 &\in A(k,\varepsilon)|V = v_k)P(V = v_k)\beta\delta/2
\end{aligned}$$

where the inequality follows from (%) and the last equality from the choice of ε.

Thus, the payoff of the strategy $b(x)$ is positive and bounded away from 0, independently of n. Since the sum of all payoffs is bounded by v_m, for large enough n there must be a bidder who could profit from deviating to $b(x)$ in contradiction to the equilibrium assumption.

It follows that (%%1) does not hold for any k. Therefore, $P\left(W_n < v_k - \delta|V = v_k\right) \to 0$. This together with (%%2) imply that there are large enough n's for which the sum of payoffs is negative. This means that for large enough n's there is some bidder whose expected profit is negative and would benefit from deviating to bidding 0.

Thus, neither (%%1) nor (%%2) hold contradicting the hypothesis that W_n does not converge to V in probability. ■

3.1 Intuitive discussion of the $m = 2$ case

Suppose that $V \in \{v_L, v_H\}$, $v_L < v_H$, with prior ρ_V. Signals $x_i \in R$ are conditionally independent with conditional distributions F_V. The likelihood ratio $\frac{f_H(x)}{f_L(x)}$ is increasing and $\lim_{x\to\infty} \frac{f_H(x)}{f_L(x)} = \infty$. That is, large realizations of the signals are more likely when the true value is v_H than when it is v_L and there are signal values which make v_L exceedingly more likely to any level.

Consider a monotone equilibrium of a first price auction with n bidders.

Let $\Pr(v_L$ |winning, $x)$ be the probability that a bidder assigns to value v_L, conditional on signal x and being the winner in that monotone equilibrium

$$\Pr(v_L \mid \text{winning}, x) = \frac{\rho_L f_L(x) [F_L(x)]^{n-1}}{\rho_L f_L(x) [F_L(x)]^{n-1} + \rho_H f_H(x) [F_H(x)]^{n-1}} = \frac{1}{1 + \frac{\rho_H}{\rho_L} \frac{f_H(x)}{f_L(x)} \frac{[F_H(x)]^{n-1}}{[F_L(x)]^{n-1}}}.$$

Observe that $\Pr(v_L \mid \text{winning}, x)$ is jointly determined by the "signal effect", $\frac{f_H(x)}{f_L(x)}$, and the "winner's curse effect", $\frac{[F_H(x)]^{n-1}}{[F_L(x)]^{n-1}}$. This is not universal terminology but just terms that are used for this intuitive explanation. The sense in which $\frac{[F_H(x)]^{n-1}}{[F_L(x)]^{n-1}}$ is the winner's curse effect is that it determines the conditional probability of v_L based on winning alone (e.g., if $\rho_V = 1/2$ and $\frac{f_H(x)}{f_L(x)} = 1$). The signal effect goes to ∞ as $x \to \infty$. For a fixed x, the winner's curse term goes to 0 as $n \to \infty$. But the more relevant question is how it behaves for signals whose receivers are more likely to win. If we focus on a sequence of signals x_n at which the probability of winning in a monotone equilibrium $[F_H(x_n)]^{n-1} > \varepsilon$, then the winner's curse effect is bounded away from 0 and the signal effect overwhelms it, $\lim_{n\to\infty} \frac{f_H(x_n)}{f_L(x_n)} \frac{[F_H(x_n)]^{n-1}}{[F_L(x_n)]^{n-1}} = \infty$. It follows that, for large n, $\Pr(v_L$ |winning auction, $x_n) \approx 0$. That is, after allowing for the winner's curse, the winner is almost certain that the good is of type v_H.

In other words if $W_n(v_H)$ converges to a value below v_H, bidders who get a sufficiently high signal that still gives them a small probablity of winning in equilibrium (higher than the fixed $\varepsilon > 0$), will feel very safe to beat $\lim W_n(v_H)$ slightly since their conditional probability of v_H is almost 1.

Remarks:

1. This analysis based on Milgrom's (Econometrica, 1979) paper which itself is a generalization and deepening of Wilson (RES, 1977).

2. In different incomplete information environments, all the relevant information is revealed in equilibrium. For example, in a separating equilibrium of a signaling model, the information is revealed in equilibrium. But there the equilibrium outcome is not the outcome that would arise in the absence of incomplete information. When we speak about perfect information aggregation (or revelation) we usually mean that, like in the above auction model, the outcome is close to what it would be in under complete information.

3. Coming up with a mechanism that will aggregate the information in any of the above environments is not a problem. So, the question addressed by the analysis is not whether such mechanism is possible, but whether such revelations occurs through "natural" trading processes.

4 Auction with endogenous solicitation

BASICS

- A single seller and N potential bidders
- State $w \in \{L, H\}$; prior$(w) = \rho_w$ $w = L, H$
- Bidders have common values,

$$v_w \in \{v_L, v_H\}, \qquad v_L < v_H$$

- Seller knows w; bidders do not.

GAME

- Seller solicits $n \leq N$ randomly drawn bidders at marginal cost $s > 0$.
- $N \geq \frac{v_H}{s}$
- Each solicited bidder privately observes signal x
- n unobservable to bidders.
- Solicited bidders submit bids simultaneously.
- Highest bidder wins; ties are broken randomly.

SIGNALS

- Bidders' signals $x \sim G_w$ with support $[\underline{x}, \bar{x}]$
  - conditional on state w, signals are i.i.d.
  - G_w atomless with density g_w strictly positive on $[\underline{x}, \bar{x}]$
- Likelihood ratio $\frac{g_H(x)}{g_L(x)}$ is weakly increasing – $\bar{x}$ most/$\underline{x}$ least favorable signal.
- Signals boundedly informative.

$$0 < \frac{g_H(\underline{x})}{g_L(\underline{x})} < 1 < \frac{g_H(\bar{x})}{g_L(\bar{x})} < \infty$$

PAYOFFS:

Let p be the winning bid and n number of solicited bidders.

Payoffs:

- Winning Bidder: $v_w - p$;
- Other Bidders: 0;
- Seller: $p - ns$

STRATEGIES

Seller's:

- A pure solicitation strategy is $\mathbf{n} = (n_L, n_H) \in \{1, ..., N\} \times \{1, ..., N\}$.

Bidders':

- A bidding strategy $\beta : [\underline{x}, \bar{x}] \to \mathbb{R}$

INTERIM BELIEFS

- Interim belief is

$$\begin{aligned}
\Pr[H|\text{ signal } x, \text{ solicited}; \mathbf{n}] &= \frac{\rho_H g_H(x) \frac{n_H}{N}}{\rho_L g_L(x) \frac{n_L}{N} + \rho_H g_H(x) \frac{n_H}{N}} \\
&= \frac{\frac{\rho_H}{\rho_L} \frac{g_H(x)}{g_L(x)} \frac{n_H}{n_L}}{1 + \frac{\rho_H}{\rho_L} \frac{g_H(x)}{g_L(x)} \frac{n_H}{n_L}}
\end{aligned}$$

Henceforth, the argument "solicited" will be omitted.

COMPOUND LR AND SOLICITATION EFFECT

- $\frac{n_H}{n_L}$ captures "solicitation effect". Solicitation is bad news if $\frac{n_H}{n_L} < 1$.
- Effective LR $= \frac{\rho_H}{\rho_L} \frac{g_H(x)}{g_L(x)} \frac{n_H}{n_L} =$ prior LR$\times$signal LR$\times$solicitationLR

BELIEFS CONDITIONAL ON WIN AND PAYOFFS

- Given common bidding strategy β,

$$\pi_w(b|\beta, n_w) \triangleq \Pr(\text{win with bid } b \text{ at state } w \mid \beta, n_w) \qquad w = L, H$$

- Belief conditional on win with b

$$\Pr[H|\text{ signal } x, \text{ win with bid } b; \mathbf{n}, \beta] = \frac{\frac{\rho_H}{\rho_L} \frac{g_H(x)}{g_L(x)} \frac{n_H}{n_L} \frac{\pi_H(b|\beta, n_H)}{\pi_L(b|\beta, n_L)}}{1 + \frac{\rho_H}{\rho_L} \frac{g_H(x)}{g_L(x)} \frac{n_H}{n_L} \frac{\pi_H(b|\beta, n_H)}{\pi_L(b|\beta, n_L)}}$$

- Bidder's interim (i.e., after x) expected payoff from bidding b, given β and $\mathbf{n} = (n_L, n_H)$,

$$U(b|x,\beta,\mathbf{n}) = \sum_{w=L,H} \frac{\rho_w g_w(x) n_w \pi_w(b|\beta,n_w)(v_w - b)}{\rho_L g_L(x) n_L + \rho_H g_H(x) n_H}$$

BIDDING GAME AND EQUILIBRIUM

A symmetric pure **bidding equilibrium** given solicitation strategy $\mathbf{n} = (n_L, n_H)$ is a bidding strategy $\beta : [\underline{x}, \bar{x}] \to \mathbb{R}$ such that for all $x \in [\underline{x}, \bar{x}]$, $b = \beta(x)$ maximizes bidder's expected payoff $U(b|x,\beta,\mathbf{n})$.

Equilibrium of standard common value auction is special case with $n_H = n_L = n$

EXAMPLE OF A BIDDING EQUILIBRIUM

- Values $v_L = 0$ and $v_H = 1$.
- Uniform prior, $\rho_H = \rho_L = \frac{1}{2}$.
- Signals $x \in [\underline{x}, \bar{x}] = [0,1]$.
- $g_H(x) = 0.8 + 0.4x$ and $g_L(x) = 1.2 - 0.4x$

Claim: Let $N = 10$, $\mathbf{n} = (n_L, n_H) = (6,2)$. For all $\bar{b} \in [1/3, 0.4]$, there is a bidding equilibrium in which

$$\beta(x) = \bar{b} \qquad \forall x \in [\underline{x}, \bar{x}].$$

Proof: Expected profit at atom nonnegative.

- Expected value conditional on winning at $\bar{b}$

$$\begin{aligned}
&\mathbb{E}[\ v \mid \text{signal } x, \text{ win at } \bar{b}; \mathbf{n}, \beta] \\
= \quad & \frac{1}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H(\bar{b}|\beta,n_H)}{\pi_L(\bar{b}|\beta,n_L)}} v_L + \frac{\frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H(\bar{b}|\beta,n_H)}{\pi_L(\bar{b}|\beta,n_L)}}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H(\bar{b}|\beta,n_H)}{\pi_L(\bar{b}|\beta,n_L)}} v_H \\
\geq \quad & \frac{1}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(\underline{x})}{g_L(\underline{x})}\frac{n_H}{n_L}\frac{\pi_H(\bar{b}|\beta,n_H)}{\pi_L(\bar{b}|\beta,n_L)}} v_L + \frac{\frac{\rho_H}{\rho_L}\frac{g_H(\underline{x})}{g_L(\underline{x})}\frac{n_H}{n_L}\frac{\pi_H(\bar{b}|\beta,n_H)}{\pi_L(\bar{b}|\beta,n_L)}}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(\underline{x})}{g_L(\underline{x})}\frac{n_H}{n_L}\frac{\pi_H(\bar{b}|\beta,n_H)}{\pi_L(\bar{b}|\beta,n_L)}} v_H \\
= \quad & \frac{\frac{2}{3}\frac{2}{6}\frac{\frac{1}{2}}{\frac{1}{6}}}{1 + \frac{2}{3}\frac{2}{6}\frac{\frac{1}{2}}{\frac{1}{6}}} 1 = 0.4 \geq \bar{b}.
\end{aligned}$$

- Recall $\pi_w(b|\beta, n_w) = \Pr[Win \text{ with } b|\ w, \beta, n_w]$
- 2nd to 3rd line uses $\frac{g_H(\underline{x})}{g_L(\underline{x})} \leq \frac{g_H(x)}{g_L(x)}$
- 3rd to 4th line uses $v_L = 0$ and substitution of values.

No Incentives to overbid atom at $\bar{b}$.

- Expected value conditional on winning at $b' > \bar{b}$

$$
\begin{aligned}
& \mathbb{E}[\ v \mid \text{signal } x, \text{ win at } b'; \mathbf{n}, \beta] \\
= \quad & \frac{1}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H(b'|\beta,n_H)}{\pi_L(b'|\beta,n_L)}} v_L + \frac{\frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H\left(b'|\beta,n_H\right)}{\pi_L(b'|\beta,n_L)}}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H(b'|\beta,n_H)}{\pi_L(b'|\beta,n_L)}} v_H \\
\leq \quad & \frac{1}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(\bar{x})}{g_L(\bar{x})}\frac{n_H}{n_L}} v_L + \frac{\frac{\rho_H}{\rho_L}\frac{g_H(\bar{x})}{g_L(\bar{x})}\frac{n_H}{n_L}}{1 + \frac{\rho_H}{\rho_L}\frac{g_H(\bar{x})}{g_L(\bar{x})}\frac{n_H}{n_L}} v_H \\
= \quad & 0 + \frac{\frac{1}{1}\frac{3}{2}\frac{2}{6}}{1 + \frac{1}{1}\frac{3}{2}\frac{2}{6}} 1 = \frac{1}{3} \leq \bar{b} < b'
\end{aligned}
$$

- 2nd to 3rd line uses: $\pi_H(b'|\beta, n_H) = \pi_L(b'|\beta, n_L) = 1$ and $\frac{g_H(\bar{x})}{g_L(\bar{x})} \geq \frac{g_H(x)}{g_L(x)}$
- 3rd to 4th line uses $v_L = 0$ and substitution of values.
- Since $\frac{1}{3} \leq \bar{b}$, profit from overbid < 0

No incentive to undercut the atom at $\bar{b}$.

- since profit at $b < \bar{b} = 0 \leq E\left(\text{profit at } \bar{b}\right)$.

This completes the proof. ■

Remark on equilibrium atom

- Equilibrium atom at $\bar{b}$ requires:
  - biding at $\bar{b}$ is profitable even for $\underline{x}$
  - overbiding at $\bar{b}$ is unprofitable even for $\bar{x}$
- Sufficient condition is

$$\mathbb{E}[\ v \mid \text{signal } \bar{x}; \mathbf{n}] < \bar{b} < \mathbb{E}[\ v \mid \text{signal } \underline{x}, \text{ win at } \bar{b}; \mathbf{n}]$$

- This is achieved by two elements:
  - Winning at $\bar{b}$ "insures" against L,

$$\frac{\pi_H\left(\bar{b}|\beta, n_H\right)}{\pi_L\left(\bar{b}|\beta, n_L\right)} = \frac{\frac{1}{n_H}}{\frac{1}{n_L}} = \frac{\frac{1}{2}}{\frac{1}{6}} = 3$$

  - The **Solicitation Curse**:

$$\frac{g_H\left(\bar{x}\right)}{g_L\left(\bar{x}\right)}\frac{n_H}{n_L} \text{ is small (in fact } < 1)$$

  that keeps $\mathbb{E}[\ v \mid \text{signal } \bar{x}; \mathbf{n}]$ low.

- Note that $\frac{g_H(\bar{x})}{g_L(\bar{x})}\frac{n_H}{n_L} < 1$ implies that $\bar{x}$ is more pessimistic than prior

$$\mathbb{E}[\ v \mid \text{signal } \bar{x}; \mathbf{n}] < \rho_L v_L + \rho_H v_H$$

Additional observations

- Whenever $\frac{n_H}{n_L} = \frac{1}{3}$ and $n_H \geq 2$ there is a bidding equilibrium where all bidders bid $\bar{b} \in [1/3, 0.4]$.

- In fact, it follows from subsequent results that, if $n_L = 3n_H$ and n_H is sufficiently large, there exists no equilibrium in strictly increasing strategies. Atoms are "unavoidable."

- Construction is just a bidding equilibrium–not a full equilibrium: Seller's solicitation strategy not optimal.

Comparison: ordinary CV auction

- For comparison: Essentially no atoms in ordinary CV auction $(n_L = n_H)$.

PROPOSITION: Suppose that $n_H \geq n_L \geq 2$ and β is a bidding equilibrium.

(i) If $\frac{g_H(x)}{g_L(x)}$ is strictly increasing over $[\underline{x}, \bar{x}]$, then β is strictly increasing.

(ii) If β has an atom, it must be over a bottom interval $[\underline{x}, \hat{x}]$, $\frac{g_H(x)}{g_L(x)}$ must be constant over $[\underline{x}, \hat{x}]$ and expected payoff of bidders with $x \in [\underline{x}, \hat{x}]$ is zero.

Full Equilibrium

$\Gamma(s)$ is the full (solicitation&bidding) game with $N = \lceil \frac{v_H}{s} \rceil$.

A symmetric (pure strategy) **equilibrium** of $\Gamma(s)$ consists of a bidding strategy $\beta : [\underline{x}, \bar{x}] \to \mathbb{R}$ and a solicitation strategy $\mathbf{n} = (n_L, n_H)$ such that

(i) β is a bidding equilibrium given solicitation strategy (n_L, n_H);

(ii) solicitation is optimal,

$$n_w \in \arg \max_{n \in \{1,\ldots,N\}} \left[\mathbb{E}\left[p|n, w, \beta\right] - ns\right].$$

- Expected winning bid, $E\left[p|w, \beta, n\right]$, non-decreasing, **concave** in n.

Equilibrium with Small Solicitation Costs

- A sequence $\{s^k\}_{k=1}^{\infty}$ such that $s^k \to 0$.
- A sequence of games $\{\Gamma(s^k)\}$
- A sequence of equilibria $\{\beta^k, \eta^k\}$ of the games $\{\Gamma(s^k)\}$.
- All associated equilibrium magitudes will be indexed by k.
- Small s – counterpart of many bidders in standard auction.

NO-AGGREGATION EQUILIBRIUM:

Pooling like in above example can arise with optimal solicitation.

- **Special case: Good News / Bad News:** $\hat{x} \in (\underline{x}, \bar{x})$

$$\frac{g_H(x)}{g_L(x)} = \begin{cases} \frac{g_H(\bar{x})}{g_L(\bar{x})} = \text{constant} > 1 & \text{if} \quad x > \hat{x}, \\ \frac{g_H(\underline{x})}{g_L(\underline{x})} = \text{constant} < 1 & \text{if} \quad x \le \hat{x}. \end{cases}$$

- A continuum of signals but from information perspective only 2 signals.
- Also assume $\frac{1}{G_L(\hat{x})} < \frac{g_H(\bar{x})}{g_L(\bar{x})}$

Let $s^k \to 0$ and $\Gamma(s^k)$ the corresponding sequence of games.

PROPOSITION.
There exists a sequence of equilibria $\{\beta^k, \mathbf{n}^k\}$ such that $(n_L^k, n_H^k) \to \infty$ and for some $\bar{b}$ and k large enough,

$$\beta^k(x) = \bar{b} < \rho_L v_L + \rho_H v_H \qquad \forall x > \hat{x}.$$

MAIN STEPS of PROOF:

- For s^k small, let
$$\beta^k(x) = \begin{cases} \bar{b} & \text{if} \quad x > \hat{x}, \\ \underline{b}^k & \text{if} \quad x \le \hat{x}. \end{cases}$$
where $\underline{b}^k < \bar{b}$, and $\lim_{k\to\infty} \underline{b}^k < \bar{b}$.

- $\bar{b}$ and $\underline{b}^k$ are not arbitrary: they have to be chosen appropriately.

Preliminary fact:

$$\pi_w(\bar{b}|\beta, n_w) = \frac{1 - G_w(\hat{x})^{n_w}}{n_w(1 - G_w(\hat{x}))} \tag{1}$$

Step 0: $n_w \to \infty, \ \ w = L, H$.

Step 1:
$$\frac{g_H(\bar{x})}{g_L(\bar{x})} \lim_{k\to\infty} \frac{n_H^k}{n_L^k} < 1.$$

Proof of Step 1:

- Ignoring integer problem, solicitation (n_H^k, n_L^k) optimality:

$$\begin{aligned}(G_L(\hat{x}))^{n_L^k}(1-G_L(\hat{x}))\left(\bar{b}-\underline{b}^k\right) &= s^k \\ (G_H(\hat{x}))^{n_H^k}(1-G_H(\hat{x}))\left(\bar{b}-\underline{b}^k\right) &= s^k\end{aligned}$$

- Substituting out s^k, making a logarithmic transformation, rearranging and then taking limits we get

$$\lim_{k\to\infty}\frac{n_H^k}{n_L^k}=\frac{\ln G_L(\hat{x})}{\ln G_H(\hat{x})}$$

- The facts $\frac{g_H(\bar{x})}{g_L(\bar{x})}=\frac{1-G_H(\hat{x})}{1-G_L(\hat{x})}$ and $\frac{1-z}{\ln z}$ decreasing in z imply the result. $\square$

Step 2: Bidder $\bar{x}$'s value conditional on winning at $\bar{b}$ = ex-ante E(v).

$$\lim_{k\to\infty}\mathbb{E}[v|\text{ signal }\bar{x},\text{ win at }\bar{b};\beta,\mathbf{n}]=\rho_L v_L+\rho_H v_H$$

Proof of Step 2: Recall $\pi_w(p|\beta,n_w)\triangleq\Pr(\text{win with bid }p|\ w,\beta,n_w)$.

$$\mathbb{E}[\ v\mid\text{signal }x,\text{ win at }p;\beta,\mathbf{n}]=\frac{v_L+\frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H(p|\beta,n_H)}{\pi_L(p|\beta,n_L)}v_H}{1+\frac{\rho_H}{\rho_L}\frac{g_H(x)}{g_L(x)}\frac{n_H}{n_L}\frac{\pi_H(p|\beta,n_H)}{\pi_L(p|\beta,n_L)}}\tag{2}$$

Now, (1) together with $\frac{g_H(\bar{x})}{g_L(\bar{x})}=\frac{1-G_H(\hat{x})}{1-G_L(\hat{x})}$ imply the result. $\square$

Step 3: For small s, bidder $\bar{x}$'s value conditional on winning at $b>\bar{b}$ is bounded away below ex-ante E(v)

$$\mathbb{E}[v|\text{signal }\bar{x},\text{win at }b>\bar{b};\beta,\mathbf{n}]\approx\frac{\rho_L v_L+\rho_H\frac{g_H(\bar{x})}{g_L(\bar{x})}\lim\frac{n_H}{n_L}v_H}{\rho_L+\rho_H\frac{g_H(\bar{x})}{g_L(\bar{x})}\lim\frac{n_H}{n_L}}<\rho_L v_L+\rho_H v_H$$

Proof of Step 3: Substitutte $x=\bar{x}$ and $\frac{\pi_H(b|\beta,n_H)}{\pi_L(b|\beta,n_L)}=1$ in (2) to get

$$\mathbb{E}[\ v\mid\text{signal }\bar{x},\text{win at }b>\bar{b};\beta,\mathbf{n}]=\frac{v_L+\frac{\rho_H}{\rho_L}\frac{g_H(\bar{x})}{g_L(\bar{x})}\frac{n_H}{n_L}v_H}{1+\frac{\rho_H}{\rho_L}\frac{g_H(\bar{x})}{g_L(\bar{x})}\frac{n_H}{n_L}}$$

Apply $\lim_{s\to 0}$ to both sides and recall from Step 2 that $\lim\frac{g_H(\bar{x})}{g_L(\bar{x})}\frac{n_H}{n_L}<1$ to get the result. $\square$.

Step 4: $\exists\ \bar{b}$ s.t., for small s, winning at $\bar{b}$ is profitable and overbidding is not.

Proof of Step 4: Choose $\bar{b}$ to satisfy

$$\frac{\rho_L v_L + \rho_H \frac{g_H(\bar{x})}{g_L(\bar{x})} \lim \frac{n_H}{n_L} v_H}{\rho_L + \rho_H \frac{g_H(\bar{x})}{g_L(\bar{x})} \lim \frac{n_H}{n_L}} < \bar{b} < \rho_L v_L + \rho_H v_H$$

Then Steps 2 and 3 imply the result. □

Step 5: No incentives to undercut $\bar{b}$.

Proof of Step 5:

- Since $\bar{b} < \rho_L v_L + \rho_H v_H$, by Step 2 profit at $b = \bar{b}$ (before the limit) > 0
- As $k \to \infty$, prob(winning at $b = \bar{b}$) $>>>$ prob(winning at $b < \bar{b}$) which overwhelmes the difference in price.
  - By (), for large n_w, $\pi_w(\bar{b}|\beta, n_w)$ is on the order of $\frac{1}{n_w}$, while $\pi_w(b|\beta, n_w)$ is on the order of $G_w(\hat{x})^{n_w}$. Hence, $\frac{\pi_w(b|\beta,n_w)}{\pi_w(\bar{b}|\beta,n_w)} \to 0$. □

The next step concern the optimality of bidding $\underline{b}^k$ by $x \leq \hat{x}$.

Let

$$\underline{b}^k = \mathbb{E}[v|\text{signal } \underline{x}, \text{win at } b = \underline{b}^k]$$

Step 6: We have to establish the following

- $\underline{b}^k$ so defined $< \rho_L v_L + \rho_H v_H$ so that we can have $\bar{b} > \underline{b}^k$.
- No incentives for $x \leq \hat{x}$ to overbid $\underline{b}^k$:
  - with $b \in (\underline{b}^k, \bar{b})$.
  - with $b = \bar{b}$
  - with $b > \bar{b}$
- The latter is obvious, since $\mathbb{E}[v|$ signal $\underline{x}$, win at $b > \bar{b}] < \mathbb{E}[v|$ signal $\bar{x}$, win at $b > \bar{b}] < \bar{b}$ by choice of $\bar{b}$.
- Each of the other two requires a calculation resembling the one we performed. The unprofitability of $b = \bar{b}$ uses the special condition on G.

■

Pooling Equilibrium–Summary

- In the limit, price is $\bar{b}$ in both states.
- Key: optimal sampling results in $\lim_{k\to\infty} \frac{n_H^k}{n_L^k}$ s.t. $\frac{g_H(\bar{x})}{g_L(\bar{x})} \lim_{k\to\infty} \frac{n_H^k}{n_L^k} < 1$.
- Argument almost did not use the two (effective) signal assumption. Can be fairly straighforwardly extended to any FINITE number of (effective) signals (which we do in the paper).
- Additional condition on G_w was used, though not "particularly strong."
- No information aggregation – auction does not become “competitive”
- $G_H(\hat{x})$ can be arbitrarily small and $G_L(\hat{x})$ arbitrarily large, i.e., signals can be arbitrarily informative
- Based on Lauermann-Wolinsky (2013).

5 Search with adverse selection

BASICS

- A single buyer and a continuum of identical potential sellers
- Buyer is looking to complete a single transaction.
- Buyer's value of the transaction is u.
- State $w \in W = \{L, H\}$. Prior(w)= ρ_w.
- A seller's cost of providing the service c_w, $c_H > c_L$.
- Buyer knows w; sellers do not.

GAME

- Buyer samples sellers sequentially (and randomly).
- Sampling (per draw) cost $s > 0$.
- Sampled seller observes signal $x \in [\underline{x}, \overline{x}]$
  - $x \backsim F_w$, $w = L, H$, continuous density f_w
  - increasing f_H/f_L: lower x's more indicative of low cost.
  - Conditional on w, signal is independent across sellers.
- Buyer-seller "bargain" over price:
  - nature draws a price p from distribution G over $[0, u]$.
    - G: full support, strictly positive differentialble density g.
  - Seller announces whether accepts p
  - Then buyer annouces
  - If both accept, transact at p – END
  - If not, disengage and buyer continues search.
- **Outcome**
  - A history of the process records the sequence of all encountered sellers, signal realizations, prices, and acceptance decisions up to a certain point.
  - A terminal history is a history that ends with a trade or an infinite history with no trade.
  - A finite terminal history determines a terminal outcome (n^t, p^t, x^t, j^t)

- **Payoffs**
  - Buyer type w's payoff after a finite terminal history is
$$u - p^t - n^t s;$$
    the payoff after an infinite history is $-\infty$.
  - Seller j^t's payoff from transacting with buyer type w is
$$p^t - c_w.$$
    Payoffs are zero for all other sellers.

STRATEGIES AND BELIEFS

- **Information**
  - Seller: x and p. Nothing else.
  - Buyer: w and entire history of search (does'nt matter whether observes x's).
- **Strategies**
  - Buyer's: $B = (B_L, B_H)$
    where $B_w(\varphi) \subset [0, u]$ are prices accepted by $w = L, H$ after history φ.
  - Seller j's: $A_j(x) \subset [0, u]$ set of accepted prices after signal x.
  - Profile $(B, \mathcal{A}) = ((B_w)_{w \in W}, (A_j)_{j \in [0,1]})$.
- **Beliefs**
  - $\Pi(w|x) \equiv \Pi(w|x; B, \mathcal{A})$ denote a seller's belief that the buyer's type is w, conditional on being sampled and observing signal x, when the strategy profile is $(B, \mathcal{A})$

PAYOFFS AS FUNCTION OF STRATEGIES AND BELIEFS

Profile $(B, \mathcal{A}) = ((B_w)_{w \in W}, (A_j)_{j \in [0,1]})$ together with the prior over the set of types W induce a distribution on the set of terminal histories and, hence, over terminal outcomes.

- **Buyer's expected payoff**
$$V_w \equiv V_w(B, \mathcal{A}) = u - E\left[p^t | w; B, \mathcal{A}\right] - sE\left[n^t | w; B, \mathcal{A}\right],$$
  where expectation is w.r.t the said distribution

– ASSUMPTION: s is small enough such that

$$u \geq \int_{c_H}^{u} p \frac{dG(p)}{1 - G(c_H)} + \frac{s}{1 - G(c_H)}.$$

Thus, $V_w > 0$ even if the sellers accept only prices that are above c_H.

- **Seller's expected payoff**
  – Trade at p after signal x yields

$$p - E[c|x, p \text{ is accepted by buyer}; B, \mathcal{A}]$$

where the expectation is w.r.t. to the belief over w conditional on x and acceptance of p.

EQUILIBRIUM

An (perfect-Bayesian) **equilibrium** consists of a strategy profile, $(B, \mathcal{A})$, and belief Π such that: (i) after any history φ, B_w maximizes the expected payoff of the buyer of type w given $\mathcal{A}$. (ii) for any signal realization x, $A_j(x)$ maximizes seller j's expected profit given B and $\Pi(w|x)$. (iii) $\Pi(w|x)$ is consistent with Bayesian updating.

EQUILIBRIUM STRATEGIES AND BELIEFS

- Buyer's
  – $B_w(\varphi) = [0, u - V_w]$, for all φ.
- Seller's
  – expected cost, given $W' \subseteq W$

$$E[c|x, W'] \equiv E[c|x, W'; B, \mathcal{A}] = \frac{\sum_{w \in W'} \Pi(w|x) c_w}{\sum_{w \in W'} \Pi(w|x)}.$$

  – Let $W(p) = \{w | \ p \in B_w\}$. For $p \in \cup B_w$, optimality of $A_j(x)$ requires that $p \in A_j(x)$ iff $p \geq E[c|x, W(p)]$.
  – Because $E[c|x, W(p)]$ is independent of j, we will drop the subscript j and write $A(x)$.[2] Consequently, the equilibrium strategy profile $\mathcal{A}$ will be identified with the individual strategy A which will denote the profile as well.

[2]For $p \notin \cup B_w$, any acceptance decision is optimal. To simplify the exposition, we assume for $p \notin \cup B_w$ that, if $p < c_1$, then $p \notin A_j(x)$ for all x and, if $p > c_m$, then $p \in A_j(x)$ for all x. This assumption has no consequences for the equilibrium outcome.

- **Additional notation**
  - Set of all signal-price pairs that result in trade given type w

$$\Omega_w \equiv \Omega_w(B, A) = \{(x, p) : p \in B_w \cap A(x)\}.$$

  - $\Gamma_w(Q) = (F_w \times G)(Q) =$ Probability (a meeting of a seller and buyer w yields $(x, p) \in Q$).
    Thus, $\Gamma_w(\Omega_w) =$ Probability (a given meeting between a seller and buyer w ends in trade).
  - The expected number of sampled sellers is

$$n_w = n_w(B, A) = E\left[n^t | w; B, A\right] = \frac{1}{\Gamma_w(\Omega_w)}.$$

- **Equilibrium Beliefs**.

Claim: Using the an extension of Bayes formula that conditions on the 0-probability events,

$$\Pi(w|x) = \frac{\rho_w f_w(x) n_w}{\rho_L f_L(x) n_L + \rho_H f_H(x) n_H}. \tag{3}$$

- **Intuitive explanation of beliefs formula:** Suppose a finite number N of sellers but that behavior is described by stationary and symmetric acceptance strategies, B and A.[3] If the buyer samples uniformly without replacement from N sellers with success probability $\Gamma_w = \Gamma_w(\Omega_w)$ and $\Gamma_w > 0$, then

$$\begin{aligned}\Pr[j \text{ sampled}|w;\ N] &= \frac{1}{N} + \frac{N-1}{N}\frac{1-\Gamma_w}{N-1} + \cdots + \frac{N-1}{N}\frac{N-2}{N-1}\cdots\frac{1}{2}(1-\Gamma_w)^{N-1} \\ &= \frac{1-(1-\Gamma_w)^N}{N\Gamma_w} = \frac{n_w}{N}\left(1-(1-\Gamma_w)^N\right),\end{aligned}$$

using $n_w = 1/\Gamma_w$. Therefore,

$$\begin{aligned}\Pr[w|j \text{ sampled}, x;\ N] &= \frac{\rho_w f_w(x)\frac{n_w}{N}\left(1-(1-\Gamma_w)^N\right)}{\rho_L f_L(x)\frac{n_L}{N}\left(1-(1-\Gamma_w)^N\right) + \rho_H f_H(x)\frac{n_H}{N}\left(1-(1-\Gamma_w)^N\right)} \\ &\underset{N\to\infty}{\longrightarrow} \frac{\rho_w f_w(x) n_w}{\rho_L f_L(x) n_L + \rho_H f_H(x) n_H}.\end{aligned}$$

which coincides with (3).

EQUILIBRIUM – Existence & Characterization

[3] In the appendix, we also allow the sellers' strategies to be not symmetric.

- **Fact:** An equilibrium exists.

- **Fact:** If (B, A) an equilibrium, then $V_L(B, A) > V_H(B, A)$

EQUILIBRIUM BEHAVIOR

- Let $E[c|x] \equiv E[c|x, W]$
- In equilibrium
$$B_w = [0, u - V_w] \qquad w = L, H$$
and since $V_L > V_H$,
$$A(x) = [E(c|x), u - V_L] \cup [c_H, u]$$
- Therefore, in equilibrium
$$\begin{aligned} \Omega_L &= \{(p, x) : p \in [E(c|x), u - V_L]\} \\ \Omega_H &= \Omega_L \cup [c_H, u - V_H]. \end{aligned}$$
hence
$$\Omega_L = \Omega_H \qquad \text{if } V_H \geq u - c_H.$$
- Define $\xi = \xi(B, A)$
$$\begin{cases} E_I(c|\xi) = u - V_L & \text{if} \quad V_L \geq u - E_I(c|\overline{x}) \\ \xi = \overline{x} & \text{if} \quad V_L < u - E_I(c|\overline{x}) \end{cases}$$

- Equilibrium is of the form
  - L searches to (x, p) s.t.
$$x \leq x_* \text{ and } p \in [E_I(c|x), E_I(c|x_*)]$$

  - H stops after same (x, p) and also after (x, p) s.t. $p \in [c_H, u - V_H]$

INFORMATION AGGREGATION

- To what extent do prices aggregate information when s is small?

- Maximal aggregation if price paid by buyer w is close to c_w;
- Minimal when both buyer types pay the same price(s).

- In 1st price CV auction version of our model (Wilson(1977) and Milgrom(1979)):

  $p_w \to c_w$ when #(bidders)$\to \infty$ iff $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$.

- Here counterpart of increasing number of bidders is small s.
- Sequence $s^k \to 0$ & associated equilibrium sequence $\left(B^k, A^k\right)$
- Ω_w^k, x_*^k, V_w^k, n_w^k, $E_I^k()$, etc. magnitudes associated with $\left(B^k, A^k\right)$
- p_w^k = expected price paid by w in $\left(B^k, A^k\right)$; $\overline{p}_w = \lim_{k \to \infty} p_w^k$
- S_w^k = w's expected search cost in $\left(B^k, A^k\right)$; $\overline{S}_w = \lim_{k \to \infty} S_w^k$.

- Why expect information aggregation?

Intuitively, the good type L might search till it generates a low enough signal that will enable trading at relatively low price. If it is too costly for H to mimic this behavior, and it settles quickly for c_H the prices might aggregate information. If however H mimics L's behavior prices may fail to aggregate information.

5.1 Information Aggregation with Boundedly Informative Signals

In the case of boundedly informative signals,

$$\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty \tag{4}$$

even the most favorable signal carries only limited information. The limit equilibrium outcome, as $s^k \to 0$, is complete pooling: all types pay the same price which is in turn equal to the ex-ante expected cost.

Proposition 1 *Suppose that* $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$. *Then:*

1. $\overline{p}_w = E[c] \qquad \forall w \in W$,
2. $\overline{S}_w = 0 \qquad \forall w \in W$.

The intuition behind the proof is most transparent when the set of possible signal values is finite (with $\underline{x}$ the lowest such value and $f_w(\underline{x}) > 0$ its probability). Suppose to the contrary that $\bar{S}_L > 0$. Since $E^k[c|\underline{x}]$ is the lowest price a seller will ever accept, sequential rationality implies that type L accepts any $p \in [E^k[c|\underline{x}], E^k[c|\underline{x}] + \frac{\bar{S}_L}{2}]$ and hence sellers also accept such p's after signal $\underline{x}$, i.e., $\{\underline{x}\} \times [E^k[c|\underline{x}], E^k[c|\underline{x}] + \frac{\bar{S}_L}{2}] \subset \Omega_L^k$. Type L's probability of realizing signal $\underline{x}$ and such p is bounded away from 0, i.e., $\Gamma_L(\{\underline{x}\} \times [E^k[c|\underline{x}], E^k[c|\underline{x}] + \frac{\bar{S}_L}{2}]) \geq \gamma > 0$. Therefore, $\Gamma_L(\Omega_L^k) \geq \gamma > 0$, implying $\bar{S}_L \equiv \lim \frac{s^k}{\Gamma_L(\Omega_L^k)} \leq \lim \frac{s^k}{\gamma} = 0$.

A similar argument establishes that $\bar{p}_L = \lim E^k[c|\underline{x}]$: If $\bar{p}_L > \lim E^k[c|\underline{x}]$, it would be profitable for type L to wait for $(\underline{x}, p)$ s.t. p is between $E^k[c|\underline{x}]$ and $\bar{p}_L$, which occur with strictly positive, non-vanshing probability and hence involves negligible search cost as $s^k \to 0$.

It is then immediate from $\frac{f_L(\underline{x})}{f_H(\underline{x})} < \infty$ that every other type can mimick type L at no cost. This is because MLRP implies that

$$\frac{s^k}{\Gamma_H(\Omega_L^k)} = \frac{\Gamma_L(\Omega_L^k)}{\Gamma_H(\Omega_L^k)} \frac{s^k}{\Gamma_L(\Omega_L^k)} \leq \frac{f_L(\underline{x})}{f_H(\underline{x})} \frac{s^k}{\Gamma_L(\Omega_L^k)}.$$

Therefore, $\lim \frac{s^k}{\Gamma_H(\Omega_L^k)} \leq \frac{f_L(\underline{x})}{f_H(\underline{x})} \bar{S}_L = 0$.

Since H can costlessly mimick type L, it follows that $\overline{p}_H = \overline{p}_L = \lim E^k[c|\underline{x}]$. Since both types trade after realizing signal $\underline{x}$ it follows that $\lim E^k[c|\underline{x}] = E[c]$.

The formal proof presents the corresponding arguments for a continuum of signals.

Proof of Proposition 1:

Step 1. For any $\delta > 0$, there exists $x(\delta) > \underline{x}$ s.t. for k sufficiently large

$$E^k[c|x(\delta)] - E^k[c|\underline{x}] < \delta.$$

Proof. Follows immediately from $E^k[c|x] = \frac{c_H + \frac{\rho_L}{\rho_H} \frac{f_L(x)}{f_H(x)} \frac{n_L^k}{n_H^k} c_L}{1 + \frac{\rho_L}{\rho_H} \frac{f_L(x)}{f_H(x)} \frac{n_L^k}{n_H^k}}$ and $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$. □

Step 2.

$$\bar{S}_L \equiv \lim_{k \to \infty} \frac{s^k}{\Gamma_L^k(\Omega_L^k)} = 0.$$

Proof. Suppose $\bar{S}_L > 0$. By step 1, there is a signal $x' = x(\bar{S}_L/3) > \underline{x}$ such that

$$E^k[c|x'] - E^k[c|\underline{x}] < \bar{S}_L/3$$

for all k. Since $V_L^k \leq u - E^k[c|\underline{x}] - 2\bar{S}_L/3$ for all k large enough, $V_L^k \leq u - E^k[c|x'] - \bar{S}_L/3$. Hence,

$$\Omega_L^k \supseteq [\underline{x}, x'] \times [E^k[c|x'], E^k[c|x'] + \bar{S}_L/3].$$

But then

$$\Gamma_L^k\left(\Omega_L^k\right) \geq F\left(x'\right)\left(G(E^k[c|x'] + \bar{S}_L/3) - G(E^k[c|x'])\right).$$

Since the RHS stays strictly positive, $\lim_{k\to\infty} \frac{s^k}{\Gamma_L^k\left(\Omega_L^k\right)} = 0$. □

Step 3.

$$\bar{p}_L = \lim_{k\to\infty} E^k[c|\underline{x}].$$

Proof. Since $E^k[c|x]$ is increasing in x, obviously $\bar{p}_L \geq \lim_{k\to\infty} E^k[c|\underline{x}]$. Suppose to the contrary that $\bar{p}_L - \lim_{k\to\infty} E^k[c|\underline{x}] = \delta > 0$. By Step 1 there is $x'' = x(\frac{\delta}{3}) > \underline{x}$ such that

$$E^k[c|x''] - E^k[c|\underline{x}] < \frac{\delta}{3}$$

for all k. Define

$$\tilde{\Omega}^k = [\underline{x}, x''] \times [E^k[c|x''], E^k[c|x''] + \frac{\delta}{3}].$$

and observe that, for k large enough, $\tilde{\Omega}^k \subset \Omega_L^k$ and $\lim \Gamma_L^k\left(\tilde{\Omega}^k\right) > 0$. Therefore, by the optimality L's equilibrium stgrategy,

$$V_L^k \geq u - E^k[c|x''] - \frac{\delta}{3} - \frac{s^k}{\Gamma_L^k\left(\tilde{\Omega}^k\right)} > u - E^k[c|\underline{x}] - \frac{2\delta}{3} - \frac{s^k}{\Gamma_L^k\left(\tilde{\Omega}^k\right)}.$$

This and Step 2 together imply

$$u - \bar{p}_L = \lim V_L^k \geq u - \lim E^k[c|\underline{x}] - \frac{2\delta}{3} - \lim \frac{s^k}{\Gamma_L^k\left(\tilde{\Omega}^k\right)} = u - \lim E^k[c|\underline{x}] - \frac{2\delta}{3}.$$

where the last equality is due to $\lim \frac{s^k}{\Gamma_L^k\left(\tilde{\Omega}^k\right)} = 0$. It follows that $\bar{p}_L < \lim E^k[c|\underline{x}] + \delta$ – contradiction. □

Step 4.

$$\lim_{k\to\infty} \frac{s^k}{\Gamma_w\left(\Omega_L^k\right)} = 0 \qquad \forall w \in W$$

and

$$\lim E[p|\left(p,x\right) \in \Omega_L^k, w] = \lim E^k\left[c|\underline{x}\right] \qquad \forall w \in W.$$

Proof. Rewriting,

$$\lim_{k\to\infty} \frac{s^k}{\Gamma_w\left(\Omega_L^k\right)} = \lim_{k\to\infty} \frac{\Gamma_L\left(\Omega_L^k\right)}{\Gamma_w\left(\Omega_L^k\right)} \frac{s^k}{\Gamma_L\left(\Omega_L^k\right)} \leq \lim_{k\to\infty} \frac{f_L\left(\underline{x}\right)}{f_w\left(\underline{x}\right)} \frac{s^k}{\Gamma_L\left(\Omega_L^k\right)} = \frac{f_L\left(\underline{x}\right)}{f_w\left(\underline{x}\right)} \bar{S}_L = 0.$$

Where the inequality stems from the MLRP and the final equality from Step 1.

Consider p s.t. $(p,x) \in \Omega_L^k$ for some x. From sellers' optimality $p \geq E^k[c|x]$ and from MLRP $E^k[c|x] \geq E^k[c|\underline{x}]$. By buyer's optimality, $p \leq u - V_L^k$. Hence,

$$E^k[c|\underline{x}] \leq E[p|(p,x) \in \Omega_L^k, w] \leq u - V_L^k \qquad \text{for } w \in W.$$

From Steps 2 and 3, $\lim E^k[c|\underline{x}] = u - \lim V_L^k$, which establishes the claim. □

Step 5.

$$\bar{p}_w = \lim E^k[c|\underline{x}] \qquad \forall w \in W.$$

Proof. Lemma **??** and Steps 2 and 3 imply $\lim V_w^k \leq \lim V_L^k = u - \lim E^k[c|\underline{x}]$ for all w. The optimality of type w's equilibrium strategy and Step 4 imply

$$\lim_{k\to\infty} V_w^k \geq u - \lim E\left[p|(p,x) \in \Omega_L^k, w\right] - \lim_{k\to\infty} \frac{s^k}{\Gamma_w\left(\Omega_L^k\right)} = u - \lim E^k[c|\underline{x}].$$

Thus, $\lim V_w^k = u - \lim E^k[c|\underline{x}]$. Now, $\Omega_w^k \supseteq \Omega_L^k$ and Step 4 imply $\overline{S}_w = 0$ and hence $\lim V_w^k = u - \bar{p}_w$. Therefore, $u - \bar{p}_w = u - \lim E^k[c|\underline{x}]$ implying the result. □

Step 6.

$$\lim E^k[c|\underline{x}] = E[c].$$

Proof. Recall $W^k(p) = \left\{w|(p,x) \in \Omega_w^k \text{ for some } x\right\}$. From the law of iterated expectations,

$$E[c] = E^k[c|\text{ trade }] = \rho_L E\left[E^k\left[c|x, w \in W^k(p)\right] \mid \Omega_L^k\right] + \rho_H E\left[E^k\left[c|x, w \in W^k(p)\right] \mid \Omega_H^k\right]. \tag{5}$$

Since by Lemma **??** and the definition of Ω_w^k, $W^k(p)$ is of the form $\{L,H\}$ or $\{H\}$, MLRP implies $E^k\left[c|x, W^k(p)\right] \geq E^k\left[c|\underline{x}, W^k(p)\right] \geq E^k[c|\underline{x}]$. This and the definition of Ω_w^k imply that, if $(p,x) \in \Omega_w^k$, then $u - V_w^k \geq E^k\left[c|x, W^k(p)\right] \geq E^k[c|\underline{x}]$. By Steps 4 and 5, $u - \lim V_w^k = \lim E^k[c|\underline{x}]$. Hence,

$$\lim E\left[E^k\left[c|x, w \in W^k(p)\right] \mid \Omega_i^k\right] = \lim E^k[c|\underline{x}].$$

Therefore, taking the limit of RHS(5) gives the result. □

This concludes the proof of Proposition 1: Steps 5 and 6 establish Part 1. Step 4 and $\Omega_w^k \supseteq \Omega_L^k$ establish Part 2. ■

An alternative argument uses two facts about the equilibrium that seem intuitively obvious but require further work to prove formally. First, when k is large enough, type H mimics L in the sense that $\Omega_L^k = \Omega_H^k$. Second, the cutoff $\xi^k \to \underline{x}$. These two observations together imply that

$$\lim_{k\to\infty} \frac{n_L^k}{n_H^k} = \lim_{k\to\infty} \frac{\Gamma_H\left(\Omega_L^k\right)}{\Gamma_L\left(\Omega_L^k\right)} = \frac{f_H(\underline{x})}{f_L(\underline{x})}.$$

So, the relative probability of being sampled is inversely related to the relative probability of the signals. Therefore

$$\lim_{k\to\infty} \frac{\rho_L}{\rho_H} \frac{f_L\left(\xi^k\right)}{f_H\left(\xi^k\right)} \frac{n_L^k}{n_H^k} = \frac{\rho_L}{\rho_H},$$

the posterior likelihood ratio conditional on $x \in \left[\underline{x}, \xi^k\right]$ and conditional on being sampled is equal to the prior likelihood ratio. Thus, when s^k is small, $E^k[c|x] \cong E[c]$ for all $x \in \left[\underline{x}, \xi^k\right]$.

BOUNDEDLY INFORMATIVE SIGNALS

Proposition: Suppose $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$. Consider $s^k \to 0$ and an associated sequence of equilibria $\left\{B^k, A^k\right\}$. Then,

$$\begin{aligned} \bar{p}_L &= \rho_L c_L + \rho_H c_H = \bar{p}_H \\ \overline{S}_H &= \overline{S}_L = 0 \end{aligned}$$

- No information aggregation: price = EX-ANTE expected cost.
- As we know L searches for a signal below x_*^k. Here, H mimics L and both end up trading after the same low signals and in the limit at the same low price.
- Different from outcome of CV auction with many bidders.

Proof: From observations above

$$\lim_{k\to\infty} \left(u - V_L^k - E_I^k[c|\underline{x}]\right) = 0.$$

Since $V_H^k < V_L^k$, $\lim V_H^k \le \lim V_L^k$. Also,

$$\begin{aligned} V_H^k &\ge E_{(p,x)}\left[u - p|\,(p,x) \in \Omega_L^k, H\right] - \frac{s^k}{\Pi_H\left(\Omega_L^k\right)} \\ &\ge V_L^k - \frac{\Pi_L\left(\Omega_L^k\right)}{\Pi_H\left(\Omega_L^k\right)} \frac{s^k}{\Pi_L\left(\Omega_L^k\right)} \\ &\ge V_L^k - \frac{f_L(\underline{x})}{f_H(\underline{x})} \frac{s^k}{\Pi_L\left(\Omega_L^k\right)} \to \lim V_L^k \end{aligned}$$

where 2nd inequality owes to $u - p \ge V_L^k$ for all $(p,x) \in \Omega_L^k$, and final step owes to previous observation $\lim \frac{s^k}{\Pi_L\left(\Omega_L^k\right)} = 0$. Therefore, $\lim V_H^k \ge \lim V_L^k$ and the previous observation $\lim V_L^k = u - \lim E_I^k[c|\underline{x}]$ implies $\lim V_H^k = u - \lim E_I^k(c|\underline{x})$ as well.

Since in the limit BOTH H and L trade after $\underline{x}$, it must be that $\lim E_I^k(c|\underline{x})$ is equal to the ex-ante cost $\rho_L c_L + \rho_H c_H$.■

Intuitive Explanation

- When signals are boundedly informative and s^k is small, both types search for $(x,p) \in \Omega_L^k$.
- Hence, $\frac{n_L^k}{n_H^k} = \frac{\Pi_L\left(\Omega_L^k\right)}{\Pi_H\left(\Omega_L^k\right)}$.
- $\Pi_w\left(\Omega_L^k\right)$ is roughly $F_w(x^*)$ (times some factor that depends on some average of $g(p)$ for $p \in \left(E^k\left[c|\underline{x}\right], E^k\left[c|x_*\right]\right)$)
- Therefore $\frac{\Pi_L\left(\Omega_L^k\right)}{\Pi_H\left(\Omega_L^k\right)} \approx \frac{F_H\left(x_*^k\right)}{F_L\left(x_*^k\right)} \to \lim_{k\to\infty} \frac{f_H\left(x_*^k\right)}{f_L\left(x_*^k\right)} = \frac{f_H(\underline{x})}{f_L(\underline{x})}$
- This implies

$$\begin{aligned}
\lim_{k\to\infty} E^k\left[c|x_*\right] &= \lim_{k\to\infty} E^k\left[c|\underline{x}\right] \\
&= \frac{c_H + c_L \frac{\rho_L}{\rho_H} \frac{f_L(\underline{x})}{f_H(\underline{x})} \frac{f_H(\underline{x})}{f_L(\underline{x})}}{1 + \frac{\rho_L}{\rho_H} \frac{f_L(\underline{x})}{f_H(\underline{x})} \frac{f_H(\underline{x})}{f_L(\underline{x})}} = \rho_L c_L + \rho_H c_H
\end{aligned}$$

- Hence,

$$\bar{p}_L = \bar{p}_H = \rho_L c_L + \rho_H c_H$$

Remarks.

1. Result holds for any finite number of states.
2. Extension to $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$
3. Based on Lauermann-Wolinsky (2013)

6 Other literature connections

1. We know from traditional demand-supply analysis that replication of the Myerson-Satterthwaite problem essentially removes the inefficiency (Gresik and Satterthwaite [1987] show this in the context of mechanism design framework and inquire about the rate of convergence).

2. The analysis of information aggregation in the CV auction, established that under certain conditions on the signal, a large auction also aggregates the information well.

3. Akerlof's problem is already presented in the context of a large market and the inefficiency is present there (one can show that it would not be removed by other mechanisms as well). So, it is not the market size alone or the private vs common values distinction alone that make the difference. What distinguishes Akerlof from the other scenarios is that in them, there are many "small" agents who jointly have "most" of the relevant information. This is not the case in Akerlof in which only the owner has the relevant information and is therefore not "small" with respect to her own car.

4.Gul and Postlewaite[Econometrica, 1992] look at a hybrid problem and point out that, when the commonality of values is "localized," replication of the problem may diminish their effect, so that such markets may become nearly efficient as the number of participants grow.