Matching Auctions
Supplementary Material

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Abstract

This document establishes that results similar to those in Theorems 1-4 in the main body obtain when match values evolve endogenously, under appropriate conditions. Section S.1 describes the environment and the main result, whereas Section S.2 contains the proof of the main result.

S.1 Endogenous Match Values

We now show how matching auctions similar to those in the main body, but with "forward-looking" scores, can be used in certain markets in which the evolution of the agents' match values is endogenous. For this purpose, consider a setting similar to the one in the main body, but where the kernels governing the evolution of the match values now depend on past allocations. To isolate the novel effects due to the endogeneity of the match values, assume for simplicity that all agents arrive in period 0.

As in the baseline model, consider a representative agent i from side A with the understanding that the same properties apply also to each agent l from side B. For each period and each pair of agents $(i, j) \in N^{AB}$, agent i's period-t horizontal value ε^A_{ijt} for agent j is drawn from the kernel $H^A_{ijt}(\varepsilon^A_{ijt} \mid \varepsilon^A_{ijt-1}, x^{t-1}_{ij})$ where x^{t-1}_{ij} is the history of past interactions between the two agents. These kernels satisfy the following properties:

1. for any $j, j' \in N^B$, $j' \neq j$, given the matches x, the sequence of horizontal types $\varepsilon^A_{ij} \equiv (\varepsilon^A_{ijt})_{t=1,..,\infty}$ is drawn independently from the sequence $\varepsilon^A_{ij'} \equiv (\varepsilon^A_{ij't})_{t=1,..,\infty}$;

2. whenever $x_{ijt-1} = 1$, the dependence of the kernel $H^A_{ijt}(\varepsilon^A_{ijt} \mid \varepsilon^A_{ijt-1}, x^{t-1}_{ij})$ on x^{t-1}_{ij} is only through $\sum_{s=1}^{t-1} x_{ijs}$;

3. whenever, instead, $x_{ijt-1} = 0$, H^A_{ijt} is a Dirac measure at $\varepsilon^A_{ijt} = \varepsilon^A_{ijt-1}$, i.e., $H^A_{ijt}(\varepsilon^A_{ijt} \mid \varepsilon^A_{ijt-1}, x^{t-1}_{ij}) = \mathbf{1}_{\{\varepsilon^A_{ijt} \geq \varepsilon^A_{ijt-1}\}}$;

4. there exists a sequence $(\omega^A_{ijs})_{s=1,\dots,\infty} \in \mathbb{R}^\infty$ drawn from an exogenous distribution, such that, for any number R_{ij} of past interactions between agent $i \in N^A$ and agent $j \in N^B$, ε^A_{ijt} is given by a deterministic function of $(\omega^A_{ijs})_{s=1}^{R_{ij}}$, uniformly over t.[1]

The above assumptions imply that match values are drawn independently across relationships and change only upon interacting with partners. Furthermore, the processes governing the agents' match values are Markov time-homogeneous and their dependence on past matches is only through the number of past interactions. As we show below, when paired with appropriate conditions on the capacity constraints, such assumptions guarantee that the match dynamics under both profit and welfare maximization continue to be implementable by matching auctions similar to those in the previous sections but with scores taking the form of indexes (more below).

In addition to the above changes, assume the period-t costs $c_{ijt}(x^{t-1}_{ij})$ the platform incurs for matching the pair (i, j) also depend on the history of past matches x^{t-1}_{ij} through the number of past interactions $\sum_{s=1}^{t-1} x_{ijs}$.[2]

The following scenario is consistent with the aforementioned assumptions.

Example S.1 (**Gaussian Learning**). Each agent $i \in N^A$ derives a constant utility $v^A_{ijt} = \theta^A_{ij} u^A_{ij}$ for interacting with each agent $j \in N^B$. Such utility is unknown to the platform and to all agents. Agent i starts with a prior belief that $u^A_{ij} \sim N(\varepsilon^A_{ij1}, \tau^A_{ij})$, where the variance τ^A_{ij} is common knowledge but where the initial prior mean ε^A_{ij1} is the agent's private information. The agent's prior mean ε^A_{ij1} is drawn from a distribution H^A_{ij1}. Each time agent i is matched to agent j, agent i receives a conditionally i.i.d. private signal $\xi^A_{ij} \sim N(u^A_{ij}, \vartheta^A_{ij})$ about his idiosyncratic appreciation for agent j, u^A_{ij}, and updates his expectation of u^A_{ij} using Bayes rule.[3] In this setting, the vector of period-t horizontal types $\varepsilon^A_{it} \equiv (\varepsilon^A_{ijt})_{j \in N^{-k}}$ thus corresponds to the collection of the agent's posterior means about the attractiveness of potential partners from the opposite side.

Another scenario consistent with the aforementioned assumptions is the following one:

Example S.2 (**Preference for Variety or Habit Formation**). The value each agent $i \in N^A$ derives from interacting with each agent $j \in N^B$ decreases (alternatively, increases) with the number of past interactions with agent j. Precisely, for all $t \geq 1$, all $(\varepsilon^A_{ijt-1}, \varepsilon^A_{ijt})$, $H^k_{ijt}(\varepsilon^A_{ijt} \mid \varepsilon^A_{ijt-1}, x^{t-1}_{ij})$ is non-decreasing (alternatively, non-increasing) in $\sum_{s=1}^{t-1} x_{ijs}$. For example, in the case of preference for variety, agents gradually lose interest in partners they already interacted with, whereas in the case of habit formation, the value each agent assigns to each partner increases with the number of past interactions.[4]

[1] In addition to assumptions (1)-(4), as in the baseline model, we continue to assume that, for any $i \in N^A$, there exists a constant $E^A_i > 0$ such that, for any sequence of matches x, $\mathbb{E}\left[\sum_{t=1}^{\infty} \delta^t \sum_{j \in N^B} |\varepsilon^A_{ijt}| \cdot x_{ijt}\right] \leq E^A_i$, where the expectation is taken with respect the distribution over $\mathcal{E}$ generated by the kernels G, under the matches x.

[2] A special case of interest is when the costs vanish after the first interaction, which corresponds to a situation where the only relevant costs are those that the platform incurs to "link" the agents.

[3] Specifically, each signal ξ^A_{ij} can be written as $\xi^A_{ij} = u^A_{ij} + \varsigma^A_{ij}$, with the innovations ς^A_{ij} drawn from a Normal distribution with mean 0 and variance ϑ^A_{ij}, independently from all other random variables.

[4] A special form of the preference for variety is when each agent demands a fixed number of interactions with each

Now consider matching auctions similar to those in the main body but where, in each period $t \geq 1$, each agent $l \in N^k$, from each side $k = A, B$, in addition to submitting bids for each possible partner from the opposite side, is offered the possibility to revise his membership status by selecting $\theta_{lt}^k \in \Theta_l^k$. The reason why the platform allows the agents to revise their status over time, despite the fact that the vertical types are perfectly persistent, is that this favors equilibria in which the agents bid myopically over time, irrespective of past bids and membership selections. In fact, while the agents' bids convey information about the agents' total match values v_{ijt}^k, with endogenous processes, they are not sufficient statistics with respect to the agents' overall private information, when it comes to predicting future values. For instance, a high bid b_{ijt}^k by agent i for agent j may either reflect a persistent high value for interacting with all agents from the opposite side (i.e., a high vertical type θ_i^k), or a high temporary appreciation for interacting with agent j (i.e., a high horizontal type, ε_{ijt}^k). When the evolution of the agents' match values is endogenous, being able to collect all the information that is necessary to predict the evolution of the agents' match values is key to a proper selection of the matches under profit- and welfare-maximizing auctions, as we show next.

Let $\lambda_{ij}|\theta_t, b_t, x^{t-1}$ denote the stochastic process over pair (i,j)'s current and future match values (v_{ijs}^A, v_{ijs}^B) that one obtains under any matching rule that matches the pair (i,j) at all periods $s \geq t$, when the pair's vertical types (θ_i^A, θ_j^B) are the ones revealed by the period-t membership choices θ_t, and when the horizontal types are given by

$$\varepsilon_{ijt}^k = \begin{cases} b_{ijt}^k/\theta_{it}^k & \text{if } b_{ijt}^k/\theta_{it}^k \in \mathcal{E}_{ijt}^k \\ \arg\min\limits_{\hat{\varepsilon}_{ijt}^k \in \mathcal{E}_{ijt}^k} \left\{ |b_{ijt}^k/\theta_{it}^k - \hat{\varepsilon}_{ijt}^k| \right\} & \text{otherwise,} \end{cases} \tag{S.1}$$

for $k = A, B$.

Definition S.1 (Index Scores). An index scoring rule $S^{I;\beta}$ (with weights β) is one in which, for each $t \geq 1$, each pair $(i,j) \in N^{AB}$, each (θ_t, b_t, x^{t-1}), the period-t score for the (i,j)-match is given by

$$S_{ijt}^{I;\beta} \equiv \max_{\tau} \left\{ \frac{\mathbb{E}^{\lambda_{ij}|\theta_t, b_t, x^{t-1}} \left[\sum_{s=t}^{\tau} \delta^{s-t} \left(\beta_i^A(\theta_{i0}^A) v_{ijs}^A + \beta_j^B(\theta_{j0}^B) v_{ijs}^B - c_{ijs} \right) \right]}{\mathbb{E}^{\lambda_{ij}|\theta_t, b_t, x^{t-1}} \left[\sum_{s=t}^{\tau} \delta^{s-t} \right]} \right\}, \tag{S.2}$$

where τ denotes a stopping time.

The period-t index score $S_{ijt}^{I;\beta}$ for the match between agent i from side A and agent j from side B thus corresponds to a Gittins index for a process for which the "rewards" are given by the flow weighted total surplus of the match (i,j), with the weights given by the functions β of the agents' period-0 membership statuses. Note that each score $S_{ijt}^{I;\beta}$ depends on x^{t-1} only through the total number of past interactions $\sum_{s=1}^{t-1} x_{ijs}$ between the pair (i,j). Also note that, contrary to the myopic scores in the auctions in the previous sections, the indexes $S_{ijt}^{I;\beta}$ depend on the entire history

partner, which amounts to assuming that, for each agent $i \in N^A$, each partner $j \in N^B$, there exists a number $\alpha_{ij}^A \in \mathbb{N}$ such that, whenever $\sum_{s=1}^{t-1} x_{ijs} > \alpha_{ij}^A$, $G_{ijt}^A(\varepsilon_{ijt}^A \mid \varepsilon_{ijt-1}^A, x^{t-1}) = 1$ for all $\varepsilon_{ijt}^A \geq 0$. Note that a market where each agent is able to provide only up to a fixed number of services to each partner is also a special case of the above specification.

of past and current membership choices. However, while the dependence on the period-0 and on the period-t choice is direct, the dependence on other periods' choices is only through the number of past interactions.

Now consider auctions that select in each period the matches for which the sum of the index scores is the highest, subject to individual and aggregate capacity constraints, and where the payments are defined as follows. Let

$$W_t \equiv \mathbb{E}^{\lambda[\chi^{I;\beta}]|\theta_t,b_t,x^{t-1}} \left[\sum_{s=t}^{\infty} \delta^{s-t} \sum_{(i,j)\in N^{AB}} \left(\beta_i^A(\theta_{i0}^A) v_{ijs}^A + \beta_j^B(\theta_{j0}^B) v_{ijs}^B - c_{ijs} \right) \cdot \chi_{ijs} \right] \tag{S.3}$$

denote the continuation weighted surplus, that is, the discounted present value of the weighted sum of match values, net of the platform's costs, when all agents follow straight-forward strategies, i.e., when they bid truthfully their myopic values v_{ijs}^k and select the membership status corresponding to their true vertical types, at all periods $s \geq t$. Here $\lambda[\chi^{I;\beta}]|\theta_t, b_t, x^{t-1}$ denotes the stochastic process over (θ_s, v_s, x^{s-1}), $s \geq t$, when the selected period-t membership statuses are θ_t, the period-t bids are b_t, all agents follow straight-forward strategies from period $s > t$ onward, the true vertical types are the ones corresponding to the selected period-t membership statuses (i.e., $\theta_l^k = \theta_{lt}^k$, $l \in N^k$, $k = A, B$), and the true period-t horizontal types are given by (S.1).

Similarly, let $W_t^{-l,k}$ denote the continuation weighted surplus, as defined in (S.3), in a fictitious market without agent l from side k. Next, let $R_{lt}^k \equiv W_t - W_t^{-l,k}$ denote the *contribution* of agent $l \in N^k$ to the continuation weighted surplus and

$$r_{lt}^k \equiv R_{lt}^k - \delta \mathbb{E}^{\lambda[\chi^{I;\beta}]|\theta_t,b_t,x^{t-1}} \left[R_{lt+1}^k \right] \tag{S.4}$$

the corresponding *flow marginal contribution*. In each period $t \geq 1$, the payment asked to each agent $i \in N^k$ is given by

$$\psi_{it}^k = \sum_{j\in N^{-k}} b_{ijt}^k \cdot \chi_{ijt} - \frac{1}{\beta_i^k(\theta_{i0}^k)} r_{it}^k. \tag{S.5}$$

The above payments are similar to those in Bergemann and Valimaki (2010) but adapted to account for the fact that the platform's objective may differ from profit maximization (see also Kakade et al. (2013) for a similar construction). The period-0 payments, instead, are given by

$$\bar{\psi}_l^k(\theta) = \theta_l^k D_l^k(\theta;\chi) - \int_{\underline{\theta}_l^k}^{\theta_l^k} D_l^k(y, \theta^{-l,k};\chi)dy - \mathbb{E}\left[\sum_{s=t+1}^{\infty} \delta^{s-t} \psi_{ls}^{k,\beta} \mid \theta \right] - Q_l^k \tag{S.6}$$

with $\theta \equiv (\theta_l^k)_{l\in N^k}^{k=A,B}$ denoting the complete profile of vertical types reported in period 0, and

$$D_l^k(\theta;\chi) \equiv \begin{cases} \mathbb{E}^{\lambda[\chi]|,\theta,} \left[\sum_{t=1}^{\infty} \delta^t \sum_{h\in N^B} \varepsilon_{lht}^A \chi_{lht} \right] & \text{if } k = A \\ \mathbb{E}^{\lambda[\chi]|\theta} \left[\sum_{t=1}^{\infty} \delta^t \sum_{h\in N^A} \varepsilon_{hlt}^B \chi_{hlt} \right] & \text{if } k = B \end{cases},$$

now denoting the expected match quality in the auctions where the scores are the ones in (S.2), the entire profile of vertical types is θ_0, and the expectation is computed under the endogenous process $\lambda[\chi^{I;\beta}]|\theta$.[5]

We then have the following result:

Theorem S.1. *Suppose that either $M = 1$, or none of the capacity constraints binds. Then conclusions analogous to those in Theorems 1-4 in the main body apply to the matching auctions where the scores are given by (S.2) and the payments are given by (S.5) and (S.6). Furthermore, the same conclusions as in parts (1) and (2) of Theorem 4 in the main body hold (with $M = 1$, for part 2).*

The proof in the next section follows from steps similar to those establishing Theorems 1-4 in the main body, but adjusted to account for the fact that the maximization of continuation weighted surplus is not separable over time.

That, under the proposed auctions, at each period after the initial one, agents have incentives to remain in the mechanism, select the membership status designed for their true vertical type, and bid truthfully follows from arguments similar to those in Bergemann and Valimaki (2010) and Kakade et al. (2013). The key difficulty is showing that they also have the right incentives to participate in period zero and select the period-0 membership status designed for their true types. As in the baseline model, this is done by showing that, under truthful bidding, the match quality each agent expects when joining the platform is non-decreasing in both the selected membership status, for given true vertical type, and in the agent's true vertical type when the latter coincides with the selected one.

With endogenous processes, the problem of maximizing continuation weighted surplus is a *multi-armed bandit problem*. Such a problem is known to admit an index solution only under relatively stringent conditions (namely, when rewards evolve independently across arms, arms are frozen when not activated, and a single arm, or any number of arms, can be activated in each period). Despite the obvious limitations imposed by the conditions guaranteeing the optimality of index scores, we believe the matching auctions introduced above capture many of the relevant trade-offs that platforms face in the design of their dynamic matching protocols.

A key difficulty when agents learn endogenously their match values by interacting with other agents is that the private value of experimentation need not align across partners (a problem that does not emerge in standard auctions for physical goods). For example, after a few interactions, agent i from side A may have learned his value for agent j from side B, while agent j may face residual uncertainty about his value for agent i. The scores $S^{I;\beta}$ in the proposed auctions are thus different from a simple combination of the Gittins indexes corresponding to the agents' own private values for experimentation. They are constructed to internalize the cost and benefits of cross-subsidization, as perceived by the platform, while also accounting for the costs of the agents' private information.

[5] This is the endogenous distribution over match values that obtains when the scores are those in (S.2), agents follow straight-forward strategies from period $t = 1$ onward, and the true vertical types are those corresponding to the period-0 membership statuses, θ.

One of the appeals of the proposed auctions is that they admit simple equilibria in which agents bid myopically their match values in each period. In other words, the agents do not need to know how to solve complex dynamic-programming problems, or be able to compute the indexes (although, nowadays, there is software that does so). Once the scoring and payments rules are understood, it is in the agents' interest to bid truthfully in all periods, irrespective of the beliefs they may have about other agents' past and current types, and irrespective of their own, as well as other agents', past behavior. As in other market design settings, however, it is important that the market designer educates the bidders by carefully explaining the structure of the scoring and payment rules and why the proposed auctions admit such simple equilibria.

S.2 Proof of Theorem S.1

The proof is in 4 parts. Part 1 establishes that conclusions analogous to those in Theorem 1 in the main body. Part 2 establishes that conclusions analogous to those in Theorem 2 in the main body hold. Part 3 establishes that conclusions analogous to those in Theorem 3 hold. Finally, Part 4 establishes that the same conclusions as in parts (1) and (2) of Theorem 4 hold (with $M = 1$, for part 2).

Part 1. We want to show that the following claims are true*: "Any matching auction in which (a) the scores are given by the indexes* $S^{I;\beta}_{ijt}$*, with arbitrary weights* β*, (b) the period-*t *payments,* $t > 0$*, are the ones in (S.5), and (c) the period-*0 *payments are as in (S.6), with* Q^k_l *large enough, all* $l \in N^k$*,* $k = A, B$*, admits an equilibrium in which all agents follow straight-forward strategies (i.e., join the auctions in period* 0 *and never leave thereafter and in in each period they select the membership status corresponding to their true vertical type and then bid truthfully their myopic values for all matches). Furthermore, such straight-forward equilibria are periodic ex-post (that is, the agents' strategies are sequentially rational, regardless of the agents' beliefs about other agents' past and current types)."*

Proof of Part 1. The proof parallels the one for Theorem 1 in the main text. Step 1 shows that, when all agents follow straight-forward strategies, the matches implemented in equilibrium maximize continuation weighted surplus, starting from any period-t history, any $t \geq 1$. It then uses this property to establish that, when the period-t payments are the ones in (*S.5*), any $t \geq 1$, then remaining in the auctions and following straight-forward strategies constitutes a periodic ex-post continuation equilibrium, after any period-t history, any $t \geq 1$. Step 2 completes the proof by showing that the matches sustained under straight-forward strategies satisfy dynamic monotonicity conditions analogous to those in the main text. It then shows that such monotonicities imply that, when the period-0 payments are those in (S.6), with Q^k_l large enough, all $l \in N^k$, $k = A, B$, any agent who expects all other agents to participate in each period and select the membership status equal to their true vertical type finds it optimal
to do the same, regardless of the agent's beliefs about the other agents' vertical types.

Step 1. We first establish that, when either $M = 1$, or none of the capacity constraints binds, the rule $\chi^{I;\beta}$ that selects in each period the matches for which the index scores $S_{ijt}^{I;\beta}$ are the highest, among those for which the scores are non-negative, maximizes continuation weighted surplus. That is, irrespective of the particular history that led to the selection of the period-0 membership statuses θ_0 and of the past matches x^{t-1}, in the continuation game that starts with period $t \geq 1$, when the true vertical type profile is θ_t, the true profile of horizontal types is ε_t —with ε_t obtained from θ_t and b_t by letting, for each $(i,j) \in N^{AB}$ and $k = A, B$,

$$\varepsilon_{ijt}^k = \begin{cases} b_{ijt}^k/\theta_{it}^k & \text{if } b_{ijt}^k/\theta_{it}^k \in \mathcal{E}_{ijt}^k \\ \arg\min\limits_{\hat{\varepsilon}_{ijt}^k \in \mathcal{E}_{ijt}^k} \left\{ |b_{ijt}^k/\theta_{it}^k - \hat{\varepsilon}_{ijt}^k| \right\} & \text{otherwise} \end{cases} \tag{S.7}$$

—and all agents follow straight-forward strategies from period t onwards, then the matches under $\chi^{I;\beta}$ maximize continuation weighted surplus

$$W_t \equiv \mathbb{E}^{\lambda[\chi]|\theta_t, b_t, x^{t-1}} \left[\sum_{s=t}^{\infty} \delta^{s-t} \sum_{(i,j)\in N^{AB}} \left(\beta_i^A(\theta_{i0}^A) v_{ijs}^A + \beta_j^B(\theta_{j0}^B) v_{ijs}^B - c_{ijs} \right) \cdot \chi_{ijs} \right]$$

over the entire set $\mathcal{X}$ of feasible matching rules χ. To see this, note that the problem of maximizing W_t can be viewed as a multi-armed bandit problem, with each arm corresponding to a potential match, and with the flow period-t reward of activating each arm (i,j) given by the myopic score

$$S_{ijt}^{m;\beta} \equiv \beta_i^A(\theta_{i0}^A) v_{ijt}^A + \beta_j^B(\theta_{j0}^B) v_{ijt}^B - c_{ijt}.$$

When $M = 1$, that $\chi^{I;\beta}$ maximizes W_t at all histories follows from known results (see, e.g., Whittle (1982)). When none of the capacity constraints bind, the platform's problem can be viewed as a collection of $n^A \cdot n^B$ separate two-armed bandit problems, one for each potential pair of agents, with the reward from matching the pair (i,j) given by $S_{ijt}^{m;\beta}$ and the reward from activating the "safe arm" identically equal to zero. It is again well known that the solution to each such problems consists in activating the risky arm if the index $S_{ijt}^{I;\beta} > 0$ and the safe arm if $S_{ijt}^{I;\beta} < 0$.

Now fix the weights β and denote by $\tilde{\chi}$ a matching rule that maximizes the continuation weighted surplus at all histories, and by $\tilde{\chi}^{-l,k}$ a matching rule that does so in the absence of agent l from side $k \in \{A, B\}$ (equivalently, that maximizes continuation weighted surplus when the myopic score of any match that involves agent l from side k is identically equal to zero, in which case $\tilde{\chi}^{-l,k}$ can be assumed to never implement any match involving agent l). From Step 1 above such rules are index rules.

Denote by $\tilde{\psi}_{t\geq 1} \equiv (\tilde{\psi}_s)_{s\geq 1}$ the collection of payment functions defined by (S.5) in the main text, when the matching rule is $\tilde{\chi}$. Henceforth, the weighted surpluses W_t and $W_t^{-l,k}$, as well as the resulting marginal and flow contributions to weighted surplus, as defined in the main text, unless otherwise specified, are with respect to the matching rules $\tilde{\chi}$ and $\tilde{\chi}^{-l,k}$, respectively. Note that, because the weights β are held fixed, to ease the notation we drop them from all functions below, when there is no risk of confusion.

Lemma S.1. Consider an auction in which the matching rule is $\tilde{\chi}$ and the payments from period 1 onwards are as in (S.5). In such an auction, participating and following straight-forward strategies (i.e., in each period, selecting the membership status corresponding to the true vertical type and then bidding truthfully the myopic values for all matches) constitutes a periodic ex-post continuation equilibrium, after any period-t history, $t \geq 1$.

Proof of Lemma S.1. We show that, in the continuation game that starts in period $t \geq 1$, irrespective of the history of past play, of the true vertical type profile θ, and of the history of past and current horizontal types $\varepsilon^t \equiv (\varepsilon_s)_{s=1}^t$, any agent $l \in N^k$, $k = A, B$, who expects all other agents to participate and follow straight-forward strategies from period t (included) onwards, finds it optimal to do the same.

Consider agent l from side A (the problem for any agent from side B is similar). Suppose that the true profile of vertical types is θ, the true profile of period-t match values is v_t, the profile of period-0 membership choices is θ_0, and the history of past matches is x^{t-1}. Denote by

$$\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta,v_t,x^{t-1};(\hat{\theta}^A_{lt},b^A_{lt})}\left[R^A_{lt+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right]$$

the expected contribution of agent l to continuation weighted surplus from period $t+1$ onwards, when, in period t, the agent selects the period-t membership status $\hat{\theta}^A_{lt}$, submits the period-t bids b^A_{lt}, follows a straight-forward strategy from period $t+1$ onwards, and expects all other agents to follow straight-forward strategies at all periods $s \geq t$.[6] Note that, when the agent follows the straight-forward strategy also in period t (i.e., when $\hat{\theta}^A_{lt} = \theta^A_l$ and $b^k_{lt} = v^A_{lt}$), then

$$\tilde{\lambda}[\tilde{\chi}]|\theta,v_t,x^{t-1};(\theta^A_l,v^A_{lt}) = \lambda[\tilde{\chi}]|\theta,v_t,x^{t-1},$$

where the process $\lambda[\tilde{\chi}]|\theta,v_t,x^{t-1}$ is as defined in the main text.

Since the agent can revise his membership status in any of the subsequent periods, any deviation from the straight-forward strategy in period t can be corrected in period $t+1$. This means that, to prove the result, it suffices to show that the agent prefers to follow the straight-forward strategy from period t onwards than deviating in period t and then reverting to the straight-forward strategy from period $t+1$ onwards.

Under the proposed auction rules, when the agent follows the straight-forward strategy from period $t+1$ onwards, his continuation payoff from period $t+1$ onwards is given by

$$\frac{1}{\beta^A_l(\theta^A_{l0})}R^A_{lt+1}(\theta_0,\theta_{t+1},b_{t+1},x^t).$$

[6] The stochastic process $\tilde{\lambda}[\tilde{\chi}]|\theta,v_t,x^{t-1};(\hat{\theta}^A_{lt},b^A_{lt})$ is here over future matches, bids and membership choices, under the rule $\tilde{\chi}$, when agent l's period-t choices are $(\hat{\theta}^A_{lt},b^A_{lt})$, the true profile of vertical types is θ, the true profile of period-t horizontal types is ε_t, with ε_t obtained from θ and v_t using (S.7), the history of past matches is x^{t-1}, the agent plans to follow a straight-forward strategy from $t+1$ onwards, and all other agents follow straight-forward strategies from period t onwards.

Therefore, it is enough to show that, for any period-t selection $(\hat{\theta}_{lt}^A, b_{lt}^A)$,

$$\begin{aligned}
&\sum_{j\in N^B} v_{ljt}^A \tilde{\chi}_{ljt}\left(\theta_0, (\theta_l^A, \theta^{-l,A}), (v_{lt}^A, v_t^{-l,A}), x^{t-1}\right) - \tilde{\psi}_{lt}^A\left(\theta_0, (\theta_l^A, \theta^{-l,A}), (v_{lt}^A, v_t^{-l,A}), x^{t-1}\right) \\
&\quad + \frac{\delta}{\beta_l^A(\theta_{l0}^A)} \mathbb{E}^{\lambda[\tilde{\chi}]|\theta_l^A, \theta^{-l,A}, v_{lt}^A, v_t^{-l,A}, x^{t-1}} \left[R_{lt+1}^A(\theta_0, \theta_{t+1}, b_{t+1}, x^t)\right] \\
&\quad \geq \sum_{j\in N^B} v_{ljt}^A \tilde{\chi}_{ljt}\left(\theta_0, (\hat{\theta}_{lt}^A, \theta^{-l,A}), (b_{lt}^A, v_t^{-l,A}), x^{t-1}\right) - \tilde{\psi}_{lt}^A\left(\theta_0, (\hat{\theta}_{lt}^A, \theta^{-l,A}), (b_{lt}^A, v_t^{-l,A}), x^{t-1}\right) \\
&\qquad + \frac{\delta}{\beta_l^A(\theta_{l0}^A)} \mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta_l^A, \theta^{-l,A}, v_{lt}^A, v_t^{-l,A}, x^{t-1}; (\hat{\theta}_{lt}^A, b_{lt}^A)} \left[R_{lt+1}^A(\theta_0, \theta_{t+1}, b_{t+1}, x^t)\right]. \qquad \text{(S.8)}
\end{aligned}$$

The left hand side of the above inequality can be rewritten in terms of the functions W_t and $W_t^{-l,A}$ as follows:

$$\frac{1}{\beta_l^A(\theta_{l0}^A)} \left[W_t\left(\theta_0, (\theta_l^A, \theta^{-l,A}), (v_{lt}^A, v_t^{-l,A}), x^{t-1}\right) - W_t^{-l,A}\left(\theta_0, (\theta_l^A, \theta^{-l,A}), (v_{lt}^A, v_t^{-l,A}), x^{t-1}\right)\right]. \qquad \text{(S.9)}$$

That is, agent l's expected continuation payoff when he follows the straight-forward strategy from period t onward is equal to his expected contribution to the maximal continuation weighted surplus, scaled by the weight $\beta_l^A(\theta_{l0}^A)$. It then suffices to show that (S.9) is weakly greater than the right hand side of (S.8).

Next, note that the flow contribution r_{lt}^k can be rewritten as

$$\begin{aligned}
r_{lt}^k(\theta_0, \theta_t, b_t, x^{t-1}) = &\sum_{i\in N^A} \sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0, \theta_t, b_t, x^{t-1}\right) \cdot \tilde{\chi}_{ijt}(\theta_0, \theta_t, b_t, x^{t-1}) \\
&- \sum_{i\in N^A} \sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0, \theta_t, b_t, x^{t-1}\right) \tilde{\chi}_{ijt}^{-l,k}(\theta_0, \theta_t, b_t, x^{t-1}) \\
&+ \delta \mathbb{E}^{\lambda[\tilde{\chi}]|\theta_t, b_t, x^{t-1}} \left[W_{t+1}^{-l,k}(\theta_0, \theta_{t+1}, b_{t+1}, x^t)\right] \\
&- \delta \mathbb{E}^{\lambda[\tilde{\chi}^{-l,k}]|\theta_t, b_t, x^{t-1}} \left[W_{t+1}^{-l,k}(\theta_0, \theta_{t+1}, b_{t+1}, x^t)\right]. \qquad \text{(S.10)}
\end{aligned}$$

Using (S.10), we can rewrite the period-t payment in the right-hand side of (S.8) as follows:

$$
\begin{aligned}
&\tilde{\psi}^A_{lt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&= \sum_{j\in N^B} b^A_{ljt}\tilde{\chi}_{ljt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (\hat{b}^A_{lt}, v_t^{-l,A}), x^{t-1}\right) - \frac{1}{\beta^A_l(\theta^A_{l0})} r^A_{lt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (\hat{b}^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&\quad = \sum_{j\in N^B} b^A_{ljt}\tilde{\chi}_{ljt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&\quad - \frac{1}{\beta^A_l(\theta^A_{l0})} \sum_{i\in N^A}\sum_{j\in N^B} S^{m;\beta}_{ijt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right)\cdot\tilde{\chi}_{ijt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&\quad - \frac{\delta}{\beta^A_l(\theta^A_{l0})}\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\hat{\theta}^A_{lt},\theta^A_{-l},b^A_{lt},v_t^{-l,A},x^{t-1}}\left[W^{-l,A}_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\
&\quad + \frac{1}{\beta^A_l(\theta^A_{l0})} W^{-l,A}_t\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&\quad = -\frac{1}{\beta^A_l(\theta^A_{l0})} \sum_{i\in N^A\setminus\{l\}}\sum_{j\in N^B} S^{m;\beta}_{ijt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right)\cdot\tilde{\chi}_{ijt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&\quad - \frac{1}{\beta^A_l(\theta^A_{l0})} \sum_{j\in N^B}\left(\beta^B_j(\theta^B_{j0})v^B_{ljt} - c_{ljt}(x^{t-1})\right)\cdot\tilde{\chi}_{ljt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&\quad - \frac{\delta}{\beta^A_l(\theta^A_{l0})}\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\hat{\theta}^A_{lt},\theta^{-l,A},b^A_{lt},v_t^{-l,A},x^{t-1}}\left[W^{-l,A}_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\
&\quad + \frac{1}{\beta^A_l(\theta^A_{l0})} W^{-l,A}_t\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right).
\end{aligned}
$$

Furthermore, note that

$$
\begin{aligned}
&\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta^A_l,\theta^{-l,A},v^A_{lt},v_t^{-l,A},x^{t-1};(\hat{\theta}^A_{lt},b^A_{lt})}\left[R^A_{lt+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\
&\quad = \mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta^A_l,\theta^{-l,A},v^A_{lt},v_t^{-l,A},x^{t-1};(\hat{\theta}^A_{lt},b^A_{lt})}\left[W_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t) - W^{-l,A}_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\
&\quad = \mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta^A_l,\theta^{-l,A},v^A_{lt},v_t^{-l,A},x^{t-1};(\hat{\theta}^A_{lt},b^A_{lt})}\left[W_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\
&\quad\quad - \mathbb{E}^{\lambda[\tilde{\chi}]|\hat{\theta}^A_{lt},\theta^{-l,A},b^A_{lt},v_t^{-l,A},x^{t-1}}\left[W^{-l,A}_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right],
\end{aligned}
$$

where the last equality uses the fact that, given x^t, $W^{-l,A}_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)$ is invariant to agent l's period-t bids and that the period-t decisions x_t are invariant to the agent's *true* types. Therefore, the right hand side of the inequality (S.8) is equal to

$$
\begin{aligned}
&\frac{1}{\beta^A_l(\theta^A_{l0})}\sum_{i\in N^A}\sum_{j\in N^B} S^{m;\beta}_{ijt}\left(\theta_0,\theta,v_t,x^{t-1}\right)\cdot\tilde{\chi}_{ijt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (\hat{b}^A_{lt}, v_t^{-l,A}), x^{t-1}\right) \\
&\quad + \frac{\delta}{\beta^A_l(\theta^A_{l0})}\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta^A_l,\theta^{-l,A},v^A_{lt},v_t^{-l,A},x^{t-1};(\hat{\theta}^A_{lt},b^A_{lt})}\left[W_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\
&\quad - \frac{1}{\beta^A_l(\theta^A_{l0})} W^{-l,A}_t\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v_t^{-l,A}), x^{t-1}\right).
\end{aligned}
$$

Since, again, $W_t^{-l,A}$ is invariant to agent l's period-t bids, to establish that the inequality in (S.8) holds, it suffices to show that

$$W_t(\theta_0, \theta, v_t, x^{t-1}) \geq \sum_{i \in N^A} \sum_{j \in N^B} S_{ijt}^{m;\beta}\left(\theta_0, \theta, v_t, x^{t-1}\right) \cdot \tilde{\chi}_{ijt}\left(\theta_0, (\hat{\theta}_{lt}^A, \theta^{-l,A}), (\hat{b}_{lt}^A, v_t^{-l,A}), x^{t-1}\right) \quad \text{(S.11)}$$
$$+ \mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta_l^A, \theta^{-l,A}, v_{lt}^A, v_t^{-l,A}, x^{t-1}; (\hat{\theta}_{lt}^A, \hat{b}_{lt}^A)}\left[W_{t+1}(\theta_0, \theta_{t+1}, b_{t+1}, x^t)\right].$$

The inequality in (S.11) follows from the fact that the matching rule $\tilde{\chi}$ maximizes continuation weighted surplus.

That it is (periodic ex-post) optimal for each agent to participate at all periods $t \geq 1$, and after all histories, follows from the fact that each agent's continuation payoff under straight-forward strategies coincides with his expected contribution to continuation weighted surplus, which is always nonnegative, scaled by a strictly positive weight.

The arguments above therefore establish that, when the matching rule is $\tilde{\chi}$ and the payments from period 1 onwards are $\tilde{\psi}_{\geq 1}$, participating and following straight-forward strategies constitutes a periodic ex-post continuation equilibrium, after any period-t history, $t \geq 1$. This completes the proof of the lemma. ■

Step 2. We now show that, when the period-0 payments are as in (S.6), participating in period zero and selecting a membership status equal to the true vertical type is optimal for any individual who expects all other agents to do the same, irrespective of the individual's beliefs about the other agents' vertical types.

Let

$$\tilde{D}_l^k(\theta_{l0}^k, \theta; \chi) \equiv \begin{cases} \mathbb{E}^{\lambda[\chi]|\theta_{l0}^k, \theta,}\left[\sum_{t=1}^{\infty} \delta^t \sum_{h \in N^B} \varepsilon_{lht}^A \chi_{lht}\right] & \text{if } k = A \\ \mathbb{E}^{\lambda[\chi]|\theta_{l0}^k, \theta}\left[\sum_{t=1}^{\infty} \delta^t \sum_{h \in N^A} \varepsilon_{hlt}^B \chi_{hlt}\right] & \text{if } k = B \end{cases},$$

denote the match quality that agent $l \in N^k$ from side $k = A, B$ expects under the rule χ when the true profile of vertical types is $\theta \in \Theta$, the agent selects the membership status θ_{l0}^k in period zero and then conforms to the straight-forward strategy from period $t = 1$ onwards, and all agents other than l (from side k) follow straight-forward strategies at each period. Note that $\lambda[\chi]|\theta$ denotes the stochastic process over matches, bids, and membership choices, when the true vertical types are θ, all agents other than agent l from side A follow straight-forward strategies from period $t = 0$ onwards and agent l from side k chooses θ_{l0}^k in period 0 and then follows a straight-forward strategy from $t = 1$ onwards. Also note that $\tilde{D}_l^k(\theta_l^k, (\theta_l^k, \theta^{-l,k}); \chi) = D_l^k(\theta; \chi)$ — hereafter we highlight the dependence of the function $D_l^k(\theta; \chi)$ on the matching rule χ to avoid possible confusion.

For any agent $i \in N^A$ (the arguments for the side-B agents are analogous), let

$$\hat{U}_i^A(\theta) \equiv \mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=1}^{\infty} \delta^t \sum_{j \in N^B} \theta_i^A \varepsilon_{ijt}^A \chi_{ijt}(\theta, \theta_t, b_t, x^{t-1})\right] - \mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=0}^{\infty} \delta^t \psi_{it}^A(\theta, \theta_t, b_t, x^{t-1})\right]$$

denote the payoff that the agent expects in the matching auctions defined by the rules (χ, ψ) when the true vertical type profile is θ and all agents follow straight-forward strategies at all periods.[7]

Let $\tilde{\chi}$ be any matching rule that maximizes continuation weighted surplus after all histories, and $\tilde{\psi} = (\tilde{\psi}_0, \tilde{\psi}_{\geq 1})$ the associated payment rule, as defined by (S.6) and (S.5) in the main text. Note that, for each agent $l \in N^k$, each profile θ of true vertical types,

$$\mathbb{E}^{\lambda[\tilde{\chi}]|\theta}\left[\tilde{\psi}_{l0}^k(\theta) + \sum_{t=1}^{\infty} \delta^t \tilde{\psi}_{lt}^k(\theta, \theta_t, b_t, x^{t-1})\right] = \theta_l^k D_l^k(\theta; \tilde{\chi}) - \int_{\underline{\theta}_l^k}^{\theta_l^k} D_l^k((y, \theta^{-l,k}); \tilde{\chi})dy - Q_l^k,$$

which guarantees that, when all agents follow straight-forward strategies in each period, including period zero, the period-zero expected payoffs are given by

$$\hat{U}_l^k(\theta) = \int_{\underline{\theta}_l^k}^{\theta_l^k} D_l^k((y, \theta^{-l,k}); \tilde{\chi})dy + Q_l^k. \tag{S.12}$$

The next lemma shows that any matching rule $\tilde{\chi}$ that maximizes continuation weighted surplus at all histories satisfies certain monotonicity conditions which play a central role in establishing the optimality of straight-forward strategies at period zero.

Lemma S.2. Suppose $\tilde{\chi}$ maximizes continuation weighted surplus W_t at all histories. For all $l \in N^k$, $k = A, B$, the following monotonicities hold: (i) $\tilde{D}_l^k\left(\theta_{l0}^k, \theta; \tilde{\chi}\right)$ is non-decreasing in θ_{l0}^k, all $\theta \in \Theta$; (ii) $D_l^k\left((\theta_l^k, \theta^{-l,k}); \tilde{\chi}\right)$ is non-decreasing in θ_l^k, all $\theta^{-l,k} \in \Theta^{-l,k}$.

Proof of Lemma S.2. Consider an arbitrary agent $i \in N^A$ from side A (the arguments for the side-B agents are analogous) and fix the profile of types $\theta^{-i,A}$ for the other agents. We prove claim (ii) first.

Claim (ii). Take any pair of types $\theta_i^A, \hat{\theta}_i^A \in \Theta_i^A$, with $\theta_i^A < \hat{\theta}_i^A$. That $\tilde{\chi}$ maximizes continuation

[7] The dependence of the payoff on the mechanism (χ, ψ) is omitted for convenience.

weighted surplus implies that[8]

$$\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|\hat{\theta}_i^A,\theta^{-i,A}\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|\hat{\theta}_i^A,\theta^{-i,A}\Bigg]$$
$$\geq\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg].$$

The left-hand side of the above inequality is the expected weighted surplus — under the weights $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$ — when all agents follow straight-forward strategies from period $t=0$ onward and the true profile of vertical types is $(\hat{\theta}_i^A,\theta^{-i,A})$. The right-hand side of the inequality is the expected weighted surplus — under the same weights $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$ — when, at the same profile of true vertical types $(\hat{\theta}_i^A,\theta^{-i,A})$, all agents other than agent i from side A follow straight-forward strategies in all periods whereas agent i from side A perfectly replicates the behavior of type θ_i^A in each period (that is, he selects the membership status θ_i^A at $t=0$ and then, at each subsequent period, given the true horizontal types $\varepsilon_{it}^A\equiv(\varepsilon_{ijt}^A)_{j=1,\dots,n^B}$, submits bids equal to $b_{ijt}^A=\theta_i^A\varepsilon_{ijt}^A$, all $j\in N^B$). The expectations are with respect to the horizontal types given the vertical types. The inequality follows from the fact that, holding the weights $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$ fixed in the computation of the flow surpluses, at each period, given the bids $\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right)$, the matches $\tilde{\chi}_t\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right)\right)$ implemented when the membership status selected by agent i leads to the same weight $\beta_i^A(\hat{\theta}_i^A)$ used to compute the flow surpluses maximize the continuation weighted surplus. Note that, when type $\hat{\theta}_i^A$ replicates the same behavior of type θ_i^A in each period, the expectation over the horizontal types when the true profile of vertical types is $(\hat{\theta}_i^A,\theta^{-i,A})$ is the same as when the true profile of vertical types is $(\theta_i^A,\theta^{-i,A})$. This follows from the independence of the horizontal types from the vertical ones. The reason why, in the right-

[8]To ease the notation, we drop from the inequality below the various measures under which the expectations are taken. The expectations are over future bids, the selection of future membership statuses, and matches, under the processes induced by the agents playing according to the strategies described in the text after the inequality.

hand side of the above inequality we condition on $(\theta_i^A, \theta^{-i,A})$, despite the true state being $(\hat{\theta}_i^A, \theta^{-i,A})$, is that this facilitates the comparison with the inequality we establish below by inverting the roles of θ_i^A and $\hat{\theta}_i^A$.

Importantly, the above inequality is not to be confused with agent i's incentive-compatibility constraints. As explained in the main text, the monotonicity of match quality in the lemma does not follow from standard arguments in screening models which are based on the combination of incentive compatibility with the supermodularity of the agents' payoffs.

Next, inverting the role of $\hat{\theta}_i^A$ and θ_i^A, we have that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A + \beta_j^B(\theta_j^B)b_{rjt}^B - c_{rjt}(x^{t-1})\right)\\
&\qquad\qquad \cdot\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B}, b_t^{-i,A}\right), x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]\\
&+\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right) + \beta_j^B(\theta_j^B)b_{ijt}^B - c_{ijt}(x^{t-1})\right)\\
&\qquad\qquad \cdot\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B}, b_t^{-i,A}\right), x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]\\
&\geq\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A + \beta_j^B(\theta_j^B)b_{rjt}^B - c_{rjt}(x^{t-1})\right)\\
&\qquad\qquad \cdot\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B}, b_t^{-i,A}\right), x^{t-1}\right)|(\hat{\theta}_i^A,\theta_{-i}^A)\Bigg]\\
&+\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right) + \beta_j^B(\theta_j^B)b_{ijt}^B - c_{ijt}(x^{t-1})\right)\\
&\qquad\qquad \cdot\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta_{-i}^A),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B}, b_t^{-i,A}\right), x^{t-1}\right)|(\hat{\theta}_i^A,\theta_{-i}^A)\Bigg].
\end{aligned}
$$

Combining the last two inequalities, we have that

$$\left(\beta_i^A(\hat{\theta}_i^A)\hat{\theta}_i^A - \beta_i^A(\theta_i^A)\theta_i^A\right)\cdot\left(D_i^A((\theta_{-i}^A,\hat{\theta}_i^A);\tilde{\chi}) - D_i^A((\theta_{-i}^A,\theta_i^A);\tilde{\chi})\right)\geq 0.$$

Because $\beta_i^A(\cdot)$ is strictly positive and non-decreasing, it must be that

$$D_i^A((\theta_{-i}^A,\hat{\theta}_i^A);\tilde{\chi})\geq D_i^A((\theta_{-i}^A,\theta_i^A);\tilde{\chi}).$$

Claim (i). Again, because $\tilde{\chi}$ maximizes continuation weighted surplus after any history, we have that for any $\theta \in \Theta$, any $\hat{\theta}_i^A, \theta_i^A \in \Theta_i^A$,[9]

$$\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$\geq\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta_{-i}^A),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg].$$

The left-hand side of the above inequality is the expected weighted surplus when the true vertical types are $(\theta_i^A,\theta^{-i,A})$, the weights used to compute the flow surpluses are given by $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$, all agents other than agent i from side A follow straight-forward strategies, and agent i selects the membership status $\hat{\theta}_i^A$ and then bids truthfully at all periods. The right-hand side of the above inequality is the expected weighted surplus under the same true vertical types $(\theta_i^A,\theta^{-i,A})$, when the weights used to compute the flow surpluses continue to be given by $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$, and all agents, including agent i from side A, follow straight-forward strategies at all periods. Once again, the inequality follows from the fact that, when the weights used to evaluate the flow surpluses are $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$, expected weighted surplus is higher when the membership status selected by agent i in period zero leads to same weight $\beta_i^A(\hat{\theta}_i^A)$ used to evaluate the flow surpluses.

[9]Again, the various expectations are under the processes induced by the agents playing according to the strategies described in the text after the inequality.

Inverting the roles of $\hat{\theta}_i^A$ and θ_i^A, we also have that

$$\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$\geq\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta_{-i}^A),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg].$$

Combining the above two inequalities, we have that, for any $\theta\in\Theta$, any $\hat{\theta}_{i0}^A,\theta_{i0}^A\in\Theta_i^A$,

$$\left(\beta_i^A(\hat{\theta}_{i0}^A)-\beta_i^A(\theta_{i0}^A)\right)\cdot\theta_i^A\cdot\left(\tilde{D}_i^A(\hat{\theta}_{i0}^A,\theta;\tilde{\chi})-\tilde{D}_i^A(\theta_{i0}^A,\theta;\tilde{\chi})\right)\geq 0.$$

Because $\theta_i^A>0$ and $\beta_i^A(\cdot)$ is strictly positive and non-decreasing, it must be that $\tilde{D}_i^A(\cdot,\theta;\tilde{\chi})$ is non-decreasing in θ_{i0}^A.[10] This completes the proof of the lemma. ■

Next, for any agent $i\in N^A$ (the arguments for the side-B agents are analogous), let

$$\tilde{U}_i^A(\theta_{i0}^A;\theta)\equiv\mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\theta_i^A\varepsilon_{ijt}^A\chi_{ijt}((\theta_{i0}^A,\theta^{-i,A}),\theta_t,b_t,x^{t-1})\right]$$
$$-\,\mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=0}^{\infty}\delta^t\psi_{it}^A((\theta_{i0}^A,\theta^{-i,A}),\theta_t,b_t,x^{t-1})\right]$$

denote the agent's expected payoff, when the true vertical type profile is θ, the agent chooses the membership status θ_{i0}^A in period zero, he follows a straight-forward strategy from period $t=1$ onwards, and all other agents follow straight-forward strategies from period $t=0$ onwards.[11] Note that $\tilde{U}_i^A(\theta_i^A;(\theta_i^A,\theta^{-i,A}))=\hat{U}_i^A(\theta_i^A,\theta^{-i,A})$, where $\hat{U}$ is as in (S.12).

From Step 1, participating and following straight-forward strategies is a periodic ex-post continu-

[10] Note that, because the only influence of period-0 membership statuses on match quality is through their impact on the weights β, if $\beta_i^A(\hat{\theta}_{i0}^A)=\beta_i^A(\theta_{i0}^A)$, then $\tilde{D}_i^A(\hat{\theta}_{i0}^A,\theta;\tilde{\chi})=\tilde{D}_i^A(\theta_{i0}^A,\theta;\tilde{\chi})$.

[11] Once again, the dependence of the payoff on (χ,ψ) is omitted for convenience.

ation equilibrium starting from any period-1 history (including those reached off path, by deviations in period zero). Standard arguments can then be used to show that the following envelope condition must be satisfied for all $l \in N^k$, $k = A, B$, all $\theta^k_{l0} \in \Theta^k_l$, all $\theta \in \Theta$,[12]

$$\tilde{U}^k_l(\theta^k_{l0};\theta) = \tilde{U}^k_l(\theta^k_{l0};(\theta^k_{l0},\theta^{-l,k})) + \int_{\theta^k_{l0}}^{\theta^k_l} \tilde{D}^k_l(\theta^k_{l0},(y,\theta^{-l,k});\tilde{\chi})dy. \tag{S.13}$$

The payoff that agent $l \in N^k$ from side $k = A, B$ obtains by selecting the membership status θ^k_{l0} when the true profile of vertical types is θ is thus given by

$$\begin{aligned}
&\tilde{U}^k_l(\theta^k_{l0};\theta) \\
&= \tilde{U}^k_l(\theta^k_{l0};(\theta^k_{l0},\theta^{-l,k})) + \int_{\theta^k_{l0}}^{\theta^k_l} \tilde{D}^k_l(\theta^k_{l0},(y,\theta^{-l,k});\tilde{\chi})dy \\
&\le \tilde{U}^k_l(\theta^k_{l0};(\theta^k_{l0},\theta^{-l,k})) + \int_{\theta^k_{l0}}^{\theta^k_l} \tilde{D}^k_l(y,(y,\theta^{-l,k});\tilde{\chi})dy = \hat{U}^k_l(\theta^k_{l0},\theta^{-l,k}) + \int_{\theta^k_{l0}}^{\theta^k_l} D^k_l((y,\theta^{-l,k});\tilde{\chi})dy \\
&= \hat{U}^k_l(\theta),
\end{aligned}$$

where the first equality follows from (S.13), the inequality follows from part (i) in Lemma S.2, and the other equalities follow from (S.12) and the definition of the interim expected payoffs. Hence, given θ, the agent is better off following a straight-forward strategy from period zero onwards than deviating in period zero and then following a straight-forward strategy from period one onwards.

Finally, since for any $l \in N^k$, $k = A, B$, any $\theta \in \Theta$, D^k_i is uniformly bounded by E^k_i, participation constraints are satisfied when $Q^k_i \ge (\bar{\theta}^k_i - \underline{\theta}^k_i)E^k_i$.

Combining the results in Step 2 with those in Step 1, we thus have that, when Q^k_l is large enough, all $l \in N^k$, $k = A, B$, participating in the auctions and following a straight-forward strategy is a periodic ex-post equilibrium in the entire game. This completes the proof of Part 1. ■

Part 2. We want to show that the following is true: "*When the weights are those in Theorem 2 and each agent's expected match quality under straight-forward strategies is such that* $D^k_l((\underline{\theta}^k_l,\theta^{-l,k});\beta^P) \ge 0$*, all* $l \in N^k$*,* $k = A, B$*, and* $\theta^{-l,k} \in \Theta^{-l,k}$*, then the matching auctions in which (a) the scores are the indexes* $S^{I;\beta^P}_{ijt}$ *and (b) the payments are those in (S.6) and (S.5), with weights* β^P *and with* $Q^k_l = 0$*, all* $l \in N^k$*,* $k = A, B$*, are profit-maximizing*"

Proof of Part 2. The arguments parallel those establishing Theorem 2 in the main text. Consider any feasible mechanism Γ and any BNE σ of the game induced by Γ. Denote by $\hat{\chi} = (\hat{\chi}_t(\theta,\varepsilon^t))^{\infty}_{t=1}$ and $\hat{\psi} = (\hat{\psi}_t(\theta,\varepsilon^t))^{\infty}_{t=0}$ the matching and payment rules, as a function of the true state, induced by

[12] Condition (S.13) follows from the fact that, in a fictitious problem where agent l's period-0 membership status is exogenously fixed at θ^k_{l0}, and else the agent is free to choose any strategy he wants from $t = 1$ onwards, the payoff $\tilde{U}^k_l(\theta^k_{l0};\theta)$ the agent expects by following a straight-forward strategy from $t = 1$ onwards coincides with the value function for the aforementioned fictitious problem (this follows directly from the fact that a straight-forward strategy maximizes the agent's payoff starting from any history). Condition (S.13) then follows from the above observation along with the fact that the value function of the aforementioned fictitious problem is Lipschitz continuous in the agent's true vertical type θ^k_l, with derivative equal to $\tilde{D}^k_l(\theta^k_{l0},(\theta^k_l,\theta^{-l,k}))$.

σ in Γ.

The platform's profits under (Γ, σ) are equal to

$$\mathbb{E}^{\lambda[\hat{\chi}]}\left[\sum_{k=A,B}\sum_{l\in N^k}\sum_{t=0}^{\infty}\delta^t\hat{\psi}_{lt}^k(\theta,\varepsilon^t)-\sum_{t=1}^{\infty}\delta^t\sum_{(i,j)\in N^{AB}}c_{ijt}(\hat{\chi}^{t-1}(\theta,\varepsilon^{t-1}))\hat{\chi}_{ijt}(\theta,\varepsilon^t)\right], \tag{S.14}$$

where $\lambda[\hat{\chi}]$ denotes the endogenous process over vertical and horizontal types under the matching rule $\hat{\chi}$ induced by the strategies σ in Γ. Alternatively, (S.14) can be rewritten as follows:

$$\mathbb{E}^{\lambda[\hat{\chi}]}\left[\sum_{t=1}^{\infty}\sum_{(i,j)\in N^{AB}}\delta^t\left(\left(\theta_i^A\varepsilon_{ijt}^A+\theta_j^B\varepsilon_{ijt}^B-c_{ijt}(\hat{\chi}^{t-1}(\theta,\varepsilon^{t-1}))\right)\hat{\chi}_{ijt}(\theta,\varepsilon^t)\right)-\sum_{k=A,B}\sum_{l\in N^k}U_l^k(\theta_l^k)\right], \tag{S.15}$$

where $U_l^k(\theta_l^k)$ denotes the period-0 interim expected payoff of agent $l \in N^k$ from side $k = A, B$ when his true vertical type is θ_l^k, under the equilibrium σ in the mechanism Γ. Note that we denote the interim payoffs by U_l^k to differentiate them from the interim payoff functions $\hat{U}_l^k$ in the matching auction.

The period-0 participation constraints are satisfied if for all $l \in N^k$, $k = A, B$, $\theta_l^k \in \Theta_l^k$, $U_l^k(\theta_l^k) \geq 0$.

Following steps similar to those in Pavan, Segal and Toikka (2014, Theorem 1), one can show that the period-0 (interim) expected payoff of each agent $l \in N^k$, $k = A, B$ must satisfy the following envelope condition

$$U_l^k(\theta_l^k)=U_l^k(\underline{\theta}_l^k)+\int_{\underline{\theta}_l^k}^{\theta_l^k}\mathbb{E}\left[D_l^k(\theta;\hat{\chi})|y\right]dy, \tag{S.16}$$

where the expectation is taken over the entire profile of vertical types θ given agent l's own vertical type. This envelope condition, together with integration by parts, yields the following representation of the platform's profits,

$$\mathbb{E}^{\lambda[\hat{\chi}]}\left[\sum_{t=1}^{\infty}\delta^t\sum_{(i,j)\in N^{AB}}\left(\left(1-\frac{1-G_i^A(\theta_i^A)}{G_i^A(\theta_i^A)\theta_i^A}\right)\theta_i^A\varepsilon_{ijt}^A+\left(1-\frac{1-G_j^B(\theta_j^B)}{G_j^B(\theta_j^B)\theta_j^B}\right)\theta_j^B\varepsilon_{ijt}^B\right.\right.$$
$$\left.\left.-c_{ijt}(\hat{\chi}^{t-1}(\theta,\varepsilon^{t-1}))\right)\hat{\chi}_{ijt}(\theta,\varepsilon^t)\right]-\sum_{k=A,B}\sum_{l\in N^k}U_l^k(\underline{\theta}_l^k). \tag{S.17}$$

The first term, which is only a function of the matching rule $\hat{\chi}$, is the *expected dynamic virtual surplus* (DVS) generated by the matching rule $\hat{\chi}$. Clearly, because such a representation applies to the matching rule generated by *any* BNE of *any* mechanism, the above representation applies also to the state-contingent matching rules generated by the straight-forward strategies in the matching auctions.

Now observe that, when the weights are given by $\beta^P \equiv (\beta_l^{k,P}(\cdot))_{l\in N^k}^{k=A,B}$, and either (i) $M = 1$, or (ii) $M \geq n^A \cdot n^B$ and $m_l^m \geq n^{-k}$, all $l \in N^k$, $k = A, B$, the matches implemented under the straight-forward equilibria of the matching auction with scores $S_{ijt}^{I;\beta^P}$ maximize DVS. This is because (a) these matches have been shown to maximize the continuation weighted surplus W_t for any vector of strictly positive and non-decreasing weights $\beta = (\beta_l^k(\cdot))_{l \in N^k}^{k=A,B}$ (step 1 in the proof of Part 1 above), (b) the ex-ante weighted expected surplus when the weights are given by β^P coincides with DVS, and (c) the weights β^P are strictly positive and non-decreasing.

Also note that, while we have restricted attention to (Γ, σ) that yield a deterministic matching rule $\hat{\chi}$, the platform cannot increase its profits by selecting a pair (Γ, σ) that induces a stochastic matching rule. This is because the matches under any such pair (Γ, σ) can also be induced through a deterministic direct mechanism that conditions on the type reports of a fictitious agent. The platform's profits under any such a mechanism thus continue to be given by the expression in (S.17), but with the matching rule conditioning on the behavior of such fictitious agent. This means that the platform's profits are equal to the weighted average of the platform's profits under the *deterministic* matching rules obtained by conditioning on the various reports of the fictitious agent. Because (S.17) is maximized over all possible deterministic rules under the equilibria in straight-forward strategies of the proposed matching auctions, we thus have that any pair (Γ, σ) yielding a stochastic matching rule can never improve upon the straight-forward equilibria of the proposed matching auctions when it comes to the platform's profits.

Next, let ψ^{β^P} be the payment scheme defined by Conditions (6) and (15) in the main text, when the scores are $S_{ijt}^{I;\beta^P}$ and $Q_l^k = 0$, all $l \in N^k$, $k = A, B$. It is then easy to see that, in the proposed auctions, the payoff expected, in equilibrium, by the lowest vertical type of each agent is exactly equal to zero (this follows from (S.12) and the fact that, given any $\theta^{-l,k} \in \Theta^{-l,k}$, $\hat{U}_l^k(\underline{\theta}_l^k, \theta^{-l,k}) = 0$, all $l \in N^k, k = A, B$). This means that the straight-forward equilibria of the above matching auctions maximize both terms of (S.17). Provided all the period-0 participation constraints are satisfied (something we verify below), we then have that the platform's profits are maximized under the straight-forward equilibria of the proposed matching auctions.

To verify that, under the conditions in Part 2, all period-0 participation constraints are satisfied it suffices to observe that, for all $\theta^{-l,k}$, $l \in N^k$, $k = A, B$, $D_l^k((\theta_l^k, \theta^{-l,k}); \beta^P)$ is non-decreasing in θ_l^k, as established in part (ii) of Lemma S.2 above. That $D_l^k((\underline{\theta}_l^k, \theta^{-l,k}); \beta^P) \geq 0$, all $l \in N^k$, $k = A, B$, all $\theta^{-l,k}$ then guarantees that $D_l^k(\theta, \hat{\beta}) \geq 0$, all $\theta \in \Theta$, $l \in N^k$, $k = A, B$. Because the period-0 interim payoffs satisfy the envelope condition

$$\hat{U}_l^k(\theta) = \int_{\underline{\theta}_l^k}^{\theta_l^k} D_l^k((y, \theta^{-l,k}); \tilde{\chi}) dy$$

we then have that $\hat{U}_l^k(\theta) \geq 0$, all $\theta \in \Theta$, $l \in N^k$, $k = A, B$. This means that all the period-0 participation constraints are satisfied (in a periodic ex-post sense, i.e., for any θ, and not just in expectation over $\theta^{-l,k}$ given θ_l^k). This completes the proof of Part 2. ■

Part 3. We want to show that the following is true: *"(i) The matching auctions in which (a) the scores are $S_{ijt}^{I;\beta^W}$, with weights β^W as defined in Theorem 3, and (b) the payments are as in (S.6) and (S.5), with weights β^W and with Q_l^k large enough, all $l \in N^k$, $k = A, B$, are welfare maximizing. (ii) Suppose that, when agents follow straight-forward strategies in the auctions with scores $S_{ijt}^{I;\beta^W}$,*

$D_l^k((\underline{\theta}_l^k, \theta^{-l,k}); \beta^W) \geq 0$*, all* $l \in N^k$*,* $k = A, B$*, and all* $\theta^{-l,k}$*. Then the matching auctions in which the scores are* $S_{ijt}^{I;\beta^W}$ *and the payments are given by (S.6) and (S.5), with* $Q_l^k = 0$*, all* $l \in N^k$*,* $k = A, B$*, admit ex-post periodic equilibria in which agents participate and follow straight-forward strategies at all histories. Furthermore, the straight-forward equilibria of such auctions maximize the platform's profits over all BNE of all feasible mechanisms implementing welfare-maximizing matches and inducing the agents to join the platform in period zero."*

Proof of Part 3. The arguments parallel those in the proof of Theorem 3 in the main text. Claim (i) follows from the fact that, when $\beta = \beta^W$, the matches implemented under straight-forward strategies maximize the sum of all agents' expected payoffs, net of the platform's costs, at all histories (this follows directly from the arguments establishing Part 1 in the present document). Claim (ii) follows from the fact that (a) the platform's expected profits under any BNE of any feasible mechanism Γ implementing the welfare-maximizing matches and inducing all agents to join at $t = 0$ satisfy the representation in (S.17), with $U_l^k(\underline{\theta}_l^k) \geq 0$, (b) each agent's period-0 expected payoff satisfies Condition (S.16), (c) in the proposed auctions, $U_l^k(\underline{\theta}_l^k) = 0$ if, and only if, the payments defined by Conditions (6) and (15) in the main text (for $\beta = \beta^W$) are such that $Q_l^k = 0$, all $l \in N^k$, $k = A, B$, and (d) when the payments are the ones defined by Conditions (6) and (15) in the main text (for $\beta = \beta^W$) with $Q_l^k = 0$, all $l \in N^k$, $k = A, B$, all agents' period-0 participation constraints are satisfied, regardless of their beliefs over other agents' types, if, and only if, $D_l^k((\underline{\theta}_l^k, \theta^{-l,k}); \beta^W) \geq 0$. The latter property in turn follows from the fact that expected match quality $D_l^k((\cdot, \theta^{-l,k}); \beta^W)$ under the straight-forward equilibria of the proposed auctions is non-decreasing in the agents' true vertical type, which was shown in part (ii) of Lemma S.2 above. This completes the proof of Part 3 in the theorem. ■

Part 4. We want to show the following: *"Suppose all agents derive a nonnegative utility from interacting with all other agents from the opposite side (formally,* $\varepsilon_{ijt}^k \geq 0$*, all* $(i,j) \in N^{AB}$*,* $k = A, B$*,* $t \geq 1$*).*

1. *If none of the capacity constraints binds (i.e., if* $M \geq n^A \cdot n^B$*, and* $m_l^m \geq n^{-k}$*, all* $l \in N^k$*,* $k = A, B$*), then, for all* $(i,j) \in N^{AB}$*, all* $t \geq 1$*,* $\chi_{ijt}^P = 1 \Rightarrow \chi_{ijt}^W = 1$*.*
2. *If* $M = 1$*, then, for all* $t \geq 1$*,* $\sum_{(i,j) \in N^{AB}} \chi_{ijt}^W \geq \sum_{(i,j) \in N^{AB}} \chi_{ijt}^P$*."*

Proof of Part 4. Let $\chi_t^P(\theta, \omega)$ and $\chi_t^W(\theta, \omega)$ denote the state-contingent matches implemented in period $t \geq 1$, under the straight-forward equilibria of, respectively, the profit-maximizing and the welfare-maximizing auctions described in Parts 2 and 3 in the proof above. Note that the arguments of these functions are the exogenous vertical types θ and the sequences of exogenous innovations $\omega \equiv (\omega_{ijs}^k)_{(i,j) \in N^{AB}, k=A,B}^{s=1,\ldots,\infty}$ that, along with the matches implemented in previous periods generate the horizontal types ε. Because the match values are endogenous, such a representation favors the comparison of the matches sustained under the two auctions by making the "state variables" exogenous, thus eliminating the confusion that may originate from the fact that the histories of horizontal types need not coincide under the two auctions.

Similarly, let $S^{I;P}_{ijt}(\theta,\omega)$ and $S^{I;W}_{ijt}(\theta,\omega)$ denote the period-t indexes under the straight-forward equilibria of the profit-maximizing and the welfare-maximizing auctions, respectively. Because (θ,ω) are exogenous and time-invariant, they are dropped from all the functions χ^P, χ^W, $S^{I;P}$, and $S^{I;W}$ below.

First, observe that, because $\beta^{k,P}_l(\theta^k_l) \leq 1 = \beta^{k,W}_l(\theta^k_l)$, all $\theta^k_l \in \Theta^k_l$, $l \in N^k$, $k = A, B$, and because the evolution of the match values is time-autonomous and the horizontal types are nonnegative, for any $(i,j) \in N^{AB}$, any $t, \tau \geq 1$,

$$\sum\nolimits_{s=1}^{t-1} \chi^W_{ijs} = \sum\nolimits_{s=1}^{\tau-1} \chi^P_{ijs} \;\Rightarrow\; S^{I;W}_{ijt} \geq S^{I;P}_{ij\tau}. \tag{S.18}$$

Claim 1. When none of the capacity constraints binds, in each period $t \geq 1$, the matches implemented under the equilibria of the profit-maximizing auctions (alternatively, the welfare-maximizing auctions) are all those for which the index $S^{I;P}_{ijt} \geq 0$ (alternatively, $S^{I;W}_{ijt} \geq 0$). The result then follows from the fact that, for any $(i,j) \in N^{AB}$, $t \geq 1$, $S^{I;W}_{ijt} \geq S^{I;P}_{ijt}$. The last property, in turn, follows by induction. First observe that the property is necessarily true at $t = 1$, given (S.18) and the fact that, in period $t = 1$, the number of past interactions is necessarily the same under profit and welfare maximization. Now suppose the result holds for all $1 \leq s < t$. Note that any match for which $S^{I;P}_{ijt} \geq 0$ has been active at each preceding period $s < t$, both under profit maximization and under welfare maximization. The result then follows again from (S.18), which implies that $S^{I;W}_{ijt} \geq S^{I;P}_{ijt}$.

Claim 2. First observe that, under the equilibria of the profit-maximizing auction, if at some period $t \geq 1$, $\chi^P_{ijt} = 0$, all $(i,j) \in N^{AB}$, then $\chi^P_{ijs} = 0$, all $s > t$, all $(i,j) \in N^{AB}$. The same property holds for χ^W. Next, observe that, if matching stops at period t under profit maximization (alternatively, welfare maximization), then $S^{I;P}_{ijt} < 0$ all $(i,j) \in N^{AB}$ (alternatively, $S^{I;W}_{ijt} < 0$ all $(i,j) \in N^{AB}$). Now suppose that, under profit maximization, matching is still active in period t (meaning, there exists $(i,j) \in N^{AB}$ such that $\chi^P_{ijt} = 1$). Then there are two cases. (1) Either $\sum_{s=1}^{t-1} \chi^W_{ijs} = \sum_{s=1}^{t-1} \chi^P_{ijs}$, all $(i,j) \in N^{AB}$, in which case (S.18) implies that $S^{I;W}_{ijt} \geq S^{I;P}_{ijt}$ for all $(i,j) \in N^{AB}$, which implies the result. Or, (2) there exists $(i,j) \in N^{AB}$ such that $\sum_{s=1}^{t-1} \chi^W_{ijs} < \sum_{s=1}^{t-1} \chi^P_{ijs}$. To see this, recall that the fact that matching is still active in period t under profit maximization implies it must have been active in each preceding period as well. That $\sum_{s=1}^{t-1} \chi^W_{ijs} < \sum_{s=1}^{t-1} \chi^P_{ijs}$ in turn implies there must exist $\tau < t$ such that $\sum_{s=1}^{\tau-1} \chi^P_{ijs} = \sum_{s=1}^{t-1} \chi^W_{ijs}$, and $\chi^P_{ij\tau} = 1$. That $\chi^P_{ij\tau} = 1$ in turn implies $S^{I;P}_{ij\tau} \geq 0$. That $\sum_{s=1}^{\tau-1} \chi^P_{ijs} = \sum_{s=1}^{t-1} \chi^W_{ijs}$ in turn implies that $S^{I;W}_{ijt} \geq S^{I;P}_{ij\tau}$, where the result follows again from (S.18). From the discussion above, that $S^{I;W}_{ijt} \geq S^{I;P}_{ij\tau}$ in turn implies that matching must be active in period t also under welfare maximization. This completes the proof of Part 4 and of the theorem. ■

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