Persuasion in Global Games
with Application to Stress Testing*

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Abstract

We study robust/adversarial information design in global games, where agents are endowed with exogenous private information and the designer is constrained to disclose the same information to all market participants, as in the case of stress testing. First, we show that the optimal policy removes any “strategic uncertainty,” i.e., coordinates all market participants on the same course of action, but without fully revealing the state. Second, we identify conditions under which the optimal policy is a simple “pass/fail” test, with no further information disclosed. Third, we show that the optimal test need not be monotone in fundamentals, but also identify conditions under which it is monotone. Finally, we discuss how the model can be used to study the effects of an increase in market uncertainty on the toughness of the optimal stress test, and how these effects depend on the securities issued by the banks.

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1 Introduction

Differing opinions on how stress tests should be undertaken are welcome and important...We need to move away from simple pass/fail policies (Piers Haben, Director, European Banking Authority, Financial Times, August 1, 2016).

Coordination plays a major role in many socio-economic environments. The damages to society of miscoordination can be severe and often call for government intervention. Think of the situation faced in 2016 by Monte dei Paschi di Siena, the oldest bank on the planet and the Italian third largest, trying to raise capital from multiple investors (mutual funds, creditors, and other major financial institutions), despite concerns about the size of the bank’s nonperforming loans. A default by an institution such as MPS can trigger a collapse in financial markets, and ultimately a deep recession in the Eurozone and beyond (The Economist, July 7, 2016).

Confronted with such prospects, governments and supervising authorities have incentives to intervene. However, a government’s ability to calm the market by injecting liquidity into a troubled bank can be limited. For example, in Europe, legislation passed in 2015 prevents Eurozone member states from rescuing banks by purchasing assets or, more generally, by acting on the banks’ balance sheets. In such situations, interventions aimed at influencing market beliefs, for example through the design of stress tests, or other targeted information policies, play a fundamental role. The questions policy makers face in designing such information policies are the following: (a) What disclosures minimize the risk of default? (b) Should all the information collected through the stress tests be passed on to the market, or should the supervising authorities commit to coarser policies, for example, a simple announcement of whether or not a bank under scrutiny passed the tests? (c) Should stress tests pass institutions with strong fundamentals and fail the rest, or are there benefits to non-monotone rules? (d) What are the effects of an increase in market uncertainty on the structure of the optimal tests, and how do they depend on the banks’ recapitalization strategies, i.e., on the type of security issued by the banks?

In this paper, we study the design of optimal information policies in markets in which a large number of receivers (e.g., investors) must choose whether to play an action favorable to the designer (e.g., pledging funds to a bank, or refraining from speculating against it), or an “adversarial” action (e.g., refraining from pledging, or engaging in predatory trading, for example by short-selling securities linked to the bank’s assets, or buying credit-default swaps, which are known to put strain on illiquid banks). Market participants are endowed with heterogeneous private information about relevant economic fundamentals, such as a bank’s nonperforming loans, the long-term profitability of its assets, or other elements of the bank’s balance sheet not in the public domain. A cash-constrained policy maker (e.g., a supervising authority such as the European Banking Authority) can influence the market’s beliefs (for example, by designing a stress test), but is constrained in its ability to use
financial instruments to shape directly the market outcome.\footnote{For an account of the key institutional details of the stress tests conducted in Europe, see, for example, Henry and Christoffer (2013) and Homar et al. (2016).}

While motivated by the design of stress tests, the analysis delivers results that are relevant also for other applications, including currency crises, technology and standards adoption, and political change.\footnote{For example, in the context of currency crises, the policy maker may represent a central bank attempting to convince speculators to refrain from short-selling the domestic currency by releasing information about the bank’s reserves and/or about domestic economic fundamentals. Alternatively, the policy maker may represent the owners of an intellectual property, or more broadly the sponsors of an idea, choosing among different certifiers in the attempt to persuade heterogeneous market users (buyers, developers, or other technology adopters) of the merits of a new product, as in Lerner and Tirole (2006)’s analysis of forum shopping.} We explicitly account for the role that coordination plays among multiple, heterogeneously informed, receivers. Coordination plays a key role in the funding of solvent but illiquid banks (see, among others, Diamond and Dybvig (1983) and Goldstein and Pauzner (2015) for runs on deposits, Copeland et al. (2014) and Gorton and Metrick (2012) for runs on repos, Covitz et al. (2013) for runs on asset-backed commercial paper, and Pérignon et al. (2018) for dry-ups on certificates of deposit).

The backbone of our analysis is a canonical \textit{global game of regime change} in which, prior to receiving information from the information designer (the policy maker), each agent is endowed with an exogenous private signal about the strength of the underlying fundamentals. In the absence of additional information, such a game admits a unique rationalizable strategy profile, whereby agents play the action favorable to the policy maker (i.e., pledge to the bank) if, and only if, they assign sufficiently high probability to the underlying fundamentals being strong, and whereby regime change (i.e., default) occurs only for sufficiently weak fundamentals. In such settings, the design of the optimal persuasion strategy must account for the effects of information disclosure not just on the agents’ first-order beliefs, but also on their higher-order beliefs (that is, the agents’ beliefs about other agents’ beliefs, their beliefs about other agents’ beliefs about their own beliefs, and so on). Equivalently, the optimal policy must be derived by accounting for how different information disclosures affect both the agents’ structural uncertainty (i.e., their beliefs about the underlying economic fundamentals), and the agents’ strategic uncertainty (i.e., the agents’ beliefs about other agents’ behavior).

We take a “robust approach” to the design of the optimal information policy. We assume that, when multiple rationalizable strategy profiles are consistent with the information disclosed, the policy maker expects the agents to play according to the “most aggressive” strategy profile (the one that minimizes the policy maker’s payoff over the entire set of rationalizable profiles). This is an important departure from both the mechanism design and the persuasion literature, where the designer is typically assumed to be able to coordinate the market on the course of action most favorable to her (among those consistent with the assumed solution concept). Given the type of applications the analysis is meant for, such “robust approach” appears more appropriate.\footnote{If the designer trusted the market to coordinate on the course of action most favorable to her, she would fully...}
Our first result shows that the optimal policy has the “perfect coordination property.” It induces all market participants to take the same action, irrespective of the heterogeneity in the agents’ first- and higher-order posterior beliefs. In other words, the optimal policy completely removes any strategic uncertainty. Under the optimal policy, each agent is able to predict the actions of any other agent, but not the beliefs that rationalize such actions. In the context of our application, an investor who is induced to pledge need not be able to predict whether other investors pledge because they expect the bank’s fundamentals to be so strong that the bank will never collapse, irrespective of what over investors do, or because they expect other investors to pledge. We show how the optimal policy leverages on belief heterogeneity in a way that makes the favorable action dominant based on first-order beliefs only for some agents, while the rest of the agents rely on higher-order beliefs to arrive at the unique rationalizable action through the iterative deletion of interim strictly dominated strategies.\(^4\)

The optimality of policies satisfying the perfect coordination property should not be taken for granted given the robustness requirement. When the designer trusts the receivers to follow her recommendations, the optimality of the perfect coordination property is straightforward and follows from arguments similar to those establishing the Revelation Principle (e.g., Myerson (1986)). This is not the case under adversarial design, for information policies that facilitate perfect coordination among the receivers may also open the door to rationalizable profiles in which some of the agents play adversarially to the designer (in the stress testing application, refrain from pledging).

Our second result shows that, when all agents follow monotone (i.e., cut-off) strategies, the optimal policy takes the form of a simple “pass/fail” test, with no further information disclosed to the market. In turn, we show that the policy maker’s confidence in the agents following monotone strategies is justified when fundamentals and beliefs co-move in the sense that states of Nature in which the fundamentals are strong are also states in which most agents expect the fundamentals to be strong, expect other agents to expect the fundamentals to be strong, and so on.\(^5\) This property is consistent with what is typically assumed in the literature on coordination under incomplete information. Importantly, we show by means of an example that, when such a property is not satisfied, the policy maker may be strictly better off disclosing information to the agents in addition to whether or not the bank passed the test.\(^6\)

\(^4\)In other words, the optimal policy does not ensure that pledging is the unique rationalizable action based on first-order beliefs for all agents. It relies on a contagion argument through higher-order beliefs to achieve the desired property of all agents pledging under the worst rationalizable profile.

\(^5\)Formally, when the agents’ beliefs are parametrized by a uni-dimensional statistics, this amounts to assuming that the distribution from which the signals are drawn is log-supermodular or, equivalently, satisfies the *monotone likelihood ratio property*.

\(^6\)This is another point of departure with respect to the pertinent literature. When the designer trusts her ability to coordinate the receivers on the course of action most favorable to her, optimal policies always take the form of action recommendations (and hence pass/fail policies are optimal, irrespective of the agents’ primitive beliefs). This is not...
The above two results contribute to the debate about the (sub)optimality of European stress tests. Such tests have been criticized for not disclosing the results of the simulations (see, e.g., “Stress tests do little to restore faith in European banks,” Financial Times, August 1, 2017). Our results indicate that simple pass/fail policies might actually be optimal. Importantly, optimal stress tests should be transparent, in the sense of facilitating coordination among the relevant receivers, but should not generate consensus among market participants about the soundness of the financial institutions under scrutiny. Preserving heterogeneous beliefs over a bank’s fundamentals is instrumental to the minimization of default risk.

Our third result is about the optimality of monotone rules that pass with certainty institutions whose fundamentals are strong and fail with certainty those whose fundamentals are weak. We identify conditions under which such policies are optimal. These conditions relate the policy maker’s preferences over the fundamentals of the banks saved to the distribution of the agents’ primitive beliefs and the agents’ payoffs. We show that these conditions are fairly sharp in the sense that, when violated, non-monotone rules may strictly outperform monotone ones. We also explain that the conditions guaranteeing the optimality of monotone rules are more stringent when the policy maker faces multiple privately-informed receivers than when she faces either a single (possibly privately-informed) receiver, or multiple receivers who possess no exogenous private information.

The reason why, under adversarial design, non-monotone policies may outperform monotone ones is that they make it more difficult for the agents to commonly learn the precise fundamentals when hearing that a bank passed the test and hence help reduce the risk of the market responding adversarially to the disclosed information. In turn, this permits the policy maker to give a pass grade to more banks, while guaranteeing that, after a pass grade is announced, the unique rationalizable strategy profile features all agents pledging.

In the Online Supplement, we show how the results extend to settings with more general payoffs and in which the policy maker faces uncertainty about the fate of the financial institutions under scrutiny, for example because default may be determined also by variables orthogonal to, or imperfectly correlated with, those measurable by the policy maker (e.g., by the behavior of noisy/liquidity traders, or by macroeconomic events only imperfectly correlated with the banks’ fundamentals). We then use these extensions to conduct comparative statics analysis in a family of simple economies in which banks issue equity or debt to fund their short-term liquidity obligations, and where the (market-clearing) price of the securities is endogenous and depends on the information revealed through the stress tests. In particular, we investigate the effects of an increase in market uncertainty on the toughness of the optimal stress tests and show how the latter depends on the type of security issued by the banks.

Throughout the analysis, we restrict attention to situations in which the agents possess primitive private information before hearing from the policy maker and where the latter is constrained to disclose the same information to all market participants, which is the relevant case in practice. In the case under adversarial/robust design.
the Online Supplement, however, we also briefly discuss why, when feasible, discriminatory disclosures may improve upon non-discriminatory ones.

**Organization.** The rest of the paper is organized as follows. Below, we wrap up the introduction with a brief review of the most pertinent literature. Section 2 presents the model. Section 3 contains all the results about properties of optimal policies (perfect-coordination, pass/fail, monotonicity). Section 4 discusses extensions of the baseline model and robustness of the key results to richer settings. It also discusses how such extensions permit one to study the comparative statics of the optimal stress test in a class of economies in which banks issue debt or equity to fund their obligations and the price of the securities is endogenous (the formal analysis corresponding to this part is in the Online Supplement). Section 5 concludes. All proofs are either in the Appendix at the end of the document or in the Online Supplement.

**(Most) pertinent literature.** The paper is related to different strands of the literature. The first and most relevant strand is the literature on information design (see Bergemann and Morris (2019) and Kamenica (2019) for overviews). This literature traces back to Myerson (1986), who introduced the idea that, in a general class of multi-stage games of incomplete information, the designer can restrict attention to private incentive-compatible action recommendations to the agents. Recent developments include Rayo and Segal (2010), Kamenica and Gentzkow (2011), Gentzkow and Kamenica (2016), Ely (2017), and Dworczak and Martini (2019). These papers consider persuasion with a single receiver. The case of multiple receivers is less studied. Calzolari and Pavan (2006a) consider an auction setting in which the sender is the initial owner of a good and where the different receivers are privately-informed bidders in an upstream market who then resell in a downstream market (see also Dworczak (2020) for an analysis of persuasion in other mechanism design environments with aftermarket). More recent papers with multiple receivers include Alonso and Camara (2016a), Bardhi and Guo (2017), Basak and Zhou (2019), Che and Hörner (2018), Doval and Ely (2020), Galperti and Perego (2020), Li et al. (2021), Mathevet et al. (2019), Morris et al. (2020) and Taneva (2019). In particular, Li et al. (2021) and Morris et al. (2020) consider adversarial design in a coordination setting similar to the one in the present paper. These papers, which are subsequent to ours, assume that (a) the receivers possess no exogenous private information prior to receiving the information from the designer, and (b) the designer can design the receivers’ private information, that is, she can engage in discriminatory disclosures that inform the receivers asymmetrically about the relevant state. In contrast, we assume that the receivers are endowed with exogenous private information and that the designer is constrained to disclose the same information to all the receivers, which appears the most relevant case for the type of applications the analysis is meant for (e.g., stress testing). Persuasion with privately-informed receivers has been examined primarily in settings with

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7Related is also Calzolari and Pavan (2006b). That paper studies information design in a model of sequential contracting with multiple principals, where upstream principals play the role of senders persuading downstream principals (the receivers).
a single receiver (see, among others, Kolotilin et al. (2017), Alonso and Camara (2016b), Chan et al. (2019), and Guo and Shmaya (2019)). See Laclau and Renou (2017), Gitmez and Molavi (2020), and Heese and Lauermann (2021) for recent papers with multiple privately-informed receivers. These papers though do not look at the implications of (adversarial) coordination for the structure of the optimal policy, which is the focus of the present paper. In a coordination setting with two privately-informed receivers and two states, Alonso and Zachariadis (2021) show that, when the precision of the receivers’ exogenous information is sufficiently high, private and public information are complements in that an increase in the precision of the agents’ private information leads to the provision of more accurate public information. Goldstein and Huang (2016) study persuasion in a coordination setting similar to ours but restricting the designer to monotone pass/fail policies. Our results show that the optimal policy need not be monotone in their setting, but also identify primitive conditions under which, in richer settings, monotone policies are optimal. In a similar vein, Galvão and Shalders (2020) look at the design of policies in a global game similar to ours but restricting the designer to monotone partitional rules (whereby if two types receive the same grade then all types between these two also receive the same grade).

The present paper contributes to this strand of the literature by identifying properties of optimal (non-discriminatory) policies when the receivers are privately informed and play adversarially.

The second strand is the literature on global games with endogenous information. Angeletos et al. (2006), and Angeletos and Pavan (2013) consider settings whereby a policy maker, endowed with private information, engages in costly actions to influence the agents’ behavior. Edmond (2013) considers a similar setting but assumes the cost of policy interventions is zero and agents receive noisy signals of the policy maker’s action. Angeletos et al. (2007) consider a dynamic model in which agents learn from the accumulation of private signals over time and from the (possibly noisy) observation of past outcomes. Cong et al. (2016) consider a dynamic setting similar to the one in Angeletos et al. (2007) but allowing for policy interventions. Denti (2020), Szkup and Trevino (2015), Yang (2015) and Morris and Yang (2019) consider global games where, prior to committing their actions, agents acquire private information about payoff-relevant variables at a small cost.

The key contribution of the present paper vis-a-vis this literature is the characterization of the optimal provision of public information.

Finally, the paper is related to the literature on stress testing. While inspired by this application, our paper does not aim at capturing many institutional details that may be relevant for stress testing. Our analysis is a reduced-form account of the role that coordination plays in the design of such tests.

Bouvard et al. (2015) study a credit rollover setting where a policy maker must choose between transparency (full disclosure) and opacity (no disclosure) but cannot commit to a disclosure policy. In contrast, we assume the policy maker can fully commit to her disclosure policy and allow for flexible

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8See also Gick and Pausch (2012), Shimoji (2017), and Arieli and Babichenko (2019a). These papers focus primarily on situations without strategic interactions among the receivers. In an extension, Arieli and Babichenko (2019b) consider a model with strategic interaction.
information structures. Alvarez and Barlevy (2015) study the incentives of banks to disclose balance sheet (hard) information in a setting where the market is not able to observe how banks are exposed to each others’ risks.\textsuperscript{9} Quigley and Walther (2020) study how stress tests shape banks’ incentives to voluntarily disclose private information. See also Goldstein and Sapra (2014) for an overview of some of the early contributions and Morgan et al. (2014), Flannery et al. (2017), and Petrella and Resti (2013) for an empirical analysis of the information provided by stress tests conducted in the US and the EU. Goldstein and Leitner (2018) study the design of stress tests by a regulator facing a competitive market, where agents have homogeneous beliefs about the bank’s balance sheet.\textsuperscript{10} Orlov et al. (2020) and Inostroza (2021) consider the joint design of stress tests and capital requirements. The latter paper also considers the interplay between information disclosures and the policy maker’s role as a lender of last resort.\textsuperscript{11}

Our paper contributes to this literature along the following dimensions: (a) it shows that optimal stress tests should not create conformism in market beliefs about banks’ fundamentals but should be sufficiently transparent to eliminate any ambiguity about the market response to the tests; (b) it identifies conditions under which simple pass/fail announcements are optimal; (c) it provides conditions for optimal tests to be monotone; and (d) it discusses how the toughness of optimal tests relates to the type of securities issued by the banks.

2 Model

To illustrate the key ideas in the simplest possible terms, we consider a stylized global game of regime change in the spirit of Rochet and Vives (2004). The game abstracts from many institutional details but highlights the effects of (adversarial) coordination among privately-informed receivers on the design of the optimal policy. Motivated by the application to stress testing, the model features a policy maker persuading investors to pledge to a bank.\textsuperscript{12} The analysis, however, can be adapted easily to many other games of regime change.

Players and Actions. A policy maker designs a stress test, i.e., an information policy that evaluates the profitability of a representative bank and communicates information based on the results of such evaluations to the market. To meet its short-term liquidity obligations, the bank may need funding from the market. The latter is populated by a (measure-one) continuum of investors.

\footnote{See also Corona et al. (2017) for an analysis of how stress tests disclosures may favor banks’ coordinated risk taking in the spirit of Farhi and Tirole (2012).}

\footnote{See also Williams (2017) for a related analysis of stress test design in a bank-run model a’ la Allen and Gale (1998), with homogenous investors.}

\footnote{See also Faria-e Castro et al. (2016) and Garcia and Panetti (2017) for a joint analysis of stress tests and government bailouts.}

\footnote{Rochet and Vives (2004) consider a three-period economy a’ la Diamond and Dybvig (1983) but with heterogenous investors, in which banks can liquidate assets to boost liquidity and may fail early or late. As shown in that paper, the full model admits a reduced-form version similar to the one considered here.}
distributed uniformly over \([0, 1]\). Each investor may either take a “friendly” action, \(a_i = 1\), or an “adversarial” action, \(a_i = 0\). The friendly action is interpreted as the decision to pledge funds to the bank (alternatively, to abstain from speculating against the bank or by engaging in predatory trading, e.g., by purchasing credit-default swaps). The adversarial action is interpreted as the decision to not pledge (alternatively, to speculate against the bank). We denote by \(A \in [0, 1]\) the size of the aggregate pledge.

**Fundamentals and Exogenous Information.** The bank’s fundamentals are parameterized by \(\theta \in \mathbb{R}\). Before the bank is scrutinized, it is commonly believed (by the policy maker and the investors alike) that \(\theta\) is drawn from a distribution \(F\), absolutely continuous over \(\Theta \subseteq [0, 1]\), with a smooth density \(f\) strictly positive over \(\Theta\). In addition, each investor \(i \in [0, 1]\) is endowed with private information summarized in a uni-dimensional signal \(x_i \in \mathbb{R}\) drawn independently across agents (given \(\theta\)) from an absolutely continuous cumulative distribution function \(P(x|\theta)\) with smooth density \(p(x|\theta)\) strictly positive over an (open) interval \(\varrho_\theta \equiv (\varrho_\theta, \bar{\varrho}_\theta)\) containing \(\theta\), with \(\varrho_\theta, \bar{\varrho}_\theta\) monotone in \(\theta\). The bounds \(\varrho_\theta, \bar{\varrho}_\theta\) can be either finite or infinite. For example, when \(x_i = \theta + \sigma \epsilon_i\), with \(\epsilon_i\) drawn from a uniform distribution over \([-1, +1]\), then, for any \(\theta\), \(\varrho_\theta = \theta - \sigma\) and \(\bar{\varrho}_\theta = \theta + \sigma\). When, instead, \(x_i = \theta + \sigma \epsilon_i\), with \(\epsilon_i\) drawn from a standard Normal distribution, then, for any \(\theta\), \(\varrho_\theta = -\infty\) and \(\bar{\varrho}_\theta = +\infty\). Furthermore, in this latter case, \(P(x|\theta) = \Phi((x - \theta)/\sigma)\), where \(\Phi\) is the cumulative distribution function of the standard Normal distribution.\(^{13}\) We denote by \(\mathbf{x} \equiv (x_i)_{i \in [0, 1]}\) a profile of private signals and by \(\mathbf{X}(\theta)\) the collection of all \(\mathbf{x} \in \mathbb{R}^{[0,1]}\) that are consistent with the fundamentals being equal to \(\theta\). As usual, we assume that any pair of signal realizations \(\mathbf{x}, \mathbf{x}' \in \mathbf{X}(\theta)\) has the same cross-sectional distribution of signals, with the latter equal to \(P(x|\theta)\).

**Default.** The bank’s fundamentals \(\theta\) parametrize the critical size of the aggregate pledge that is necessary for the bank to avoid default. If \(A > 1 - \theta\), the bank meets all its short-term obligations and avoids default. If, instead, \(A < 1 - \theta\), the bank is in distress and defaults. We denote by \(r = 0\) the event that the bank defaults, and by \(r = 1\) the complement event in which the bank avoids default.\(^{15}\)

**Dominance Regions.** For any \(\theta \leq 0\), the bank defaults, whereas for any \(\theta > 1\) the bank avoids default, irrespective of the size of the aggregate pledge. For \(\theta \in (0, 1]\), instead, whether or not the bank defaults is determined by the behavior of the market.

**Payoffs.** Each investor’s payoff differential between the friendly and the adversarial action is equal to \(g(\theta) > 0\) in case the bank avoids default and \(b(\theta) < 0\) otherwise. The policy maker’s payoff is equal to \(W(\theta)\) in case default is avoided and \(L(\theta)\) in case of default, with \(W(\theta) > L(\theta)\) for all \(\theta\). When \(W\) and \(L\) are invariant in \(\theta\), the policy maker’s objective reduces to minimizing the probability

\(^{13}\)Formally, \(\varrho_\theta \equiv \text{supp}[P(\cdot|\theta)]\).

\(^{14}\)The uniform and Gaussian distributions are the ones considered in most of the literature.

\(^{15}\)The model assumes that, given \(A\) and \(\theta\), the regime outcome is binary: either the bank defaults or it survives. The case in which default is “partial” is qualitatively similar, from a strategic standpoint, to the case where, given \(A\) and \(\theta\), the regime outcome is stochastic and determined by variables that are not observable by the policy maker at the time of the stress test (see the discussion in Section 4 and the formal analysis in in Section S5 in the Online Supplement).
of default. The functions $b$, $g$, $W$, and $L$ are all bounded. For any $(\theta, A) \in \Theta \times [0, 1]$, then let

$$u(\theta, A) \equiv g(\theta)I(A > 1 - \theta) + b(\theta)I(A \leq 1 - \theta)$$

and

$$U^P(\theta, A) \equiv W(\theta)I(A > 1 - \theta) + L(\theta)I(A \leq 1 - \theta)$$

denote the payoffs of a representative agent and of the policy maker, respectively, when the fundamentals are $\theta$ and the aggregate size of the pledge is $A$.

**Stress Tests.** Let $\mathcal{S}$ be a compact metric space defining the set of possible signal realizations (think of these as grades or scores given to the bank under examination). A stress test $\Gamma = (\mathcal{S}, \pi)$ consists of the set $\mathcal{S}$ along with a mapping $\pi : \Theta \to \Delta(\mathcal{S})$ specifying, for each $\theta$, a (probability distribution over the) score given to type $\theta$.

**Timing.** The sequence of events is the following:

1. The policy maker publicly announces the policy $\Gamma = (\mathcal{S}, \pi)$ and commits to it.
2. The fundamentals $\theta$ are drawn from the distribution $F$ and the agents’ exogenous signals $x \in X(\theta)$ are drawn from the distribution $P(x|\theta)$.
3. The score $s$ is drawn from $\pi(\theta)$ and publicly announced.
4. Agents simultaneously choose whether or not to pledge.
5. The fate of the bank is determined and payoffs are realized.

**Adversarial Play and Robust Design.** The policy maker does not trust the market to follow her recommendations and play according to the strategy profile that is most advantageous to her (i.e., pledge to the bank whenever the latter is solvent, i.e., whenever $\theta > 0$). Instead, the policy maker adopts a robust approach to the design of the stress test. She evaluates any policy $\Gamma$ under the “worst-case” scenario. That is, given any policy $\Gamma$, the policy maker expects the market to play according to the rationalizable profile most adversarial to her.

**Definition 1.** Given any policy $\Gamma$, the **most aggressive rationalizable profile** (MARP) consistent with $\Gamma$ is the strategy profile $a^\Gamma \equiv (a^\Gamma_i)_{i \in [0, 1]}$ that minimizes the policy maker’s ex-ante expected payoff over all profiles surviving *iterated deletion of interim strictly dominated strategies* (henceforth IDISDS).

In the IDISDS procedure leading to MARP, agents update their beliefs about the fundamentals $\theta$ and the other agents’ exogenous signals $x \in X(\theta)$ using the common prior, $F$, the signal distribution,

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16 Here we assume that, through the stress test, the policy maker learns all information that is relevant for the fate of the bank, for the policy maker’s payoff, and for the payoffs of all market participants. We relax these assumptions in Section S5 in the Online Supplement.

17 If she did, a simple monotone test revealing whether or not $\theta > 0$ would be optimal.
$P(x|\theta)$, the disclosure policy, $\Gamma$, and Bayes rule. Under MARP, given $(x,s)$, each agent $i \in [0,1]$, after receiving exogenous information $x$ and endogenous information $s$, then refrains from pledging whenever there exists at least one conjecture over $(\theta,A)$ consistent with the above Bayesian updating and supported by all other agents playing strategies surviving IDISDS, under which refraining from pledging is a best response for the individual.

Remarks. Hereafter, we confine attention to policies $\Gamma$ for which MARP exists.$^{18}$ Because the game among the agents is supermodular, the strategy profile $a^\Gamma$ is also a Bayes-Nash equilibrium (BNE) of the continuation game among the agents, and minimizes the policy maker’s payoff state-by-state, and not just in expectation.

Furthermore, given a policy $\Gamma = (S,\pi)$, when describing the agents’ behavior, we do not distinguish between pairs $(x,s)$ that are mutually consistent given $\Gamma$ (meaning that the joint density of $(x,s)$ is positive, i.e., $\int_{\theta,s \in \text{supp}(\pi(\theta))} p(x|\theta) dF(\theta) > 0$) and those that are not. Because the policy maker commits to the policy $\Gamma$, the abuse is legitimate and permits us to ease the exposition. Any claim about the optimality of the agents’ behavior, however, should be interpreted to apply to pairs $(x,s)$ that are mutually consistent given $\Gamma$.

3 Properties of optimal policies

We discuss three key properties of optimal policies.

3.1 Perfect-coordination property

Definition 2. A policy $\Gamma = (S,\pi)$ satisfies the perfect-coordination property (PCP) if, for any $\theta \in \Theta$, any exogenous information $x \in X(\theta)$, any $s \in \text{supp}(\pi(\theta))$, and any pair of individuals $i,j \in [0,1]$, $a^\Gamma_i(x_i,s) = a^\Gamma_j(x_j,s)$, where $a^\Gamma = (a^\Gamma_i)_{i \in [0,1]}$ is the most aggressive rationalizable profile (MARP) consistent with the policy $\Gamma$.

A disclosure policy has the perfect-coordination property if it induces all market participants to take the same action, after any information it discloses. For any $\theta \in \Theta$, any $s \in \text{supp}(\pi(\theta))$, let $r^\Gamma(\theta,s) \in \{0,1\}$ denote the default outcome when investors play according to $a^\Gamma$ (i.e., $r^\Gamma(\theta,s)$ is the fate of the bank that prevails at $(\theta,s)$, when the agents play according to MARP consistent with $\Gamma$).$^{19}$ Hereafter, we say that the policy $\Gamma$ is regular if MARP under $\Gamma$ is well-defined and the default outcome under $a^\Gamma$ is measurable in $(\theta,s)$.$^{20}$

$^{18}$Because the state is continuous, in principle, one can think of policies $\Gamma$ for which the agents’ common posterior is not well defined or, when combined with the agents’ exogenous information, is such that the agents’ hierarchies of beliefs are not well defined, in which case MARP may not exist. While we cannot exclude such a possibility, we did not construct examples in which MARP does not exist.

$^{19}$Because the cross-sectional distribution of signals is uniquely pinned down by $P(x|\theta)$, the fate of the bank under MARP is the same across any pair of signal profiles $x,x' \in X(\theta)$ and hence depends only on $\Gamma, \theta, \text{ and } s$.

$^{20}$Because the game has infinitely many states and players, these properties, while fairly natural, cannot be guaranteed for arbitrary policies.
Theorem 1. Given any (regular) policy $\Gamma$, there exists another (regular) policy $\Gamma^*$ satisfying the perfect coordination property and such that, for any $\theta$, the default probability under $\Gamma^*$ is the same as under $\Gamma$.

Proof of Theorem 1: See the Appendix.

The policy $\Gamma^*$ is obtained from the original policy $\Gamma$ by disclosing, for each $\theta$, in addition to the score $s \in \text{supp}(\pi(\theta))$ disclosed by the original policy $\Gamma$, a second piece of information that reveals to the market whether at $(\theta, s)$, under $\alpha^\Gamma$, the agents’ expected payoff differential (between pledging and not pledging) is positive or negative. Because in this simple economy, the sign of this differential is given by the regime outcome, this additional piece of information takes the form of $r^\Gamma(\theta, s) \in \{0, 1\}$. That, under $\Gamma^*$, it is rationalizable for all agents to pledge when the policy discloses the information $(s, 1)$ and to refrain from pledging when the policy discloses the information $(s, 0)$ is straight-forward. In fact, the announcement of $(s, 1)$ (alternatively, of $(s, 0)$) makes it common certainty among the agents that $\theta > 0$ (alternatively, that $\theta \leq 1$).

The interesting part of the result is that, in the continuation game that starts after the policy $\Gamma^*$ announces $(s, 1)$, pledging is the unique rationalizable action for any agent, irrespective of his signal. When, under the original policy $\Gamma$, $r^\Gamma(\theta, s)$ is increasing in $\theta$, the new piece of information that $\theta$ is such that $r^\Gamma(\theta, s) = 1$ is equivalent to the announcement that $\theta > \hat{\theta}(s)$, for some threshold $\hat{\theta}(s)$. In this case, agents update their first-order beliefs about $\theta$ upward when receiving the additional information that $r^\Gamma(\theta, s) = 1$. That each agent is more optimistic about the strength of the fundamentals, however, does not guarantee that MARP under the new policy is less aggressive than under the original policy. In fact, the new piece of information changes not only the agent’s first-order beliefs about $\theta$ but also his higher-order beliefs and the latter matter for the determination of the most-aggressive rationalizable profile. Furthermore, in general, $r^\Gamma(\theta, s)$ need not be monotone in $\theta$.

This is because MARP, under the original policy $\Gamma$, need not entail strategies that are monotone in $x$. As a result, the announcement that $r^\Gamma(\theta, s) = 1$ need not trigger an upward revision of the agents’ beliefs.\footnote{In richer settings, the fate of the bank may depend also on variables other than $\theta$ for which both the policy maker and the market have imperfect information about. Furthermore, the agents’ payoffs may depend on $A$ beyond the effect that this variable has on the fate of the bank. See Section S5 in the Online Supplement for how, in these richer economies, perfect coordination may still be optimal and continues to be attained by announcing to the market the sign of the agents’ expected payoff differential at $(\theta, s)$, but the latter is not pinned down by the fate of the bank.}

The result in Theorem 1 holds irrespectively of whether or not, given $s$, $r^\Gamma(\theta, s)$ is monotone in $\theta$. It follows from the fact that, at any stage $n$ of the IDISDS procedure, any agent who, under the original policy $\Gamma$ would have pledged under the most aggressive strategy profile surviving $n-1$ rounds of deletion, continues to do the same under the new policy $\Gamma^*$. In the Appendix, we show that this last property in turn follows from the game being supermodular along with the fact that Bayesian updating preserves the likelihood ratio of any two states that are consistent with no default under the original policy $\Gamma$. Formally, for any $s \in \text{supp}(\pi(\Theta))$, any pair of states $\theta'$ and $\theta''$ such that (a)
$s \in \text{supp } \pi(\theta') \cap \text{supp } \pi(\theta'')$, and (b) $r^\Gamma(\theta', s) = r^\Gamma(\theta'', s) = 1$, the likelihood ratio of such two states under $\Gamma^*$ is the same as under the original policy $\Gamma$. This property, together with the announcement that default would have not occurred under MARP consistent with the original policy $\Gamma$, guarantees that, for any agent for whom pledging was optimal under MARP consistent with the original policy $\Gamma$, pledging is the unique rationalizable action under the new policy $\Gamma^*$.

The policy $\Gamma^*$ thus eliminates any strategic uncertainty. When $(s, 1)$ (alternatively, $(s, 0)$) is announced, each agent knows that all other agents will pledge (alternatively, will refrain from pledging), irrespective of their exogenous private information. Importantly, while the policy $\Gamma^*$ removes any strategic uncertainty, it preserves heterogeneity in the agents’ posterior beliefs about $\theta$. To avoid default at certain fundamentals, it is essential that agents who pledge are uncertain as to whether other agents pledge because they find it dominant to do so, or because they expect other agents to pledge, which requires heterogeneity in posterior beliefs.

When it comes to stress testing, the proof of Theorem 1 implies that optimal stress tests should combine the announcement of a pass/fail result (captured by $r \in \{0, 1\}$) with the disclosure of additional information (captured by $s$) whose role is to guarantee that, even if the market were to play adversarially, when a pass grade is given, the bank under scrutiny receives enough funds to continue operating. This structure appears broadly consistent with common practice. The theorem, however, says more. It indicates that optimal stress test should leave no room to ambiguity as to whether or not a bank receiving a pass grade will succeed in raising the liquidity it needs. Importantly, optimal tests should not be expected to create conformism in market beliefs about the soundness of a bank under scrutiny. Preserving different views among participants is key to sparing more banks from default.

### 3.2 Pass/Fail

**Theorem 2.** Suppose that $p(x|\theta)$ is log-supermodular. Then, given any policy $\Gamma$ satisfying the perfect coordination property, there exists a binary policy $\Gamma^* = (\{0, 1\}, \pi^*)$ that also satisfies the perfect coordination property and such that, for any $\theta$, the probability of default under $\Gamma^*$ is the same as under $\Gamma$.

**Proof of Theorem 2:** See the Appendix.

Take any policy $\Gamma = (S, \pi)$ satisfying the perfect coordination property. Given the result in Theorem 1, without loss of generality, assume that $\Gamma = (S, \pi)$ is such that $S = \{0, 1\} \times S$, for some $s \in S$. Formally, as we show in the Appendix, the above two properties jointly imply that each agent’s posterior beliefs after $r^\Gamma(\theta', s) = 1$ is announced are a “truncation” of the agent’s beliefs under the original policy $\Gamma$, with the truncation eliminating from the support of the agent’s beliefs states $\theta$ at which, under the most aggressive profile surviving $n$ rounds of IDISDS under the original policy $\Gamma$, the agent’s payoff from pledging would have been negative. The truncation thus contributes to making the agent more willing to pledge.

The property that $p(x|\theta)$ is log-supermodular means that, for any $x', x'' \in \mathbb{R}$, with $x' < x''$, and any $\theta', \theta'' \in \Theta$, with $\theta'' > \theta'$, then $p(x'|\theta'')p(x'|\theta') \geq p(x'|\theta')p(x'|\theta'')$. 

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measurable set \( S \), and is such that, under MARP, (a) when the policy discloses any signal \((s, 1)\), all agents pledge and default does not happen, whereas (b) when the policy discloses any signal \((s, 0)\), all agents refrain from pledging and default happens. Given the policy \( \Gamma \), let \( U^\Gamma(x, (s, 1)|k) \) denote the expected payoff differential of an agent with exogenous information \( x \) who receives information \((s, 1)\) from the policy maker and who expects all other agents to pledge if and only if their exogenous signal exceeds a cut-off \( k \). In the Appendix, we show that, no matter the shape of the policy \( \Gamma \), when \( p(x|\theta) \) has the monotone likelihood ratio property (in short, MLRP), MARP associated with the policy \( \Gamma \) is in monotone (i.e., cut-off) strategies.\(^{24} \) Hence, each agent’s expected payoff differential when all other agents play according to MARP can be written as \( U^\Gamma(x, (s, 1)|k) \) for some \( k \) that depends on \( s \). That the original policy \( \Gamma \) satisfies the perfect-coordination policy in turn implies that, for any \( s \) and \( k \) such that \((k, (s, 1))\) are mutually consistent,\(^{25} \) \( U^\Gamma(k, (s, 1)|k) > 0 \). That is, the expected payoff differential of any agent whose private signal \( x \) coincides with the cutoff \( k \) must be strictly positive.

If this were not the case, the continuation game would also admit a rationalizable profile (in fact, a continuation equilibrium) in which some of the agents refrain from pledging, contradicting the fact that pledging irrespectively of \( x \) is the unique rationalizable profile following the announcement of \((s, 1)\).

Now consider a policy \( \Gamma^* \) that, for any \( \theta \), discloses the same outcome \( r^\Gamma(\theta, s) \) as the original policy \( \Gamma \) but conceals the additional information \( s \). By the law of iterated expectations, for all \( k \) such that \((k, (s, 1))\) are mutually consistent, because \( U^\Gamma(k, (s, 1)|k) > 0 \) then \( U^{\Gamma^*}(k, 1|k) > 0 \). The last property implies that the new policy \( \Gamma^* \) also satisfies the perfect-coordination property. The policy maker can thus drop the additional signals \( s \) from the original policy \( \Gamma \) and continue to guarantee that, after \( r = 1 \) is announced, pledging is the unique rationalizable action for all agents. The result in the theorem thus implies that simple pass/fail policies are optimal.

The property that justifies restricting attention to simple pass/fail policies is the log-supermodularity of the signal distribution \( p(x|\theta) \). This property, which is formally equivalent to MLRP, is essential for the optimality of simple pass/fail policies, as the next example shows.\(^{26} \)

\textbf{Example 1.} Suppose that \( \theta \) is drawn from a uniform distribution over \([-1, 2]\). Given \( \theta \), each agent \( i \in [0, 1] \) receives an exogenous signal \( x_i \in \{x^L, x^H\} \), drawn independently across agents from a

\( ^{24} \)That \( p(x|\theta) \) satisfies MLRP implies that, at any round of the IDISDS procedure, agents follow monotone strategies, no matter \( \Gamma \). This is the only role that MLRP plays in the result. Hence, if one were to exogenously assume that, for any \( \Gamma \), agents follow monotone strategies (and justify the assumption, for example, on behavioral grounds), then MLRP of \( p(x|\theta) \) could be dispensed with without affecting the conclusion in the theorem.

\( ^{25} \)Recall that the latter means that the set \( \theta \in \Theta \) such that \((a) k \in \varrho_\theta \) (i.e., \( k \in \text{supp}(P(\cdot|\theta)) \)) and \((s, 1) \in \text{supp}(\pi(\theta)) \) has positive measure.

\( ^{26} \)The example below features signals drawn from a distribution with finite support. This, however, is only for simplicity. Conclusions similar to those in the example can be established by having the agents receive signals drawn from a continuous distribution. We thank Tommaso Denti for suggesting a related example with finite signals and Leifu Zhang for suggesting a related example with continuous signals.
Bernoulli distribution with probability

\[
Pr(x^L|\theta) = \begin{cases} 
2/3 & \text{if } \theta \in (0, 1/3) \cup [2/3, 5/6) \cup [1, 7/6) \cup [4/3, 5/3) \\
1/3 & \text{if } \theta \in [1/3, 2/3) \cup [5/6, 1) \cup [7/6, 4/3) \cup [5/3, 2]. 
\end{cases}
\]

The value of \(Pr(x^L|\theta)\) for \(\theta \in [-1, 0]\) plays no role in this example, so it can be taken arbitrarily.

Suppose that agents’ payoffs are such that \(g(\theta) = 1 - c\) and \(b(\theta) = -c\), for all \(\theta\), with \(c \in (1/2, 8/15)\).

There exits a deterministic policy that satisfies PCP and guarantees that no \(\theta > 0\) defaults, but no deterministic pass/fail policy can spare all \(\theta > 0\) from default.

**Proof of Example 1.** Figure 1 illustrates the signal structure considered in Example 1. The dash line depicts the probability of signal \(x^L\) whereas the solid line the complementary probability of signal \(x^H\), as a function of \(\theta\). Note that the agents’ posterior beliefs under the signal structure of Example 1 can be ranked according to FOSD. Each agent observing \(x^H\) has posterior beliefs that dominate those of each agent observing \(x^L\). Nonetheless, the ratio \(p(x^H|\theta)/p(x^L|\theta)\) is not increasing in \(\theta\) over the entire domain, meaning that \(p(x|\theta)\) is not log-supermodular. Also note that, under the payoff specification in the example, pledging is rationalizable if the probability of default is no greater than 1 - \(c\), whereas not pledging is rationalizable if such a probability is at least 1 - \(c\).

To see that there exists no pass/fail policy sparing all \(\theta > 0\) from default, note that, by virtue of Theorem 1, if such a policy existed, there would also exist a binary policy satisfying PCP and such that \(\pi(1|\theta) = 0\) for all \(\theta \leq 0\) and \(\pi(1|\theta) = 1\) for all \(\theta > 0\), with \(\pi(1|\theta)\) denoting the probability that the policy discloses signal \(s = 1\) when the fundamentals are \(\theta\). Under such a policy, after hearing that \(s = 1\), no matter the private signal \(x\), each agent assigns probability 1/2 to \(\theta \in [0, 1]\) and probability 1/2 to \(\theta \in [1, 2]\). Because \(c > 1/2\), each agent expecting all other agents to refrain from pledging (and hence default to occur for all \(\theta \in [0, 1]\)) then finds it optimal to do the same. Hence, under
MARP consistent with the above policy, after the signal \( s = 1 \) is announced, all agents refrain from pledging, meaning that the above policy fails to spare all \( \theta > 0 \) from default.

Next, to see that all types \( \theta > 0 \) can be spared from default under policies richer than simple pass/fail ones, consider the policy \( \Gamma = \{(0,(1,mid),(1,ext))\},\pi \) that, in addition to publicly announcing whether or not the bank passed the test, also announces whether fundamentals are extreme (i.e., \( \theta \in (0,5/6) \cup (7/6,2) \)), or intermediate (i.e., \( \theta \in [5/6,7/6] \)). Formally, for any \( \theta \in [-1,0] \), \( \pi(0|\theta) = 1 \), meaning that the policy announces with certainty \( s = 0 \) meaning that the bank failed the test. For any \( \theta \in [5/6,7/6] \), \( \pi(1,mid|\theta) = 1 \), meaning that the policy announces with certainty that the bank passed the test and that fundamentals are intermediate. Finally, for any \( \theta \in (0,5/6)\cup(7/6,2) \), \( \pi(1,ext|\theta) = 1 \) meaning that the policy announces with certainty that the bank passed the test and that fundamentals are extreme. See again Figure 1 for a graphical representation of such a policy.

Under MARP associated with such a policy, all agents pledge when they hear that the bank passed the test, no matter whether the policy announces that fundamentals are extreme or intermediate, whereas all agents refrain from pledging when hearing that the bank failed the test.

Consider first the case in which the fundamentals are extreme, i.e., \( \theta \in (0,5/6) \cup (7/6,2) \). All agents with exogenous signal \( x^H \) find it dominant to pledge when they hear that \( s = (1,ext) \). In fact, even if all other agents were to refrain from pledging, the probability that each agent with signal \( x^H \) assigns to \( \theta > 1 \) (and hence to the bank surviving) is \( Pr(\theta > 1|x^H,ext) = 8/15 > c \), making it dominant for the individual to pledge. As a consequence of this property, each agent receiving an exogenous signal \( x^L \) finds it iteratively dominant to pledge. This is because, for any \( \theta \in [1/3,5/6] \), even if all agents receiving a signal equal to \( x^L \) were to refrain from pledging, the aggregate size of the pledge from those agents receiving an \( x^H \) signal would suffice for the bank to survive. This means that the probability that each agent with signal \( x^L \) assigns to the bank surviving is at least equal to \( Pr(\theta > 1/3|x^L,(1,ext)) = 11/15 \), implying that it is optimal for the agent to pledge.

Next, consider the case in which fundamentals are intermediate, i.e., \( \theta \in [5/6,7/6] \). In this case, each agent with a signal equal to \( x^L \) assigns probability \( 2/3 > c \) to \( \theta \geq 1 \) and hence finds it dominant to pledge. Because, for any \( \theta \in (5/6,1) \), \( 1/3 \) of the agents receives an \( x^L \) signal, the minimal size of the pledge that each agent with signal equal to \( x^H \) can expect at any \( \theta \in (5/6,1) \) is thus equal to \( Pr(x^L|\theta) = 1/3 > 1 - \theta \), implying that even if all agents with signal equal to \( x^H \) were to refrain from pledging, the bank would survive. Because of the above properties, pledging is iteratively dominant for those agents receiving the \( x^H \) signal.

Hence, under the proposed policy, default does not occur for any \( \theta > 0 \). Because, under MARP, all agents pledge when they hear that the bank passed the test, no matter whether they hear that the fundamentals are extreme or intermediate, one may conjecture that the policy maker could refrain from specifying whether the fundamentals are extreme or intermediate and simply announce that the bank passed the test. However, as explained above, such a simple pass/fail policy would not induce all agents to pledge when playing according to MARP. □
The above example illustrates both the failure of the Revelation Principle (when the market is expected to play according to MARP, it is *with* loss of generality to confine attention to policies that take the form of action recommendations), as well as the sub-optimality of simple pass/fail tests, when beliefs and fundamentals do not co-move according to the monotone likelihood ratio property.\(^{27}\)

### 3.3 Monotone rules

We now turn to the optimality of policies that fail with certainty institutions with weak fundamentals and pass with certainty those with strong fundamentals. Let

\[
\bar{x}_G \equiv \sup \left\{ x \in \mathbb{R} : \int_{\Theta} u(\theta, 1 - P(x|\theta)) \mathbb{I}(\theta \geq 0) p(x|\theta) dF(\theta) \leq 0 \right\}
\]

denote the largest signal threshold \(x\) such that, when each agent pledges when receiving a signal above \(x\) and does not pledge when receiving a signal below \(x\), then the expected payoff from pledging for the marginal agent with signal \(x\) is non-positive under the additional information that \(\theta\) is non-negative. As we show in the Appendix, \(\bar{x}_G\) corresponds to an upper bound for the set of cut-offs characterizing the strategies consistent with MARP across all disclosure policies \(\Gamma\) satisfying the perfect coordination property.

Next, for any \(\theta \in (0, 1)\), let \(x^*(\theta)\) be the critical signal threshold such that, when agents follow a cut-off strategy with threshold \(x^*(\theta)\) (that is, pledge for \(x > x^*(\theta)\) and refrain from pledging for \(x < x^*(\theta)\)), then default occurs if and only if the fundamentals are below \(\theta\).\(^{28}\) For any \(\theta \in (0, 1)\), the threshold \(x^*(\theta)\) is implicitly defined by

\[
P(x^*(\theta)|\theta) = \theta. \tag{2}
\]

Let

\[
\theta^* \equiv \inf \left\{ \tilde{\theta} \geq 0 : \int_{\tilde{\theta}}^{\infty} u(\tilde{\theta}, 1 - P(x^*(\theta)|\tilde{\theta})) p\left(x^*(\theta)|\tilde{\theta}\right) dF(\tilde{\theta}) \geq 0 \quad \text{for all} \quad \theta \in \left[\tilde{\theta}, 1\right) \right\} \tag{3}
\]

be the lowest truncation point \(\tilde{\theta}\) such that, when the policy reveals that fundamentals are above \(\tilde{\theta}\), then for any possible default threshold \(\theta \in \left[\tilde{\theta}, 1\right)\), if default were to occur for fundamentals below \(\theta\) and not occur for fundamentals above \(\theta\), then the marginal agent with signal \(x^*(\theta)\) would find it optimal to pledge. Finally, for any \(x\), let \(\Theta(x) \equiv \{ \theta \in \Theta : x \in \Theta \} \) denote the set of fundamentals that, given the distribution \(P(\cdot|\theta)\) from which the agents’ exogenous signals are drawn, are consistent with signal \(x\).

---

\(^{27}\)The reader may wonder whether the optimality of pass/fail policies under MARP and MLRP is an artifact of the fact that each agent has only two actions. Clearly, the result continues to hold when actions are taken from \([0, 1]\) and payoffs are linear in \(a\) and given by \([g(\theta)](A > 1 - \theta) + b(\theta)](A \leq 1 - \theta)]a\), with \(A\) denoting the integral of the agents’ actions.

\(^{28}\)When the noise in the agents’ signals is bounded, the definition of \(x^*(\theta)\) can be extended to \(\theta = 0\) and \(\theta = 1\). When the noise is unbounded, abusing notation, one can extend the definition to \(\theta = 0\) and \(\theta = 1\) by letting \(x^*(0) = -\infty\) and \(x^*(1) = +\infty\).
Condition M. The following properties hold:

1. \( \inf \Theta(\bar{x}_G) \leq 0; \)

2. the functions \( p(x|\theta) \) and \( |u(\theta, 1 - P(x|\theta))| \) are log-supermodular, respectively, over \( \mathbb{R}^2 \) and\(^{29} \)

\[
\{(\theta, x) \in [0,1] \times \mathbb{R} : u(\theta, 1 - P(x|\theta)) \leq 0\};
\]

3. for any \( \theta_0, \theta_1 \in [0,1] \), with \( \theta_0 < \theta_1 \), and any \( x \leq \bar{x}_G \) such that (a) \( u(\theta_1, 1 - P(x|\theta_1)) \leq 0 \) and (b) \( x \in g_{\theta_0} \),

\[
\frac{U^P(\theta_1, 1) - U^P(\theta_1, 0)}{U^P(\theta_0, 1) - U^P(\theta_0, 0)} \geq \frac{p(x|\theta_1) u(\theta_1, 1 - P(x|\theta_1))}{p(x|\theta_0) u(\theta_0, 1 - P(x|\theta_0))}.
\]

Property 1 in Condition M says that the lower bound of the support of the beliefs of the marginal agent with signal \( \bar{x}_G \), where \( \bar{x}_G \) is the threshold defined in (1), is not strictly positive. Clearly, this property trivially holds when, for any \( \theta \), the agents' signals are drawn from a distribution whose support is large enough (and hence, a fortiori, when the noise in the agents' signals is drawn from a distribution with unbounded support, e.g., a Normal distribution).

Property 2 says that signals are ordered according to MLRP and that the (percentage) reduction in the agents' loss from pledging due to higher fundamentals is larger when more agents pledge. Formally, for any \( \theta' < \theta'' \) and \( x' < x'' \) such that \( u(\theta'', 1 - P(x'|\theta'')) < 0 \),

\[
\frac{u(\theta'', 1 - P(x'|\theta'')) - u(\theta'', 1 - P(x''|\theta''))}{u(\theta', 1 - P(x''|\theta''))} \leq \frac{u(\theta', 1 - P(x'|\theta')) - u(\theta'', 1 - P(x''|\theta''))}{u(\theta', 1 - P(x'|\theta'))}.
\]

Note that \( u(\theta'', 1 - P(x'|\theta'')) < 0 \) implies that \( u(\theta', 1 - P(x'|\theta')), u(\theta', 1 - P(x''|\theta'')), u(\theta'', 1 - P(x''|\theta'')) < 0 \). The left-hand side of (5) is thus the percentage reduction in the agents’ payoff loss when fundamentals improve from \( \theta' \) to \( \theta'' \) and agents pledge when receiving a signal \( x \geq x'' \). The right-hand side of (5), instead, is the percentage reduction in the agents’ payoff loss when fundamentals improve from \( \theta' \) to \( \theta'' \) and agents pledge for \( x \geq x' \). Importantly, this property is required to hold only for fundamentals \( \theta \) and thresholds \( x \) for which the agents’ expected payoffs from pledging, \( u(\theta, 1 - P(x|\theta)) \), is non-positive. Also note that this property trivially holds in the baseline model considered so far, for payoffs \( u(\theta, A) \) are invariant in \( A \) conditional on the fate of the bank. The reason for stating the condition in these more general terms is that, as we show in Section S5 in the Online Supplement, Condition M above plays a key role for the optimality of monotone policies also under richer payoff specifications in which \( u(\theta, A) \) depends on \( A \) over and above the effect that the latter has on the default outcome.

\(^{29}\)That \( |u(\theta, 1 - P(x|\theta))| \) is log-supermodular over \( \{(\theta, x) \in [0,1] \times \mathbb{R} : u(\theta, 1 - P(x|\theta)) \leq 0\} \) means that, for any \( x', x'' \in \mathbb{R} \), with \( x' < x'' \), and any \( \theta', \theta'' \in \Theta \), with \( \theta'' > \theta' \), such that \( u(\theta'', 1 - P(x''|\theta'')) \leq 0 \),

\[
u(\theta'', 1 - P(x''|\theta'')) u(\theta', 1 - P(x'|\theta')) \geq u(\theta'', 1 - P(x'|\theta')) u(\theta', 1 - P(x''|\theta')).
\]
Property 3 in turn says that the benefit that the policy maker derives from changing the agents’ behavior (inducing all agents to pledge starting from a situation in which no agent pledges) increases with the fundamentals at a sufficiently high rate, with the critical rate determined by a combination of the agents’ payoffs and beliefs (the right-hand-side of (4)). Such a property is required to hold only for fundamentals \( \theta_0 \) and \( \theta_1 \) in the critical region and for signal realizations \( x \leq \bar{x}_G \) such that
\[
(a) \quad u(\theta_1, 1 - P(x|\theta_1)) \leq 0 \quad \text{(meaning that the payoff from pledging is negative when agents pledge for signals above } x \text{ and refrain from pledging for signals below } x),
\]
and (b) \( x \in \varrho_{\theta_0} \) (meaning that signal \( x \) is consistent with the fundamentals being \( \theta_0 \)). Also note that, in the simple model of Section 2, the right-hand side of (4) is equal to \( p(x|\theta_1) b(\theta_1) / p(x|\theta_0) b(\theta_0) \). Once again, the reason for the more general condition is that the result in Theorem 3 below extends to richer payoff specifications, as we show in the Online Supplement.

We then have the following result:

**Theorem 3.** Suppose Condition M holds. Given any policy \( \Gamma \), there exists a deterministic monotone policy \( \Gamma^\hat{\theta} = ([0, 1], \pi^\hat{\theta}) \) satisfying the perfect-coordination property and yielding the policy maker a payoff weakly higher than \( \Gamma \). The policy \( \Gamma^\hat{\theta} \) is such that there exists a threshold \( \hat{\theta} \in [0, 1] \) such that, for any \( \theta \leq \hat{\theta} \), \( \pi^\theta(\theta) \) assigns probability one to \( s = 0 \), whereas for any \( \theta > \hat{\theta} \), \( \pi^\theta(\theta) \) assigns probability one to \( s = 1 \).

**Proof of Theorem 3:** See the Appendix.

When Condition M holds, the choice of the optimal policy reduces to the choice of the smallest threshold \( \hat{\theta} \) such that, when agents commonly learn that \( \theta > \hat{\theta} \), under the unique rationalizable profile, all agents pledge, irrespective of their exogenous private information. For this to be the case, it must be that, for any \( x \in \mathbb{R} \), \( \int_{\hat{\theta}}^\infty u(\theta, 1 - P(x|\theta)) dF(\theta) > 0 \). The above problem, however, does not have a formal solution, due to the lack of upper-hemicontinuity of the designer’s payoff in \( \hat{\theta} \).

Notwithstanding these complications, hereafter we follow the pertinent literature and refer to the monotone policy with cut-off \( \hat{\theta} = \theta^* \), with \( \theta^* \) as defined in (3), as the “optimal monotone policy”.

As we show in the Appendix, property 1 in Condition M guarantees that, starting from the optimal monotone policy (the one with cut-off \( \theta^* \)), one cannot perturb the policy by assigning a pass grade also to a small interval \( [\theta', \theta''] \) of fundamentals with \( 0 \leq \theta' < \theta'' < \theta^* \), while guaranteeing that all agents necessarily pledge when hearing that the bank passed the test (i.e., when hearing
that $s = 1$). This property trivially holds when the noise in the agents’ signals is large (and hence, a fortiori, when noise is unbounded), but plays a key role when the noise is drawn from a bounded interval of small size (see Example 2 below for an illustration).

Properties 2 and 3 of Condition M in turn guarantee that, given a non-monotone rule, perturbations of the original policy that swap the probability of inducing all agents to pledge from low to high fundamentals in a way that preserves the agents’ incentives to pledge (under MARP) when hearing that $s = 1$, increase the policy maker’s payoff. These conditions guarantee that the higher value the policy maker derives from saving banks with stronger fundamentals compensates for the possibility that, from an ex-ante perspective, the probability of default may be larger under monotone policies than under non-monotone ones (see Example 3 for an illustration of why non-monotone rules may permit the policy maker to save a larger set of banks).

Condition M is fairly sharp in the sense that, when violated, one can construct examples where the optimal policy is non-monotone. We provide two such examples below. Example 2 illustrates the role of property 1 in Condition M, whereas Example 3 illustrates the role of properties 2 and 3 in Condition M.

Let $\theta^{MS} \in (0, 1)$ be implicitly defined by the unique solution to

$$\int_0^1 u(\theta^{MS}, A)dA = 0. \quad (6)$$

The threshold $\theta^{MS}$ corresponds to the value of the fundamentals at which an agent who knows $\theta$ and holds Laplacian beliefs with respect to the measure of agents pledging is indifferent between pledging and not pledging.\textsuperscript{32} Importantly, $\theta^{MS}$ is independent of the initial common prior $F$ and of the distribution of the agents’ signals.

Example 2. Suppose that there exist scalars $g, b, W, L \in \mathbb{R}$, with $g > 0 > b$ and $W > L$, such that, for any $\theta$, $g(\theta) = g$, $b(\theta) = b$, $W(\theta) = W$, and $L(\theta) = L$. Suppose that $\theta$ is drawn from a uniform distribution with support $[-K, 1 + K]$, for some $K \in \mathbb{R}^+$. Finally, assume that the agents’ exogenous signals are given by $x_i = \theta + \sigma \epsilon_i$, with $\sigma \in \mathbb{R}^+$ and with each $\epsilon_i$ drawn independently across agents from a uniform distribution over $[-1, 1]$, with $\sigma < K/2$. Let $\theta^{*}_\sigma$ be the threshold defined in (3), applied to the primitives described in this example (with the subscript $\sigma$ used to emphasize the dependence on the noise in the agents’ exogenous signals). There exists $\sigma^\# \in (0, K/2)$ such that (a) $\inf \Theta(x^{*}_{\sigma^\#}(\theta^{MS})) > 0$, and (b) for all $\sigma \in (0, \sigma^\#)$, starting from the optimal monotone policy with cut-off $\theta^{*}_\sigma$ (the one saving the largest set of fundamentals over all monotone rules), there exists a deterministic non-monotone policy satisfying the perfect-coordination property and sparing the banks from default over a set of fundamentals of strictly larger probability measure than the optimal monotone policy (and hence yielding the policy maker a strictly higher payoff).

\textsuperscript{32}This means that the agent believes that the proportion of agents pledging is uniformly distributed over $[0, 1]$. See Morris and Shin (2006).
Proof of Example 2: The formal proof is in the Online Supplement. Here we sketch the key arguments for why monotone policies are not optimal under the specification considered in the example. To fix ideas, let $g = 1 - c$ and $b = -c$, with $c \in (0, 1)$, as in Example 1. For any $\theta \in [0, 1]$, let $x^*_\sigma(\theta)$ be the critical signal threshold such that, when the quality of the agents’ exogenous information is $\sigma$ and all agents pledge for $x > x^*_\sigma(\theta)$ and refrain from pledging for $x < x^*_\sigma(\theta)$, a bank survives if its fundamentals exceed $\theta$ and defaults otherwise, as defined in (2). Note that, under the specification in this example $x^*_\sigma(\theta) = (1 + 2\sigma)\theta - \sigma$. For any binary policy $\Gamma = (\{0, 1\}, \pi)$, any quality of the agents’ exogenous information $\sigma$, and any threshold $\theta \in [0, 1]$ such that $(x^*_\sigma(\theta), 1)$ are mutually consistent, let $V^\Gamma_{\sigma, \sigma}(\theta)$ be the payoff of the marginal agent with signal $x^*_\sigma(\theta)$ when each agent with signal below $x^*_\sigma(\theta)$ refrains from pledging and each agent with signal above $x^*_\sigma(\theta)$ pledges, after the policy $\Gamma$ announces that $s = 1$. That is,

$$V^\Gamma_{\sigma, \sigma}(\theta) \equiv U^\Gamma_{\sigma, \sigma}(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)),$$

where $U^\Gamma_{\sigma, \sigma}$ is the function defined after Theorem 2.

Now, for any $\hat{\theta} \in \Theta$, let $\Gamma^{\hat{\theta}} = (\{0, 1\}, \pi^{\hat{\theta}})$ be the deterministic monotone policy with cut-off $\hat{\theta}$ (that is, the policy that discloses $s = 1$ with certainty when fundamentals are above $\hat{\theta}$ and discloses $s = 0$ with certainty when fundamentals are below $\hat{\theta}$). Note that the absence of any public disclosure is equivalent to a monotone policy with cut-off $\hat{\theta} = \min \Theta = -K$ and that, under such a policy, the default threshold is $\theta^{MS} = c$.\footnote{To see this, observe that, for any $\theta \in [0, 1]$, $V^{\Gamma^{\min}}_{\sigma, \sigma}(\theta) = \text{Pr} \left( \theta > x^*_\sigma(\theta) \right) - c = \theta - c$, which is strictly positive for $\theta > \theta^{MS} = c$ and strictly negative for $\theta < \theta^{MS}$. Hence, in the absence of any public disclosure, the unique rationalizable profile features all agents pledging for $x > x^*_\sigma(\theta^{MS})$ and all agents refraining from pledging for $x < x^*_\sigma(\theta^{MS})$.}

Compared to a situation in which the policy reveals no information, the knowledge that the
fundamentals are above a threshold \( \hat{\theta} \in [0, 1] \), other things equal, increases the payoff of the marginal agent from pledging. However, because the noise in the agents’ signals is bounded, the announcement that fundamentals are above \( \hat{\theta} \) has a bite on the marginal agent’s payoff only insofar \( \theta \leq (\hat{\theta} + 2\sigma)/(1 + 2\sigma) \). In fact, when \( \theta > (\hat{\theta} + 2\sigma)/(1 + 2\sigma) \), \( x^*_\sigma(\theta) - \sigma > \hat{\theta} \) meaning that the marginal agent with signal \( x^*_\sigma(\theta) \) already knows that fundamentals are above \( \hat{\theta} \) from the observation of his own signal and thus learns nothing from learning that the bank passed the test (i.e., that fundamentals are above \( \hat{\theta} \)).

A necessary and sufficient condition for all agents to pledge under MARP associated with a monotone policy \( \Gamma^{\hat{\theta}} \) is that, for any possible default threshold \( \theta > \hat{\theta}, V^{\theta\hat{\theta}}_\sigma(\theta) > 0 \).\(^{35}\) This implies that the cut-off \( \theta^*_\sigma \) for the optimal deterministic monotone rule is \( \theta^*_\sigma = x^*_\sigma(\theta^{MS}) - \sigma \).\(^{36}\)

Now to see that the above monotone policy is improvable, assume that \( \sigma \) is small so that \( \inf \Theta(x^*_\sigma(\theta^{MS})) = x^*_\sigma(\theta^{MS}) - \sigma > 0 \). Next, pick \( \gamma, \delta > 0 \) small and let \( \theta'' \equiv x^*_\sigma(\theta^{MS} - \delta) - \sigma \) and \( \theta' \equiv \theta'' - \gamma \). That inf(\( x^*_\sigma(\theta^{MS}) \)) > 0 implies that it is possible to find \( \delta, \gamma > 0 \) small so that \( \theta'' \) and \( \theta' \) are non-negative. Next, consider a policy \( \Gamma_{\gamma,\delta} = (\{0, 1\}, \pi_{\gamma,\delta}) \) that, in addition to passing all banks with fundamentals above \( \theta^*_\sigma \), also passes those with fundamentals \( \theta \in [\theta', \theta''] \).\(^{37}\) Let \( V^{\Gamma_{\gamma,\delta}}_\sigma(\theta) \) be the payoff of the marginal agent with signal \( x^*_\sigma(\theta) \) when the policy \( \Gamma_{\gamma,\delta} \) announces that \( s = 1 \), thus revealing that fundamentals belong to \( [\theta', \theta''] \cup [\theta^*_\sigma, 1 + K] \). This payoff is represented in Figure 2 along with the payoff \( V^{\tilde{\Gamma}}_\sigma(\theta) \) under the optimal monotone rule. Provided that \( \gamma \) and \( \delta \) are small, \( V^{\Gamma_{\gamma,\delta}}_\sigma(\theta) \geq 0 \) for all \( \theta \in [\theta'/ (1 + 2\sigma), 1] \), with \( V^{\Gamma_{\gamma,\delta}}_\sigma(\theta) = 0 \) if and only if \( \theta = \theta^{MS} \).\(^{38}\) Starting from \( \Gamma_{\gamma,\delta} \), one can then further perturb the policy \( \Gamma_{\gamma,\delta} \) by giving a pass grade with certainty to banks with fundamentals in \( [\theta', \theta''] \cup [\theta^*_\sigma + \varepsilon, +\infty) \), with \( \varepsilon > 0 \) but small, and failing with certainty the others. The new policy \( \tilde{\Gamma} \) so constructed is such \( V^{\tilde{\Gamma}}_\sigma(\theta) > 0 \) for all \( \theta \in [\theta'/ (1 + 2\sigma), 1] \) meaning that, when \( \tilde{\Gamma} \) announces that \( s = 1 \), under MARP all agents pledge, no matter their signal. Hence the policy \( \tilde{\Gamma} \) so constructed satisfies the perfect-coordination property. That it strictly improves upon the original deterministic optimal monotone one follows directly from the fact that it spares a bank from default over a set of fundamentals of strictly larger probability measure. \( \square \)

The reason why the non-monotone policy \( \tilde{\Gamma} \) constructed in the proof of Example 2 spares more banks from default than the optimal deterministic monotone policy with threshold \( \theta^*_\sigma \) is that agents

\(^{35}\)Indeed, if there exists \( \theta > \hat{\theta} \) such that \( V^{\theta\hat{\theta}}_\sigma(\theta) \leq 0 \), then not pledging when receiving a signal below \( x^*_\sigma(\theta) \) is rationalizable. In this case, there must exist \( \theta' > \hat{\theta} \) such that \( V^{\theta\theta'}_\sigma(\theta') = 0 \). In the continuation game that starts after the announcement that \( s = 1 \), refraining from pledging for any \( x < x^*_\sigma(\theta') \) and pledging for any \( x > x^*_\sigma(\theta') \) is thus a continuation equilibrium. Because \( x^*_\sigma(\theta') > x^*_\sigma(\theta) \), it is then easy to see that not pledging for \( x < x^*_\sigma(\theta) \) is rationalizable.

\(^{36}\)As explained above, for any \( \theta > (\hat{\theta} + 2\sigma)/(1 + 2\sigma) \), the announcement that fundamentals are above \( \hat{\theta} \) has no effect on the payoff of the marginal agent with signal \( x^*_\sigma(\theta) \), meaning that \( V^{\mathcal{I},\hat{\theta}}_\sigma(\theta) = V^{\mathcal{I},\min}_\sigma(\theta) \). Because \( V^{\mathcal{I},\min}_\sigma(\theta) < 0 \) for \( \theta < \theta^{MS} \) we thus have that, for any \( \hat{\theta} < x^*_\sigma(\theta^{MS}) - \sigma \), and any \( \theta \in (\hat{\theta} + 2\sigma)/(1 + 2\sigma), \theta^{MS} \), \( V^{\mathcal{I},\hat{\theta}}_\sigma(\theta) = \theta - c < 0 \), meaning that not pledging is rationalizable for some agents.

\(^{37}\)Formally, \( \tau_{\gamma,\delta} \) discloses \( s = 1 \) with certainty when \( \theta \in [\theta', \theta''] \cup [\theta^*_\sigma, 1 + K] \) and discloses \( s = 0 \) with certainty when \( \theta \in (-K, \theta') \cup (\theta', \theta^*_\sigma) \).

\(^{38}\)For \( \theta < \theta'/ (1 + 2\sigma), x^*_\sigma(\theta) + \sigma < \theta' \), meaning that \( x^*_\sigma(\theta), 1 \) are not mutually consistent and hence the payoff function \( V^{\mathcal{I},\hat{\theta}}_\sigma(\theta) \) is not well defined.
receiving signals around $\theta^{MS}$ are highly sensitive to the grade the policy gives to banks with fundamentals around $\theta^{MS}$ but not so much so to the grade the policy gives to fundamentals far from $\theta^{MS}$. In the above example with bounded noise, an agent receiving a signal $x_i^*(\theta^{MS})$ is not sensitive at all to the grade the policy gives to a bank with fundamentals below $x_i^*(\theta^{MS}) - \sigma$ given that his private signal informs him that the fundamentals are above $x_i^*(\theta^{MS}) - \sigma$. Hence, while it is impossible to amend the optimal deterministic monotone policy (the one with cut-off $\theta^*_\sigma = x_i^*(\theta^{MS}) - \sigma$) by giving a pass grade also to banks with fundamentals slightly below $\theta^*_\sigma$ (say, with fundamentals $\theta \in [\theta^*_\sigma - \varepsilon, \theta^*_\sigma]$), without inducing some of the agents to refrain from pledging, it is possible to amend the optimal deterministic monotone policy by extending the pass grade to a small interval $[\theta', \theta'']$ of fundamentals with $0 \leq \theta' < \theta'' < \theta^*_\sigma$, while continuing to induce all agents to pledge under MARP. Property 1 in Condition M implies that $x_i^*(\theta^{MS}) - \sigma < 0$ thus making such perturbations unfeasible.

Next, consider the other two properties in Condition M. Given any policy $\Gamma = (\{0, 1\}, \pi)$ in which $\pi$ is deterministic (meaning that, for any $\theta$, $\pi(\theta)$ assigns probability 1 either to $s = 1$ or to $s = 0$), let $D^\Gamma = \{ (\bar{\theta}_i, \bar{\theta}_i) : i = 1, \ldots, N \}$ denote the partition of $\{0, \theta^{MS}\}$ induced by $\pi$, with $N \in \mathbb{N}$, $\bar{\theta}_1 = 0$, and $\theta_N = \theta^{MS}$.

Let $d \in D^\Gamma$ denote a generic cell of the partition $D^\Gamma$ and, for any $\theta \in [0, \theta^{MS}]$, denote by $d^\Gamma(\theta) \in D^\Gamma$ the cell that contains $\theta$. Finally, let $M(\Gamma) = \max_{i=1,\ldots,N} |\bar{\theta}_i - \bar{\theta}_i|$ denote the mesh of $D^\Gamma$, that is, the Lebesgue measure of the cell of $D^\Gamma$ of maximal Lebesgue measure.

The next example considers an economy in which the noise in the agents’ exogenous signals is drawn from an unbounded distribution (in which case, property 1 in Condition M trivially holds), but where properties 2 and 3 in Condition M are violated. It shows that any deterministic policy giving the same grade to an interval of types to the left of $\theta^{MS}$ of measure larger than $\mathcal{E}(\sigma)$, with $\mathcal{E}(\sigma)$ going to zero as $\sigma$ goes to zero, can be improved upon by a non-monotone deterministic policy with a smaller mesh. This property in turn implies that, when $\sigma$ is small, optimal policies are non-monotone.

**Example 3.** Suppose that $\theta$ is drawn from an improper uniform prior over $\mathbb{R}$ and that the agents’ signals are given by $x_i = \theta + \sigma \varepsilon_i$ with $\varepsilon_i$ drawn from a standard Normal distribution. Further assume that there exist scalars $g, b, W, L \in \mathbb{R}$, with $g > 0 > b$ and $W > L$, such that, for any $\theta$, $g(\theta) = g$, $b(\theta) = b$, $W(\theta) = W$ and $L(\theta) = L$. There exists a scalar $\bar{\sigma} > 0$ and a function $\mathcal{E} : (0, \bar{\sigma}] \rightarrow \mathbb{R}_+$, with $\lim_{\sigma \rightarrow 0^+} \mathcal{E}(\sigma) = 0$, such that, for any $\sigma \in (0, \bar{\sigma}]$, in the game in which the noise in the agents’ information is $\sigma$, the following is true: given any deterministic pass/fail policy $\Gamma = (\{0, 1\}, \pi)$ satisfying the perfect-coordination property and such that $M(\Gamma) > \mathcal{E}(\sigma)$, there

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39 That is, letting $\pi(\theta) = 0$ denote the Dirac distribution assigning probability one to $s = 0$ and $\pi(\theta) = 1$ be the Dirac distribution assigning measure one to $s = 1$, we have that $D^\Gamma$ is such that either (a) $\pi(\theta) = 0$ for all $\theta \in U_{i=2k,k=1,2,\ldots,N}(\bar{\theta}_i, \bar{\theta}_i)$ and $\pi(\theta) = 1$ for all $\theta \in U_{i=2k-1,k=1,2,\ldots,N}(\bar{\theta}_i, \bar{\theta}_i)$, or (b) $\pi(\theta) = 1$ for all $\theta \in U_{i=2k,k=1,2,\ldots,N}(\bar{\theta}_i, \bar{\theta}_i)$ and $\pi(\theta) = 0$ for all $\theta \in U_{i=2k-1,k=1,2,\ldots,N}(\bar{\theta}_i, \bar{\theta}_i)$.

40 That the prior is improper simplifies the exposition but is not important. Also note that the agents’ hierarchies of beliefs are well defined despite the improperness of the prior.
exists another deterministic pass/fail policy $\Gamma^*$ with $M(\Gamma^*) < E(\sigma)$ that also satisfies the perfect-coordination property and such that the ex-ante probability of default under $\Gamma^*$ is strictly smaller than under $\Gamma$ (and hence $\Gamma^*$ yields the policy maker a payoff strictly higher than $\Gamma$).

**Proof of Example 3:** The formal proof is in the Online Supplement. Here we sketch the key arguments. Heuristically, non-monotone policies permit the policy maker to save more banks than monotone policies by making it difficult for the agents to commonly learn the fundamentals when the latter are above 0 but below $\theta^{MS}$ and the policy maker announces that the bank passed the test. Intuitively, if the policy maker assigned a pass grade to an interval $[\theta', \theta''] \subset [0, \theta^{MS}]$ of large Lebesgue measure, when $\sigma$ is small and $\theta \in [\theta', \theta'']$, most agents would receive signals $x_i \in [\theta', \theta'']$. No matter the grade assigned to fundamentals outside the interval $[\theta', \theta'']$, in the continuation game that starts after the policy maker announces that the bank passed the test, most agents receiving signals $x_i \in [\theta', \theta'']$ would then assign high probability to the joint event that $\theta \in [\theta', \theta'']$, that other agents assign high probability to $\theta \in [\theta', \theta'']$, and so on. When this is the case, it is rationalizable for such agents to refrain from pledging. Hence, when $\sigma$ is small, the only way the policy maker can guarantee that, when $\theta \in [0, \theta^{MS}]$, the agents pledge after hearing that the bank passed the test is by dividing the subset of $[0, \theta^{MS}]$ receiving a pass grade into a collection of disjoint intervals, each of small Lebesgue measure.\footnote{Formally speaking, a highly non-monotone policy guarantees that the support of each agent’s posterior beliefs after hearing that the bank passed the test is not connected. Connectedness of the supports facilitates rationalizable profiles where some agents refrain from pledging.}

Next, suppose that the intervals $(\bar{\theta}_i, \check{\theta}_i) \subset (0, \theta^{MS})$, $i = 1, ..., N$, receiving a pass grade were far apart, implying that the policy maker fails an interval $[\theta', \theta''] \subset (0, \theta^{MS})$ of large Lebesgue measure (note that this is indeed the case under the optimal monotone deterministic rule with cut-off $\theta^*$, as defined in (3)). The proof in the Online Supplement then shows that, starting from $\Gamma$, the policy maker could assign a pass grade to some types in the middle of $[\theta', \theta'']$ and a fail grade to some types to the right of $\theta''$, in such a way that (a) pledging continues to be the unique rationalizable action for all agents after hearing that the bank passed the test, and (b) the set of fundamentals receiving a pass grade under the new policy is strictly larger than under the original one. Formally, suppose that, starting from the original policy $\Gamma$, the policy maker assigns a pass grade to types $\theta \in [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \xi]$ and a fail grade to types $\theta \in [\theta'' + \delta/2, \theta'' + \delta]$, with $\xi$ and $\delta$ small and chosen so that the ex-ante probability of a pass grade is the same as under the original policy $\Gamma$. Now take any individual with signal $x < (\theta' + \theta'')/2$. Suppose that, under the original policy $\Gamma$, the individual pledges and rationalizes such behavior by expecting all individuals with signal above his to also pledge. When $\sigma$ is small, the individual then expects default to occur only for $\theta < x$. Because the new policy assigns a pass grade to fundamentals $\theta > (\theta' + \theta'')/2$ closer to the individual’s own signal than the original policy, and because such fundamentals are associated with no default, the individual’s incentives to pledge under the new policy are stronger than under the original one.

Next consider an individual with signal $x \geq \theta'' + \delta$. Suppose again that, under the original policy
Γ, such an individual pledges and rationalizes his behavior by expecting all individuals with signal higher than his to also pledge. When \( \sigma \) is small, such an individual expects the bank to default only for \( \theta < x \). Because the new policy assigns a pass grade to types \( \theta < x \) farther away from \( x \) than the original policy, and because such fundamentals are associated with default, the individual’s incentives to pledge are again stronger under the new policy than under the original one.

In the Online Supplement, we show that the above two properties in turn imply that, for those individuals with signals \( x \not\in [(\theta' + \theta'')/2, \theta'' + \delta] \), if pledging was the unique rationalizable action under the original policy \( \Gamma \) then pledging continues to be the unique rationalizable action under the new policy.

For those agents with signal \( x \in [(\theta' + \theta'')/2, \theta'' + \delta] \), instead, the incentives to pledge may be smaller under the new policy than under the original one. However, as we show in the Online Supplement, for such individuals pledging is the unique rationalizable action under small perturbations of the original policy. Hence, provided that \( \sigma, \xi, \delta \) are small, pledging is the unique rationalizable action for such individuals as well.

The policy maker can then extend the pass grade to some types to the left of \((\theta' + \theta'')/2\) and to some types to the right of \(\theta'' + \delta/2\) while guaranteeing that pledging after hearing that the bank passed the test continues to be the unique rationalizable action for all agents. The construction sketched above can be iterated till one arrives at a new policy with a mesh smaller than \( \mathcal{E}(\sigma) \). Because default under the new policy is smaller than under the original one, the new policy improves strictly over the original one.

Finally, one can show that, when \( \sigma \) is small, a pass grade can be given to all \( \theta > \theta^{MS} + \varepsilon \), with \( \varepsilon > 0 \) small, while guaranteeing that all agents pledge after hearing that the bank passed the test.\(^{42}\)

The above properties in turn imply that, under the specification considered in the example, if the policy maker is restricted to deterministic policies (arguably, the most relevant case in practice), as the quality of the agents’ exogenous information grows large, the optimal stress test converges to one that is maximally non-monotone over \([0, \theta^{MS})\) and that passes all banks with fundamentals above \( \theta^{MS} \).

3.3.1 Discussion: role of multiplicity of receivers and exogenous private information

It is worth contrasting the above results about the sub-optimality of monotone rules (when Condition M is violated) to those for economies featuring either a single privately-informed receiver, or multiple receivers with no exogenous private information.

**Single receiver.** In this case, the optimal stress test is a simple monotone pass/fail policy with cut-off equal to \( \theta^* = 0 \). This is because, in this model, the policy maker’s and the receiver’s payoffs

\(^{42}\)Formally, for any \( \varepsilon > 0 \), there exists \( \sigma(\varepsilon) \) such that, for any \( \sigma < \sigma(\varepsilon) \), given any pass/fail policy \( \Gamma \) satisfying the perfect-coordination property, there exists another pass/fail policy \( \Gamma' \) also satisfying the perfect-coordination property that agrees with \( \Gamma \) on any \( \theta < \theta^{MS} \) and passes with certainty all banks with fundamentals \( \theta \geq \theta^{MS} + \varepsilon \).
are aligned (they both want to avoid default when possible). Things are different when preferences are misaligned. To see this, suppose the policy maker’s payoff is equal to \( W \) in case the bank avoids default, and \( L \) in case of default, with \( W > L \) as in Examples 2 and 3 above. However, now suppose that the receiver’s payoff differential between pledging and not pledging is equal to \(-g\) in case of default and \(-b\) in case of no default, with \( g > 0 > b \). Such a payoff differential may reflect the idea that the receiver is a speculator whose payoff is equal to zero when he refrains from speculating (equivalently, when he pledges). When, instead, he speculates, his payoff is positive in case speculation leads to default but negative in case the bank survives the speculative attack. Using the results in Guo and Shmaya (2019), one can then show that the optimal stress test in this case has the interval structure: each type \( x \) of the receiver is induced to play the action favorable to the policy maker (abstain from speculating) over an interval of fundamentals \([\theta_1(x), \theta_2(x)]\), with \( \theta_1(x) < 1 < \theta_2(x) \), for all \( x \), and with \( \theta_1(x) \) decreasing in \( x \) and \( \theta_2(x) \) increasing in \( x \). Such a policy requires disclosing more than two signals and hence cannot be implemented through a simple pass/fail test. In contrast, with a continuum of heterogeneously informed receivers with the same payoffs as in the variant above, the optimal stress is a pass-fail policy that is typically non-monotone in \( \theta \), as shown in Examples 2 and 3 above.\footnote{This is because, under MARP, all agents play the friendly action if and only if it is iteratively dominant for them to do so, irrespective of the alignment in payoffs.} Furthermore, when the optimal policy is not monotone, it does not have an interval structure, as each receiver with signal \( x \) is induced to pledge over a non-connected set of fundamentals. The reason for these differences is that, with a single receiver, to avoid an attack, the policy maker must persuade the receiver that the fundamentals are likely to be above 1, in which case the attack is unsuccessful. With multiple receivers, instead, the policy maker must persuade each receiver that enough other receivers are not attacking, which, as shown above, is best accomplished by a non-monotone policy that makes it difficult for the receivers to commonly learn the fundamentals, when the latter are above 0 but below \( \theta^{MS} \).

**Multiple receivers with no exogenous private information.** Because all receivers have the same posterior beliefs, under MARP, each of them plays the friendly action only if it is dominant to do so. The optimal policy is then again a simple monotone pass/fail policy with cut-off equal to \( \theta^* = 0 \) in case preferences are aligned and equal to some value \( \theta^* \in (0,1) \) in case they are mis-aligned. The reason why the optimal policy is monotone when the receivers possess no exogenous private information is that the policy maker needs to convince each of them that \( \theta \) is above 1 with sufficiently high probability to induce them to play the friendly action. Interestingly, when the receivers possess no exogenous private information, the optimality of monotone rules extends to economies in which the policy maker can disclose different information to different receivers, as shown in Li et al. (2021) and Morris et al. (2020).
4 Extensions, micro-foundations, and comparative statics

In the Online Supplement, we show how the results in Theorems 1-3 above extend to richer economies and can be used to study the comparative statics of optimal stress tests in a family of fully micro-founded economies. In this section, we briefly review these points.

4.1 Generalizations

Assume that the fundamentals are given by \((\theta, z)\), with the two variables imperfectly correlated. The variable \(\theta\) continues to parametrize the maximal information the policy maker can collect about the fundamentals. The additional variable \(z\) parametrizes risk that the agents and the policy maker face at the time of the stress test (e.g., macroeconomic variables that are only imperfectly correlated with the bank's fundamentals, and/or the exogenous supply of funds to the bank from sources other than the agents under consideration. Given \(\theta\) and \(A\), default occurs if, and only if, \(R(\theta, A, z) \leq 0\), with the function \(R\) continuous, strictly increasing in \((\theta, A, z)\), and such that \(R(\bar{\theta}, 1, z) = R(\bar{\theta}, 0, z) = 0\), for some \(\theta, \bar{\theta} \in \mathbb{R}\), with \(\theta < \bar{\theta}\). The function \(R\) thus implicitly defines the critical size of the pledge necessary for the bank to avoid default. The policy maker’s payoff is equal to

\[
\hat{U}^P(\theta, A, z) = \begin{cases} 
\hat{W}(\theta, A, z) & \text{if } R(\theta, A, z) > 0 \\
\hat{L}(\theta, A, z) & \text{if } R(\theta, A, z) \leq 0.
\end{cases}
\] (7)

whereas the agents’ payoff differential between playing the “friendly” action (pledging to the bank, or abstaining from speculating against it) and the “adversarial” action (refusing to pledge, or speculating against the bank) is equal to

\[
\hat{u}(\theta, A, z) = \begin{cases} 
\hat{g}(\theta, A, z) & \text{if } R(\theta, A, z) > 0 \\
\hat{b}(\theta, A, z) & \text{if } R(\theta, A, z) \leq 0.
\end{cases}
\]

We identify conditions on the agents’ payoffs such that, for any \(\Gamma\), MARP (a) continues to coincide with the “smallest” rationalizable profile and (b), when the agent’s signals satisfy MLRP, is in monotone (i.e., cut-off) strategies.

Next, we identify a condition on the policy maker’s payoff such that the optimal policy continues to satisfy PCP, thus generalizing Theorem 1 above. Roughly, the condition is that the loss to the policy maker from having no agent pledging in those states in which, under the policy \(\Gamma\), the agents’ expected payoff differential is negative is more than compensated by the benefit from having all agents pledging in those states in which their expected payoff differential is positive. The condition, which is trivially satisfied when \(W\) and \(L\) do not depend on \(A\) as in the baseline model, thus requires that the policy maker’s and the agents’ payoffs be not too misaligned.

Next, we show that the same condition guarantees that the optimal policy takes a pass/fail form when paired with the conditions on the agents’ payoffs that guarantee the monotonicity of their strategies, thus generalizing Theorem 2.
Finally, we show that, when, in addition to the conditions guaranteeing the optimality of pass/fail policies satisfying PCP, an analog of Condition M in the previous section holds, the optimal policy is monotone and deterministic in the component of the fundamentals $\theta$ that is measurable at the time of the stress test, thus generalizing Theorem 3 above.

4.2 Micro-foundations and comparative statics

In the Online Supplement, we use the generalizations above to study the effects of variations in the quality of market information on the toughness of the optimal stress test. We do so by considering a family of fully micro-founded economies in which banks issue equity (alternatively, debt) to finance their short-term obligations and where the price of the securities is endogenous and influenced by the outcome of the stress test. Formally, we investigate how the critical threshold $\theta^*_\sigma$ defining the optimal monotone stress tests is affected by variations in the parameter $\sigma$ in the agents’ signals $x_i = \theta + \sigma \epsilon_i$ that proxies the quality of the agents’ exogenous private information.

We show that more risk (a higher $\sigma$) leads to a reduction in the toughness of the optimal stress test when the banks finance themselves with equity and to an increase in the toughness of the optimal stress test when they finance themselves with debt.

Intuitively, the reason why risk can be beneficial to the banks in case of equity financing but not in case of debt financing is as follows. Under equity financing, investors are exposed to variations in fundamentals primarily through upside risk. Their payoff differential (between purchasing and short-selling equity) is increasing in $\theta$ in case of no default and is constant in case of default, reflecting the fact that most other claims have seniority over equity in case of default. Provided that the price of equity does not vary much with $(\theta, z)$, an increase in risk then makes investors more willing to purchase equity. The policy maker can then decrease the critical threshold $\theta^*$ below which she fails the bank while guaranteeing that, after announcing that the bank passed the test, the unique rationalizable profile continues to feature all investors pledging by purchasing equity.

Under debt financing, instead, investors are exposed to variations in fundamentals primarily through downside risk. When the liquidation value is increasing in $\theta$ and the price of debt is not too sensitive to $(\theta, z)$, in a sense made precise in the Online Supplement, the investors’ payoff differential (between purchasing and short-selling debt) is increasing in $\theta$ in case of default (reflecting the seniority of debt obligations) but constant in fundamentals in case the bank survives. An increase in risk then makes investors less willing to pledge. The policy maker must then increase the critical threshold $\theta^*$ below which she fails the bank if she wants to guarantee that, after announcing that the bank passed the test, the unique rationalizable profile features all investors pledging by purchasing the newly issued debt.

That risk is beneficial to the bank in case of equity financing but detrimental in case of debt financing need not extend to alternative specifications of the investors’ payoffs under the two securities. What appears to be true more generally is the following single-crossing property. Whenever
more risk is beneficial to the bank in case of debt financing, the same tends to be true under equity financing.

5 Conclusions

We consider the design of optimal persuasion policies in coordination settings in which the receivers cannot be trusted to play favorably to the designer (e.g., pledging to a solvent but illiquid bank). We show that the optimal policy completely removes any strategic uncertainty, while retaining structural uncertainty: each agent can perfectly predict the actions of any other agent, but not the beliefs that rationalize such actions. We identify conditions under which the optimal policy has a pass/fail structure, as well as conditions under which the optimal policy is monotone, passing with certainty institutions with strong fundamentals and failing the others.

The results are worth extending in a few directions. The analysis assumes the policy maker knows how the distribution of market beliefs co-moves with the banks’ fundamentals. Such knowledge may come from previous experience with banks of similar characteristics, polls, data on professional forecasters, the IOWA betting markets, and the like. While this is a natural starting point, in future work it would be interesting to investigate how the structure of the optimal policy is affected by the policy maker’s ambiguity about the joint distribution of the underlying fundamentals and market beliefs.44

The analysis in the present paper is static. Many applications of interest are intrinsically dynamic, with agents coordinating on multiple attacks and learning over time (see the discussion in Angeletos et al. (2007)). In future work, it would be interesting to consider dynamic extensions and investigate how the timing of information disclosures is affected by the agents’ behavior in previous periods.45

Finally, the analysis is conducted by assuming that the maximal information that the designer can collect about the fundamentals (in the paper, \( \theta \)) is exogenous. In future work, it would be interesting to accommodate for the possibility that part of the information is provided by the banks themselves. This creates an interesting screening+persuasion problem in the spirit of what is examined in the literature on privacy in sequential contacting (e.g., Calzolari and Pavan (2006a) Calzolari and Pavan (2006b), Dworczak (2020), and Inostroza (2021)).

Appendix

Proof of Theorem 1. Given any regular policy \( \Gamma = (S, \pi) \) and any \( n \in \mathbb{N} \), let \( T_{(n)}^\Gamma \) be the set of strategies surviving \( n \) rounds of IDISDS, with \( T_{(0)}^\Gamma \) denoting the entire set of strategy profiles \( a = (a_i(\cdot))_{i \in [0,1]} \), where for any \( i \in [0,1] \), \( a_i(x, s) \) denotes the probability agent \( i \) pledges, given \( (x, s) \). Let \( a_{(n)}^\Gamma = \left( a_{(n),i}(\cdot) \right)_{i \in [0,1]} \in T_{(n)}^\Gamma \) denote the most aggressive profile surviving \( n \) rounds of IDISDS.

44 For some recent work in this direction, see Dworczak and Pavan (2021).

45 For some recent work in this direction, see Basak and Zhou (2020).
(that is, the profile in $T^\Gamma_{(n)}$ that minimizes the policy maker’s ex-ante payoff). The profiles $\left(a^\Gamma_{(n)}\right)_{n \in \mathbb{N}}$ can be constructed inductively as follows. The profile $a^\Gamma_{(0),i} \equiv \left(a^\Gamma_{(0),i}(\cdot)\right)_{i \in [0,1]}$ prescribes that all agents refrain from pledging, irrespective of $(x,s)$. Next, let $U^\Gamma_i(x_i,s;a)$ denote the payoff differential between pledging and not pledging for agent $i$ when, under $\Gamma$, all other agents follow the strategy in $a$. Then, $a^\Gamma_{(n),i}(x_i,s) = 0$ if $U^\Gamma_i\left(x_i,s;a^\Gamma_{(n-1)}\right) \leq 0$ and $a^\Gamma_{(n),i}(x_i,s) = 1$ if $U^\Gamma_i\left(x_i,s;a^\Gamma_{(n-1)}\right) > 0$. MARP consistent with $\Gamma$ is then the profile $\Gamma^\cdot = (a^\Gamma_{i}(\cdot))_{i \in [0,1]}$ given by $a^\Gamma_{i}(\cdot) = \lim_{n \to \infty} a^\Gamma_{(n),i}(\cdot)$, all $i \in [0,1]$.

Next, consider the policy $\Gamma^+ = (S^+, \pi^+)$, $S^+ \equiv S \times \{0,1\}$, that, for each $\theta$, draws the score $s$ from the same distribution $\pi(\theta) \in \Delta(S)$ as the original policy $\Gamma$, and then, for each $s$ it draws, it also announces the regime outcome $\Gamma(\theta,s)$ that would have prevailed at $\theta$ when agents play according to MARP consistent with $\Gamma$; that is, for any $\theta$, and any $s \in supp(\pi(\theta))$, it announces $(s, \Gamma(\theta,s))$.

Define $T^\Gamma_{(n)}$ and $a^\Gamma_{(n)}$ analogously to $T^\Gamma_{(n)}$ and $a^\Gamma_{(n)}$, but with respect to the policy $\Gamma^+$ so defined.

The proof is in three steps. Steps 1 and 2 show that any agent $i$ who, given $(x_i,s)$, finds it dominant (alternatively, iteratively dominant) to pledge under $\Gamma$ also finds it dominant (alternatively, iteratively dominant) to pledge under $\Gamma^+$ when receiving information $(x_i,(s,1))$. Step 3 uses the above property to establish that, because the game is supermodular and $a^\Gamma$ is “less aggressive” than $a^\Gamma^+$ (meaning that any agent who, given $(x,s)$, pledges under $a^\Gamma$ also pledges under $a^\Gamma^+$ when receiving the information $(x,(s,1))$, then, under $a^\Gamma^+$, all agents pledge (alternatively, refrain from pledging) when receiving information $(s,1)$ (alternatively, $(s,0)$).

**Step 1.** First, we prove that, for each $i \in [0,1]$, 

$$\{(x_i,s) : U^\Gamma_i(x_i,s;a) > 0 \forall a\} \subseteq \{(x_i,s) : U^\Gamma^+_i(x_i,(s,1);a) > 0 \forall a\}.$$ 

That is, any agent $i$ who, under $\Gamma$, finds it dominant to pledge, given the information $(x_i,s)$, also finds it dominant to pledge under $\Gamma^+$ when receiving the information $(x_i,(s,1))$.

To see this, first use the fact that the game is supermodular to observe that

$$\{(x_i,s) : U^\Gamma_i(x_i,s;a) > 0 \forall a\} = \{(x_i,s) : U^\Gamma_i(x_i,s;a^\Gamma_{(0)}) > 0\}$$

and

$$\{(x_i,s) : U^\Gamma^+_i(x_i,(s,1);a) > 0 \forall a\} = \{(x_i,s) : U^\Gamma^+_i(x_i,(s,1);a^\Gamma^+_{(0)}) > 0\}.$$ 

Now let $\Lambda^\Gamma_i(\theta,x|x_i,s)$ denote the beliefs of agent $i \in [0,1]$ over the fundamentals, $\theta$, and the cross-sectional distribution of signals, $x \in \mathbb{R}^{[0,1]}$, when receiving information $(x_i,s) \in \mathbb{R} \times S$ under $\Gamma$, and $\Lambda^\Gamma^+_i(\theta,x|x_i,(s,1))$ the corresponding beliefs under $\Gamma^+$. Bayesian updating implies that

$$\partial \Lambda^\Gamma^+_i(\theta,x|x_i,(s,1)) = \frac{\mathbb{I}(\Gamma(\theta,s) = 1)}{\Lambda^\Gamma_i(1|x_i,s)} \partial \Lambda^\Gamma_i(\theta,x|x_i,s), \quad (8)$$

where $\mathbb{I}(\Gamma(\theta,s) = 1)$ is the indicator function, taking value 1 if $\theta$ is such that $\Gamma(\theta,s) = 1$, and 0 otherwise, and where $\Lambda^\Gamma_i(1|x_i,s) \equiv \int_{\{(\theta,x):\Gamma(\theta,s)=1\}} \Lambda^\Gamma_i(d(\theta,x)|x_i,s)$.

Next, observe that, under both $a^\Gamma_{(0)}$ and $a^\Gamma^+_{(0)}$, default occurs if, and only if, $\theta \leq 1$. Take any $i \in [0,1]$ and $(x_i,s) \in \mathbb{R} \times S$ such that

$$U^\Gamma_i\left(x_i,s;a^\Gamma_{(0)}\right) = \int_{(\theta,x)} (b(\theta)\mathbb{I}(\theta \leq 1) + g(\theta)\mathbb{I}(\theta > 1)) \Lambda^\Gamma_i(d(\theta,x)|x_i,s) > 0. \quad (9)$$

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The aforementioned property of Bayesian updating implies that
\[ U_i^{\Gamma^+}(x_i, (s, 1); a_{(0)}^i) = \frac{1}{\Lambda_i^*(1|x_i,s)} \int_{\theta,x} (b(\theta) \mathbb{I}(\theta \leq 1) + g(\theta) \mathbb{I}(\theta > 1)) \mathbb{I}(r^\Gamma(\theta, s) = 1) \Lambda_i^*(d(\theta,x)|x_i,s) \]
\[ \geq \frac{1}{\Lambda_i^*(1|x_i,s)} \int_{\theta,x} (b(\theta) \mathbb{I}(\theta \leq 1) + g(\theta) \mathbb{I}(\theta > 1)) \Lambda_i^*(d(\theta,x)|x_i,s) = \frac{1}{\Lambda_i^*(1|x_i,s)} U_i^{\Gamma}(x_i, s; a_{(0)}^i) > 0, \]
where the first equality follows from (8), the first inequality from the fact that, for all \( \theta \) such that \( r^\Gamma(\theta, s) = 0, b(\theta) \mathbb{I}(\theta \leq 1) + g(\theta) \mathbb{I}(\theta > 1) = b(\theta) < 0 \), the second equality follows from the definition of \( U_i^{\Gamma}(x_i, s; a_{(0)}^i) \), and the second inequality from (9).

This means that any agent for whom pledging was dominant after receiving information \((x_i, s)\) under \( \Gamma \), continues to find it dominant to pledge after receiving information \((x_i, (s, 1))\) under \( \Gamma^+ \).

**Step 2.** Next, take any \( n > 1 \). Assume that, for any \( 1 \leq k \leq n - 1 \), any \( i \in [0, 1] \),
\[ \left\{ (x_i, s) : U_i^\Gamma(x_i, s; a) > 0 \, \forall a \in T_{(k-1)}^\Gamma \right\} \subseteq \left\{ (x_i, s) : U_i^{\Gamma^+}(x_i, (s, 1); a) > 0, \forall a \in T_{(k-1)}^{\Gamma^+} \right\}. \tag{10} \]
Arguments similar to those establishing the result in Step 1 above imply that
\[ \left\{ (x_i, s) : U_i^\Gamma(x_i, s; a) > 0 \, \forall a \in T_{(n-1)}^\Gamma \right\} \subseteq \left\{ (x_i, s) : U_i^{\Gamma^+}(x_i, (s, 1); a) > 0, \forall a \in T_{(n-1)}^{\Gamma^+} \right\}. \tag{11} \]
Intuitively, the result follows from the combination of the following two properties: (a) because the game is supermodular, \( \{ (x_i, s) : U_i^\Gamma(x_i, s; a) > 0 \, \forall a \in T_{(n-1)}^\Gamma \} \subseteq \{ (x_i, s) : U_i^{\Gamma^+}(x_i, s; a_{(n-1)}^i) > 0 \} \) where recall that \( a_{(n-1)}^\Gamma \) is the most aggressive profile surviving \( n - 1 \) rounds of IDISDS (clearly, the same property holds for \( \Gamma^+ \)); (b) \( a_{(n-1)}^{\Gamma^+} \) is “less aggressive” than \( a_{(n-1)}^\Gamma \), in the sense that any agent who, given \((x, s)\), pledges under \( a_{(n-1)}^\Gamma \) also pledges under \( \Gamma^+ \) when receiving information \((x, (s, 1))\); and (c) the observation that \( r^\Gamma(\theta, s) = 1 \) removes from the support of the agents’ posterior beliefs states in which default would have occurred under \( a^\Gamma \) and hence under \( a_{(n-1)}^\Gamma \) as well (observe that \( a_{(n-1)}^\Gamma \) is more aggressive than \( a^\Gamma \), meaning that any agent who, given \((x, s)\), pledges under \( a_{(n-1)}^\Gamma \); also pledges under \( a^\Gamma \) when receiving the same information \((x, 1))\).

**Step 3.** Equipped with the results in steps 1 and 2 above, we now prove that, for all \( \theta \in \Theta \) and all \( s \in \text{supp}(\pi(\theta)) \) such that \( r^\Gamma(\theta, s) = 1 \), for any \( x \in X(\theta) \), and any \( i \in [0, 1] \), \( a_{(n)}^{\Gamma^+}(x_i, (s, 1)) \equiv \lim_{n \to \infty} a_{(n),i}^{\Gamma^+}(x_i, (s, 1)) = 1 \). This follows directly from the fact that, as shown above,
\[ a_i^\Gamma(x_i, s) = 1 \Rightarrow a_i^{\Gamma^+}(x_i, (s, 1)) = 1. \tag{12} \]
The announcement that \( \theta \) is such that \( r^\Gamma(\theta, s) = 1 \) thus reveals to each agent that, when agents play according to \( a^{\Gamma^+} \), default does not occur. Because the payoff from pledging is strictly positive when default does not occur, any agent \( i \) receiving information \((s, 1)\) under \( \Gamma^+ \) thus necessarily pledges, no matter \( x_i \). Under the new policy \( \Gamma^+ \), all agents thus pledge when they learn that \( \theta \) is such that \( r^\Gamma(\theta, s) = 1 \). That they all refrain from pledging when they learn that \( \theta \) is such that \( r^\Gamma(\theta, 0) = 0 \) follows from the fact that such an announcement makes it common certainty that \( \theta \leq 1 \).

We conclude that the new policy \( \Gamma^+ \) satisfies the perfect-coordination property and is such that, for any \( \theta \), the probability of default under \( \Gamma^+ \) is the same as under \( \Gamma \). The result in the theorem then follows by taking \( \Gamma^* = \Gamma^+ \). Q.E.D.
Proof of Theorem 2. The proof is in 2 steps. Step 1 shows that, when \( p(x|\theta) \) is log-supermodular, i.e., it satisfies MLRP, then, irrespective of \( \Gamma \), MARP is in cut-off strategies. Step 2 then shows that, starting from any \( \Gamma \) satisfying the perfect-coordination property, one can drop any signal other than the predicted fate of the bank without changing the agents’ behavior.

Step 1. Fix an arbitrary policy \( \Gamma = (\mathcal{S}, \pi) \) and, for any pair \( (x, s) \in \mathbb{R} \times \mathcal{S} \), let \( \Lambda^\Gamma(x|s, \theta) \) represent the endogenous posterior beliefs over \( \Theta \) of each agent receiving exogenous information \( x \) and endogenous information \( s \).\(^{46}\) Let \( u(\theta, A) \equiv g(\theta)P(A > 1 - \theta) + b(\theta)P(A \leq 1 - \theta) \) be the payoff differential between pledging and not pledging when the fundamentals are \( \theta \) and the aggregate size of the pledge is \( A \).

Next, let \( U^\Gamma(x, s|k) \equiv \int u(\theta, 1 - P(k|\theta))d\Lambda^\Gamma(x|s, \theta) \) denote the expected payoff differential of an agent with information \( (x, s) \), when all other agents follow a cut-off strategy with cut-off \( k \) (i.e., they pledge if their private signal exceeds \( k \) and refrain from pledging if it is below \( k \)). The following result establishes that, when the distribution \( p(x|\theta) \) from which the signals are drawn satisfies MLRP, no matter \( \Gamma \), MARP is in cut-off strategies:

**Lemma 1.** Suppose that \( p(x|\theta) \) is log-supermodular. Given any policy \( \Gamma = (\mathcal{S}, \pi) \), for any \( s \in \mathcal{S} \), there exists \( \xi^\Gamma:s \in \mathbb{R} \) such that MARP consistent with \( \Gamma \) is given by the strategy profile \( a^\Gamma \equiv (a^\Gamma_i)_{i \in [0,1]} \) such that, for any \( s \in \mathcal{S} \), \( x \in \mathbb{R} \), \( i \in [0,1] \), \( a^\Gamma_i(x, s) = \mathbb{I}\{x > \xi^\Gamma:s\} \) with \( \xi^\Gamma:s \equiv \sup\{x : U^\Gamma(x, s|x) \leq 0\} \). \(^{46}\)

Proof of Lemma 1. Fix the policy \( \Gamma = (\mathcal{S}, \pi) \). For any \( s \in \mathcal{S} \), let \( \xi^\Gamma:s \equiv \sup\{x : \lim_{k \to \infty} U^\Gamma(x, s|k) \leq 0\} \). Given the public signal \( s \), it is dominant for any agent with private signal \( x \) exceeding \( \xi^\Gamma:s \) to pledge. Next, recall that, for any \( n \in \mathbb{N} \), \( T^\Gamma_{(n)} \) denotes the set of strategy profiles that survive the first \( n \) rounds of IDISDS and \( a^\Gamma_{(n)} \equiv (a^\Gamma_i)_{i \in [0,1]} \) denotes the most aggressive profile in \( T^\Gamma_{(n)} \). Observe that the profile \( a^\Gamma_{(1)} \) is given by \( a^\Gamma_{(1),i}(x, s) = \mathbb{I}\{x > \xi^\Gamma:s_{(1)}\} \) for all \( (x, s) \in \mathbb{R} \times \mathcal{S} \), and all \( i \in [0,1] \), and minimizes the policy maker’s payoff not just in expectation but for any \( (\theta, s) \). This follows from the fact that, when nobody else pledges, the expected payoff differential \( \int u(\theta, 0)d\Lambda^\Gamma(x|s, \theta) \) between pledging and not pledging crosses 0 only once and from below at \( x = \xi^\Gamma:s_{(1)} \). The single-crossing property of \( \int u(\theta, 0)d\Lambda^\Gamma(x|s, \theta) \) in turn is a consequence of the fact that \( u(\theta, 0) \) crosses 0 only once from below at \( \theta = 1 \) along with Property SCB below.

**Property SCB.** Suppose that the function \( h : \mathbb{R} \to \mathbb{R} \) crosses 0 only once from below at \( \theta = \theta_0 \) (that is, \( h(\theta) \leq 0 \) for all \( \theta \leq \theta_0 \) and \( h(\theta) \geq 0 \) for all \( \theta > \theta_0 \)). Let \( g : \mathbb{R}^2 \to \mathbb{R}_+ \) be a log-supermodular function and suppose that, for any \( \theta \), there is an open interval \( \varrho_0 = (\varrho_0, \bar{\varrho}_0) \subset \mathbb{R} \) containing \( \theta \) such that \( g(x, \theta) > 0 \) for all \( x \in \varrho_0 \) and \( g(x, \theta) = 0 \) for (almost) all \( x \in \mathbb{R} \setminus \varrho_0 \), with the bounds \( \varrho_0, \bar{\varrho}_0 \) non-decreasing in \( \theta \). Choose any (Lebesgue) measurable subset \( \Omega \subset \mathbb{R} \) containing \( \theta_0 \) and, for any \( x \in \mathbb{R} \), let \( \Psi(x; \Omega) \equiv \int_\Omega h(\theta)g(x, \theta)d\theta \). Suppose there exists \( x^* \in \varrho_0 \) such that \( \Psi(x^*; \Omega) = 0 \). Then,

\(^{46}\)Because \( \Theta \subset \mathbb{R} \), \( \Lambda^\Gamma(x|s, \theta) \) can be taken to be the cdf of the agent’s posterior beliefs.
necessarily, \( \Psi(x; \Omega) \geq 0 \) for all \( x \in g_{\theta_0} \) with \( x > x^* \), and \( \Psi(x; \Omega) \leq 0 \) for all \( x \in g_{\theta_0} \) with \( x < x^* \), with both inequalities strict if (a) \( \{ \theta \in \Omega : h(\theta) \neq 0 \} \) has strict positive Lebesgue measure, (b) \( g \) is strictly log-supermodular over \( \mathbb{R}^2 \). \(^{47}\)

**Proof of Property SCB.** For any \( x \in \mathbb{R} \), let \( \Omega_x \equiv \{ \theta \in \Omega : x \in g_\theta \} \). The monotonicity of \( g_\theta \) in \( \theta \) implies that \( \Omega_x \) is monotone in \( x \) in the strong-order sense. Pick any \( x' \in g_{\theta_0} \) with \( x' > x^* \). That \( x^* \) and \( x' \) belong to \( g_{\theta_0} \) implies that \( \theta_0 \in \Omega_{x^*} \cap \Omega_{x'} \). Next, observe that

\[
\Psi(x'; \Omega) = \int_{\Omega_{x'}} h(\theta)g(x', \theta)d\theta 
= \int_{\Omega_{x'} \cap \Omega_{x^*}} h(\theta)g(x', \theta)d\theta + \int_{\Omega_{x'} \setminus \Omega_{x^*}} h(\theta)g(x', \theta)d\theta 
= \int_{\Omega_{x^*} \cap (\Omega_{x'} \cap (-\infty, \theta_0))} h(\theta)g(x^*, \theta)\frac{g(x', \theta)}{g(x^*, \theta)}d\theta + \int_{\Omega_{x^*} \cap (\Omega_{x'} \cap \theta_0, \infty)} h(\theta)g(x^*, \theta)d\theta + \int_{\Omega_{x'} \setminus \Omega_{x^*}} h(\theta)g(x', \theta)d\theta 
\geq \frac{g(x', \theta_0)}{g(x^*, \theta_0)} \Psi(x^*; \Omega) + \int_{\Omega_{x'} \setminus \Omega_{x^*}} h(\theta)g(x', \theta)d\theta \geq 0.
\]

The first equality follows from the fact that \( g(x', \theta) = 0 \) for almost all \( \theta \in \Omega \setminus \Omega_{x'} \). The second equality follows from the fact that \( \Omega_{x'} \) can be partitioned into \( \Omega_{x^*} \cap \Omega_{x'} \) and \( \Omega_{x'} \setminus \Omega_{x^*} \). The third equality follows from the fact that \( g(x^*, \theta) > 0 \) for all \( \theta \in \Omega_{x^*} \). The first inequality follows from the fact that \( g(x^*, \theta)/g(x^*, \theta) \) is increasing over \( \Omega_{x^*} \cap \Omega_{x'} \) as a consequence of \( g \) being log-supermodular, along with the fact that \( \theta_0 \in \Omega_{x^*} \cap \Omega_{x'} \) and the assumption that \( h \) crosses 0 only once from below at \( \theta = \theta_0 \). The second inequality follows from the fact that, for any \( \theta \in (\Omega_{x^*} \setminus \Omega_{x'}) \cap (-\infty, \theta_0) \), \( h(\theta) \leq 0 \), along with the fact that \( \Omega_{x^*} \cap (\theta_0, +\infty) = \Omega_{x^*} \cap \Omega_{x'} \cap (\theta_0, \infty) \), with the last property following from the fact that \( \Omega_x \) are ranked in the strong-order sense. The last inequality follows from the observation that, for any \( \theta \in \Omega_{x'} \setminus \Omega_{x^*} \), \( h(\theta) \geq 0 \), which in turn is a consequence of (i) the monotonicity of the sets \( \Omega_x \) in \( x \), (ii) the assumption that \( h \) crosses 0 only once from below at \( \theta = \theta_0 \), and (iii) the assumption that \( \theta_0 \in \Omega_{x^*} \cap \Omega_{x'} \).

Similar arguments imply that, for \( x < x^* \), \( \Psi(x; \Omega) \leq 0 \). The same arguments also imply that, when (a) \( \{ \theta \in \Omega : h(\theta) \neq 0 \} \) has strict positive Lebesgue measure and (b) \( g \) is strictly log-supermodular over \( \mathbb{R}^2 \), then \( \Psi(x; \Omega) < 0 \) for all \( x < x^* \) and \( \Psi(x; \Omega) > 0 \) for all \( x > x^* \). This completes the proof of Property SCB.\( \blacksquare \)

The facts that (a) the continuation game is supermodular, (b) the density \( p(x|\theta) \) is log-supermodular, and (c) when agents follow monotone strategies, the fate of the bank is monotone in \( \theta \) imply that,

\(^{47}\)That \( g \) is strictly log-supermodular over \( \mathbb{R}^2 \) also implies that \( g(x, \theta) > 0 \) for all \( (x, \theta) \in \mathbb{R}^2 \).
for any \( s \in \mathcal{S} \), there exists a unique sequence \( (\xi_{(n)}^\Gamma)_{n \in \mathbb{N}} \) such that, for any \( n \geq 1 \), \( a_{(n)}^\Gamma \) is such that
\[
a_{(n),i}^\Gamma(x, s) = \mathbb{I}\{x > \xi_{(n)}^\Gamma, s\}, \quad \text{all } (x, s) \in \mathbb{R} \times \mathcal{S}, \text{ all } i \in [0, 1],
\]
with each \( \xi_{(n)}^\Gamma \) as defined above, and with all other cut-offs \( \xi_{(n)}^\Gamma, n > 1, s \in \mathcal{S} \), defined inductively by
\[
\xi_{(n)}^\Gamma = \sup\{x : U^\Gamma(x, s|\cdot) \leq 0\}\] for any \( s \in \mathcal{S} \), and is such that (a) when the policy discloses any signal \( s \in \mathcal{S} \), defined inductively by
\[
\xi_{(n)}^\Gamma = \sup\{x : U^\Gamma(x, s|\cdot) \leq 0\}.\] Next, let \( T^\Gamma \equiv \cap_{n=1}^\infty T_n^\Gamma \) denote the set of strategy profiles that are rationalizable for the agents under the policy \( \Gamma \). The most aggressive strategy profile in \( T^\Gamma \) is then given by
\[
a_i^\Gamma(x, s) = \mathbb{I}\{x > \xi^\Gamma, s\}, \quad \text{all } (x, s) \in \mathbb{R} \times \mathcal{S}, \text{ all } i \in [0, 1],
\]
where, for any \( s \in \mathcal{S} \), \( \xi^\Gamma \equiv \lim_{n \to \infty} \xi_{(n)}^\Gamma \). The sequence \( (\xi_{(n)}^\Gamma)_{n} \) is monotone and its limit is given by \( \xi^\Gamma = \sup\{x : U^\Gamma(x, s|\cdot) \leq 0\} \) if \( \{x : U^\Gamma(x, s|\cdot) \leq 0\} \neq \emptyset \), and \( \xi^\Gamma = -\infty \) otherwise. This establishes the first part of the lemma. That the profile \( a^\Gamma \) is a BNE for the continuation game that starts with the announcement of the policy \( \Gamma \) follows from the fact that, given any \( s \in \mathcal{S} \), when all agents follow a cut-off strategy with cutoff \( \xi^\Gamma \), the best response for each agent \( i \in [0, 1] \) is to pledge for \( x_i > \xi^\Gamma \) and to refrain from pledging for \( x_i < \xi^\Gamma \) (he is indifferent for \( x_i = \xi^\Gamma \)). This completes the proof of the lemma. ■

**Step 2.** Now take any policy \( \Gamma = (\mathcal{S}, \pi) \) satisfying the perfect-coordination property. Given the result in Theorem 1, without loss of generality, assume that \( \Gamma = (\mathcal{S}, \pi) \) is such that \( \mathcal{S} = \{0, 1\} \times \hat{\mathcal{S}} \), for some measurable set \( \hat{\mathcal{S}} \), and is such that (a) when the policy discloses any signal \( s = (\hat{s}, 1) \), all agents pledge and default does not happen, whereas (b) when the policy discloses any signal \( s = (\hat{s}, 0) \), all agents refrain from pledging and default happens.

Equipped with the result in Lemma 1, we then show that, starting from \( \Gamma = (\mathcal{S}, \pi) \), one can construct a binary policy \( \Gamma^* = (\{0, 1\}, \pi^*) \) also satisfying the perfect-coordination property and such that the probability of default under \( \Gamma^* \) is the same as under \( \Gamma \). The policy \( \Gamma^* = (\{0, 1\}, \pi^*) \) is such that, for any \( \theta \), \( \pi^*(1|\theta) = \int_{\hat{\mathcal{S}}} \pi(d(\hat{s}, 1)|\theta) \). That is, for each \( \theta \), the binary policy \( \Gamma^* \) recommends to pledge (equivalently, announces a “pass” grade) with the same total probability the original policy \( \Gamma \) discloses signals leading all agents to pledge.\(^{48}\)

We now show that, under \( \Gamma^* \), when the policy announces that \( s = 1 \), the unique rationalizable action for each agent is to pledge. To see this, for any \( (x, 1) \) that are mutually consistent given \( \Gamma^* \), let \( U^\Gamma(x, 1|k) \) denote the expected payoff differential for any agent with private signal \( x \), when the policy \( \Gamma^* \) announces \( s = 1 \), and all other agents follow a cut-off strategy with cut-off \( k \).\(^{49}\) From the law of iterated expectations, we have that
\[
U^\Gamma(x, 1|k) = \int_{\mathcal{S}} U^\Gamma(x, (\hat{s}, 1)|k) \xi^\Gamma(d\hat{s}|x, 1)
\]
\[\text{(13)}\]

\(^{48}\)The notation \( \int_{\hat{\mathcal{S}}} \pi(d(\hat{s}, 1)|\theta) \) represents the total probability that the measure \( \pi(\theta) \) assigns to signal \((\hat{s}, r)\) such that \( r = 1 \).

\(^{49}\)Recall that \( (x, 1) \) are mutually consistent under \( \Gamma^* \) if \( p^\Gamma(x, 1) \equiv \int p(x|\theta)\pi^*(1|\theta)dF(\theta) > 0 \).
where $\zeta^\Gamma(\cdot|x,1)$ is the probability measure over $\hat{S}$ obtained by conditioning on the event $(x,1)$, under the policy $\Gamma$. For any signal $s = (\hat{s},1)$ in the range of $\pi$, MARP consistent with $\Gamma$ is such that $a^\Gamma_i(x,(\hat{s},1)) = 1$ all $x \in \mathbb{R}$, meaning that pledging is the unique rationalizable action after the policy $\Gamma$ announces $s = (\hat{s},1)$. Lemma 1 in turn implies that, for all $s = (\hat{s},1)$ in the range of $\pi$, $\hat{s} \in \hat{S}$, all $k \in \mathbb{R}$, $U^\Gamma(k,(\hat{s},1)|k) > 0$. From (13), we then have that, for all $k \in \mathbb{R}$, $U^{\Gamma^*}(k,1|k) > 0$. In turn, this implies that, given the new policy $\Gamma^*$, when $s = 1$ is disclosed, under the unique rationalizable profile, all agents pledge, that is, $a^\Gamma^*_i(x,1) = 1$ all $x$, all $i \in [0,1]$. It is also easy to see that, when the policy $\Gamma^*$ discloses the signal $s = 0$, it becomes common certainty among the agents that $\theta \leq 1$. Hence, under MARP consistent with $\Gamma^*$, after $s = 0$ is disclosed, all agents refrain from pledging, irrespective of their private signals. The new pass/fail policy $\Gamma^*$ so constructed thus (a) satisfies the perfect-coordination property, and (b) is such that, for any $\theta$, the probability of default under $\Gamma^*$ is the same as under $\Gamma$. Q.E.D.

Proof of Theorem 3. Without loss of generality, assume that the policy $\Gamma = (S,\pi)$ (a) is a (possibly stochastic) “pass/fail” policy (i.e., $S = \{0,1\}$, with $\pi(1|\theta) = 1 - \pi(0|\theta)$ denoting the probability that signal $s = 1$ is disclosed when the fundamentals are $\theta$), (b) is such that $\pi(1|\theta) = 0$ for all $\theta \leq 0$ and $\pi(1|\theta) = 1$ for all $\theta > 1$, and (c) satisfies the perfect-coordination property. Theorems 1 and 2 imply that, if $\Gamma$ does not satisfy these properties, there exists another policy $\Gamma'$ that satisfies these properties and yields the policy maker a payoff weakly higher than $\Gamma$. The proof then follows from applying the arguments below to $\Gamma'$ instead of $\Gamma$.

Suppose that $\Gamma$ is such that there exists no $\hat{\theta}$ such that $\pi(1|\theta) = 0$ for $F$-almost all $\theta \leq \hat{\theta}$ and $\pi(1|\theta) = 1$ for $F$-almost all $\theta > \hat{\theta}$.\footnote{Clearly, if the policy $\Gamma = ((0,1),\pi)$ is such that there does exist $\hat{\theta} \in [0,1]$ such that $\pi(1|\theta) = 0$ for $F$-almost all $\theta \leq \hat{\theta}$ and $\pi(1|\theta) = 1$ for $F$-almost all $\theta > \hat{\theta}$, then the deterministic monotone policy $\Gamma^\theta = ((0,1),\pi^\theta)$ with cut-off $\hat{\theta}$ (that is, the policy such that $\pi^\theta(1|\theta) = 1(\theta > \hat{\theta})$ for all $\theta$) also satisfies the perfect-coordination property and yields the policy maker the same payoff as $\Gamma$, in which case the result trivially holds.} We establish the result by showing that there exists a deterministic monotone policy $\Gamma^\theta = ((0,1),\pi^\theta)$ satisfying the perfect-coordination property that yields the policy maker a payoff strictly higher than $\Gamma$.

Recall that, for the policy $\Gamma$ to satisfy the perfect-coordination property, it must be that, when the policy discloses the signal $s = 1$, $U^\Gamma(x,1|x) > 0$ for all $x$ such that $(x,1)$ are mutually consistent, where $U^\Gamma(x,1|x)$ is the expected payoff of an agent with signal $x$ who hears that $s = 1$ and who expects all other agents to follow a cut-off policy with cut-off $x$.

Now let $\mathcal{G}$ denote the set of policies $\Gamma' = (S,\pi')$ that, in addition to properties (a) and (b) above, are such that $U^{\Gamma'}(x,1|x) \geq 0$ for all $x$ such that $(x,1)$ are mutually consistent. Observe that some policies $\Gamma'$ in $\mathcal{G}$ need not satisfy the perfect-coordination property (namely, those for which there exists $x$ such that $(x,1)$ are mutually consistent and $U^{\Gamma'}(x,1|x) = 0$). For any $\Gamma$, let $U^\Gamma[\Gamma]$ denote the policy maker’s ex-ante expected payoff under MARP consistent with the policy $\Gamma$. Denote by
arg max \( \{ U^P[\Gamma] \} \) the set of policies that maximize the policy maker’s payoff over the set \( \mathcal{G} \).

Step 1 below shows that any \( \Gamma' \in \arg max \{ U^P[\Gamma] \} \) is such that \( \pi'(1|\theta) = 0 \) for \( \mathcal{F} \)-almost all \( \theta \leq \theta^* \) and \( \pi'(1|\theta) = 1 \) for \( \mathcal{F} \)-almost all \( \theta > \theta^* \), where \( \theta^* \) is the cut-off defined in (3).

We establish the result by showing that, given any policy \( \Gamma' \in \mathcal{G} \) for which there exists no \( \bar{\theta} \) such that \( \pi'(1|\theta) = 0 \) for \( \mathcal{F} \)-almost all \( \theta \leq \bar{\theta} \) and \( \pi'(1|\theta) = 1 \) for \( \mathcal{F} \)-almost all \( \theta > \bar{\theta} \), there exists another policy \( \Gamma'' \in \mathcal{G} \) that that yields the policy maker a payoff strictly higher than \( \Gamma' \). This property, together with the fact that any policy \( \Gamma' = (\{0,1\}, \pi') \) such that \( \pi'(1|\theta) = 0 \) for \( \mathcal{F} \)-almost all \( \theta \leq \bar{\theta} \) and \( \pi'(1|\theta) = 1 \) for \( \mathcal{F} \)-almost all \( \theta > \bar{\theta} \), for some \( \bar{\theta} \), belongs to \( \mathcal{G} \) only if \( \bar{\theta} \in [\theta^*, 1] \) then gives the result.

Step 2 then shows that the policy maker’s payoff under the optimal deterministic monotone policy \( \Gamma^{\theta^*} = (\{0,1\}, \pi^{\theta^*}) \) with cut-off \( \theta^* \) can be approximated arbitrarily well by a deterministic monotone policy \( \Gamma^\theta = (\{0,1\}, \pi^\theta) \in \mathcal{G} \) that satisfies the perfect-coordination property (i.e., such that \( U^{\Gamma^\theta}(x,1|x) > 0 \) for all \( x \) such that \( (x,1) \) are mutually consistent), thus establishing the result in the theorem.

For brevity, the proof below considers the case where the prior \( \mathcal{F} \) from which the fundamentals \( \theta \) are drawn and the distribution \( \mathcal{P} \) from which the agents’ signals are drawn have unbounded support: \( \Theta = \mathbb{R} \) and \( (\omega, p_\theta) = \mathbb{R} \) for all \( \theta \in \Theta \). In the Online Supplement, we dispense with these restrictions.

**Step 1.** Take any policy \( \Gamma' \in \mathcal{G} \) for which there exists no \( \bar{\theta} \) such that \( \pi'(1|\theta) = 0 \) for \( \mathcal{F} \)-almost all \( \theta \leq \bar{\theta} \) and \( \pi'(1|\theta) = 1 \) for \( \mathcal{F} \)-almost all \( \theta > \bar{\theta} \). Let \( X^{\Gamma'} \equiv \{ x : U^{\Gamma'}(x,1|x) = 0 \} \). Clearly, if \( X^{\Gamma'} = \emptyset \), there exists another policy \( \Gamma'' \in \mathcal{G} \) that yields the policy maker a payoff strictly higher than \( \Gamma' \). Thus, assume that \( X^{\Gamma'} \neq \emptyset \), and let \( \bar{x} \equiv \sup X^{\Gamma'} \).

For any \( x \), let \( \theta_0(x) \) be the fundamental threshold such that, when agents pledge when their private signal exceeds \( x \) and refrain from pledging otherwise, then their expected payoff \( u(\theta, 1 - \mathcal{P}(x|\theta)) \) crosses zero from below at \( \theta = \theta_0(x) \). For any policy \( \Gamma = \{\{0,1\}, \pi\} \in \mathcal{G} \), let \( p^\Gamma(x,1) \equiv \int_{-\infty}^{+\infty} \pi(1|\theta)p(x|\theta)d\mathcal{F}(\theta) \) represent the joint probability density of observing the exogenous signal \( x \) and the endogenous signal \( s = 1 \). Let

\[
\theta_H \equiv \sup \{ \theta \in \Theta : \exists \delta > 0 \text{ s.t. } \pi'(1|\theta') < 1 \text{ for } \mathcal{F} \text{-almost all } \theta' \in [\theta - \delta, \theta) \},
\]

51 That \( \arg max \{ U^P[\Gamma] \} \neq \emptyset \) follows from the compactness of \( \mathcal{G} \) and the upper hemi-continuity of \( U^P \) over \( \mathcal{G} \).

52 To see this, note that, because there exists no \( \bar{\theta} \) such that \( \pi'(1|\theta) = 0 \) for \( \mathcal{F} \)-almost all \( \theta \leq \bar{\theta} \) and \( \pi'(1|\theta) = 1 \) for \( \mathcal{F} \)-almost all \( \theta > \bar{\theta} \), if \( X^{\Gamma'} = \emptyset \), there must exists a set \( (\theta^*, \bar{\theta}) \subseteq [0,1] \) of \( \mathcal{F} \)-positive probability over which \( \pi'(1|\theta) < 1 \). The policy \( \Gamma'' \) can then be obtained from \( \Gamma' \) by increasing \( \pi'(1|\theta) \) over such a set. Provided the increase is small, the new policy is such that \( U^{\Gamma''}(x,1|x) \geq 0 \) for all \( x \), and hence \( \Gamma'' \in \mathcal{G} \). Because \( U^P(\theta, 1) > U^P(\theta, 0) \) over \( [0,1] \), the new policy improves over the original one.

53 Clearly, \( \bar{x} \) depends on the policy \( \Gamma' \). We do not highlight the dependence to ease the notation.

54 Because the sign of \( u(\theta, 1 - \mathcal{P}(x|\theta)) \) is determined by the default outcome, \( \theta_0(x) \) is implicitly defined by \( \mathcal{P}(x|\theta_0(x)) = \theta_0(x) \).
\[ \theta_L \equiv \inf \{ \theta \in \Theta : \exists \delta > 0 \text{ s.t. } \pi'(1|\theta') > 0 \text{ for } F\text{-almost all } \theta' \in [\theta, \theta + \delta] \}. \]

That \( \Gamma' \in \mathbb{G} \) guarantees that \( \theta_H \) and \( \theta_L \) are well-defined. That, under \( \Gamma' \), there exists no \( \hat{\theta} \) such that \( \pi'(1|\theta) = 0 \) for \( F\)-almost all \( \theta \leq \hat{\theta} \) and \( \pi'(1|\theta) = 1 \) for \( F\)-almost all \( \theta > \hat{\theta} \) implies that \( \theta_L < \theta_H \).

Furthermore, \( u(\theta_L, 1 - P(\bar{x}|\theta_L)) \) < 0.55

We distinguish between two cases.

**Case 1:** \( \theta_0(\bar{x}) < \theta_H \).

Consider the policy \( \Gamma^{\epsilon, \delta} = (\{0, 1\}, \pi^{\epsilon, \delta}) \) defined by \( \pi^{\epsilon, \delta}(1|\theta) = \pi'(1|\theta) \) for all \( \theta \leq \theta_0(\bar{x} + \delta) \), with \( \delta > 0 \) small so that \( \theta_0(\bar{x} + \delta) < \theta_H \), and \( \pi^{\epsilon, \delta}(1|\theta) = \min\{\pi'(1|\theta) + \epsilon, 1\} \) for all \( \theta > \theta_0(\bar{x} + \delta) \), with \( \epsilon > 0 \) also small. To see that, when \( \epsilon \) and \( \delta \) are small, \( \Gamma^{\epsilon, \delta} \in \mathbb{G} \), note that, by definition of \( \theta_0(\cdot) \), for any \( x \), and any \( \theta > \theta_0(\bar{x}) \), \( u(\theta, 1 - P(x|\theta)) > 0 \). This fact, together with the monotonicity of \( \theta_0(\cdot) \), jointly imply that, for any \( x \leq \bar{x} + \delta \),

\[
U^{\Gamma_{\epsilon, \delta}}(x, 1|x)p^{\Gamma_{\epsilon, \delta}}(x, 1) = \int_{-\infty}^{\theta_0(\bar{x} + \delta)} u(\theta, 1 - P(x|\theta))\pi'(1|\theta)p(x|\theta)dF(\theta) + \int_{\theta_0(\bar{x} + \delta)}^{+\infty} u(\theta, 1 - P(x|\theta))(\pi'(1|\theta) + \epsilon, 1)p(x|\theta)dF(\theta) > U^{\Gamma'}(x, 1|x)p^{\Gamma'}(x, 1) \geq 0.
\]

The strict inequality obtains from the fact that, for any \( \theta \in [\theta_0(\bar{x} + \delta), \theta_H] \), \( \pi^{\epsilon, \delta}(1|\theta) \geq \pi'(1|\theta) \), with the inequality strict over a subset of \( [\theta_0(\bar{x} + \delta), \theta_H] \) of strictly positive \( F \)-measure, along with the fact that, because \( x \leq \bar{x} + \delta \), \( u(\theta, 1 - P(x|\theta)) > 0 \) for all \( \theta \geq \theta_0(\bar{x} + \delta) \). That, when \( \epsilon > 0 \) is sufficiently small, \( U^{\Gamma_{\epsilon, \delta}}(x, 1|x) > 0 \) also for all \( x > \bar{x} + \delta \) follows from the fact that, by definition of \( \bar{x} \), for any \( x > \bar{x} + \delta \), \( U^{\Gamma'}(x, 1|x) \) is bounded away from 0 along with the fact that, for any \( \delta > 0 \), the function family \( \left(U^{\Gamma_{\epsilon, \delta}}(\cdot, 1|\cdot)\right)_\epsilon \) is continuous in \( \epsilon \) in the sup-norm, in a neighborhood of 0.56

**Case 2:** \( \theta_0(\bar{x}) \geq \theta_H \).

Consider the monotone policy \( \Gamma^0 = \{\{0, 1\}, \pi^0\} \) such that \( \pi^0(1|\theta) \equiv I(\theta \geq 0) \). Note that, for any \( x \geq \bar{x} \),

\[
U^{\Gamma^0}(x, 1|x)p^{\Gamma^0}(x, 1) = \int_{0}^{\theta_0(x)} u(\theta, 1 - P(x|\theta))\pi(1|\theta)p(x|\theta)dF(\theta) + \int_{\theta_0(x)}^{+\infty} u(\theta, 1 - P(x|\theta))\pi'(1|\theta)p(x|\theta)dF(\theta) > \int_{0}^{\theta_0(x)} u(\theta, 1 - P(x|\theta))p(x|\theta)dF(\theta) + \int_{\theta_0(x)}^{+\infty} u(\theta, 1 - P(x|\theta))p(x|\theta)dF(\theta) = U^{\Gamma^0}(x, 1|x)p^{\Gamma^0}(x, 1),
\]

55 That \( u(\theta_L, 1 - P(\bar{x}|\theta_L)) < 0 \) follows from the fact that, by definition of \( \bar{x} \) and \( \theta_L \), \( U^{\Gamma^0}(x, 1|\bar{x}) = \frac{1}{p^{\Gamma^0}(x, 1)} \int_{\theta_L}^{+\infty} u(\theta, 1 - P(\bar{x}|\theta))\pi'(1|\theta)p(\bar{x}|\theta)dF(\theta) = 0 \), together with the single-crossing property of \( u(\theta, 1 - P(\bar{x}|\theta)) \) in \( \theta_H \).

56 This means that, for any \( z > 0 \), there exists \( \Delta > 0 \) such that, for any \( 0 \leq \epsilon < \Delta \), and all \( x \), \( |U^{\Gamma_{\epsilon, \delta}}(x, 1|x) - U^{\Gamma^0}(x, 1|x)| \leq z \), where, by definition, \( \Gamma_{\epsilon, \delta} = \Gamma^\epsilon \).

36
where the inequality follows from the fact that (i) $u(\theta, 1 - P(x|\theta)) < 0$ for any $\theta \leq \theta_0(x)$ along with (ii) the fact that $\pi'(1|\theta) = 1$ for $F$-almost all $\theta \geq \theta_0(x) \geq \theta_0(\bar{x}) \geq \bar{\theta}_H$. As a result,

$$U^{\Gamma_0}(\bar{x}, 1|\bar{x}) < U^{\Gamma'}(\bar{x}, 1|\bar{x}) \frac{p^{\Gamma'}(x, 1)}{p^{\Gamma_0}(x, 1)} = 0.$$ 

We conclude that, necessarily, $\bar{x} < \bar{x}_G$, where $\bar{x}_G$ is the threshold defined in (1). This property in turn permits us to apply part (3) of Condition M to $\bar{x}$ in the arguments below.

For any $\gamma > 0$, let $\theta_L^\gamma \equiv \theta_L + \gamma$ and $\theta_H^\gamma \equiv \theta_H - \gamma$. Pick $\gamma, \epsilon_L, \epsilon_H > 0$ small such that (i) $\pi'(1|\theta_L^\gamma) > 0$ and $\pi'(1|\theta) > 0$ for $F$-almost all $\theta \in (\theta_L^\gamma, \theta_L^\gamma + \epsilon_L)$, (ii) $\pi'(1|\theta_H^\gamma) < 1$ and $\pi'(1|\theta) < 1$ for $F$-almost all $\theta \in (\theta_H^\gamma - \epsilon_H, \theta_H^\gamma)$, and (iii) $\theta_L^\gamma + \epsilon_L < \theta_H^\gamma - \epsilon_H$.\footnote{If a single $\gamma$ satisfying properties (i)-(iii) does not exist, let $\gamma = (\gamma_L, \gamma_H) \in \mathbb{R}_{++}^2$. The arguments below then apply verbatim by letting $\theta_L^\gamma = \theta_L + \gamma_L$ and $\theta_H^\gamma = \theta_H + \gamma_H$ and noting that a $\gamma = (\gamma_L, \gamma_H)$ satisfying properties (i)-(iii) always exists.} Next, pick $\eta \in (0, \bar{x}_G - \bar{x})$ small such that $U^{\Gamma'}(x, 1|x) > \eta$ for all $x \geq \bar{x} + \eta$. Pick $\epsilon > 0$ also small and let $\delta(\epsilon, \eta)$ be implicitly defined by

$$\int_{\theta_L^\gamma}^{\theta_L^\gamma + \epsilon} u(\theta, 1 - P(\bar{x} + \eta|\theta)) \pi'(1|\theta)p(\bar{x} + \eta|\theta)dF(\theta) = \int_{\theta_H^\gamma - \delta(\epsilon, \eta)}^{\theta_H^\gamma} u(\theta, 1 - P(\bar{x} + \eta|\theta))(1 - \pi'(1|\theta))p(\bar{x} + \eta|\theta)dF(\theta).$$

Note that, for $\epsilon > 0$ small, $\theta_L^\gamma + \epsilon < \theta_H^\gamma - \delta(\epsilon, \eta)$. Consider the policy $\Gamma^{\epsilon, \gamma, \eta} = \{0, 1\}, \pi^{\epsilon, \gamma, \eta}$ defined by the following properties: (a) $\pi^{\epsilon, \gamma, \eta}(1|\theta) = \pi'(1|\theta)$ for all $\theta \notin \{\theta_L^\gamma, \theta_L^\gamma + \epsilon, \epsilon_L | \theta_H^\gamma - \delta(\epsilon, \eta)\}$; (b) $\pi^{\epsilon, \gamma, \eta}(1|\theta) = 0$ for all $\theta \in [\theta_L^\gamma, \theta_L^\gamma + \epsilon]$; and (c) $\pi^{\epsilon, \gamma, \eta}(1|\theta) = 1$ for all $\theta \in [\theta_H^\gamma - \delta(\epsilon, \eta), \theta_H^\gamma]$. Note that Condition (14) implies that $U^{\Gamma^{\epsilon, \gamma, \eta}}(\bar{x} + \eta, 1|\bar{x} + \eta) = U^{\Gamma'}(\bar{x} + \eta, 1|\bar{x} + \eta) > 0$.

We now show that, under the new policy, $U^{\Gamma^{\epsilon, \gamma, \eta}}(x, 1|x) \geq 0$ for any $x$. Recall that, for any $\theta \in (0, 1)$, $x^*(\theta)$ is the critical threshold such that, when agents pledge for $x > x^*(\theta)$ and do not pledge for $x < x^*(\theta)$, default occurs when fundamentals are below $\theta$ and does not occur when they are above $\theta$, and hence $u(\hat{\theta}, 1 - P(x^*(\theta)|\hat{\theta})$ turns from negative to positive at $\hat{\theta} = \theta$.

Clearly, for any $(\epsilon, \gamma, \eta)$, and any $x \leq x^*(\theta_L)$, $U^{\Gamma'}(x, 1|x), U^{\Gamma^{\epsilon, \gamma, \eta}}(x, 1|x) > 0$. This is because, for any such $x$, $\theta_0(x) < \theta_L$ and hence $u(\theta, 1 - P(x|\theta)) > 0$ for all $\theta > \theta_L$. The result then follows from the fact that, under both $\Gamma'$ and $\Gamma^{\epsilon, \gamma, \eta}$,

$$\int_{-\infty}^{\theta_L^\gamma} \pi'(1|\theta) dF(\theta) = \int_{-\infty}^{\theta_L^\gamma} \pi^{\epsilon, \gamma, \eta}(1|\theta) dF(\theta) = 0$$

meaning that all agents assign probability one to the event that $\theta \geq \theta_L$. Furthermore, that

$$U^{\Gamma'}(x^*(\theta_L), 1|x^*(\theta_L)) > 0$$

along with the fact that $U^{\Gamma'}(x, 1|x) > \eta$ for all $x \geq \bar{x} + \eta$ and the continuity of

$$U^{\Gamma'}(x, 1|x) p^{\Gamma'}(x, 1) = \int u(\theta, 1 - P(x|\theta)) \pi'(1|\theta)p(x|\theta)dF(\theta)$$
in \( x \) imply that there exists \( \xi > 0 \) such that, for any \( x \in [x^*(\theta_L), x^*(\theta_L) + \xi] \cup [\bar{x} + \eta, +\infty) \), \( U^{\gamma} (x,1|x)p^{\gamma} (x,1) > \xi \). Because, for any \( \eta \), the function family \( \{ U^{\gamma,\eta} (\cdot,1|\cdot)p^{\gamma,\eta} (\cdot,1) \}^{\epsilon,\gamma} \) is continuous in \( (\gamma, \epsilon) \) in the sup-norm, in a neighborhood of \((0,0)\),\(^{58}\) and \( x^* (\theta) \) is continuous in \( \theta \), there exist \( \bar{\gamma}, \bar{\epsilon} > 0 \) such that, when \( \gamma \leq \bar{\gamma} \) and \( \epsilon \leq \bar{\epsilon} \), \( U^{\gamma,\eta} (x,1|x) \geq 0 \) for any \( x \in (-\infty, x^* (\theta_L^\gamma + \epsilon)) \cup [\bar{x} + \eta, +\infty) \).

Next observe that, for any \( x \in \{ x^*(\theta_L + \epsilon), x^*(\theta_L^\gamma - \delta (\epsilon, \eta) \} \),

\[
U^{\gamma,\eta} (x,1|x)p^{\gamma,\eta} (1, x) - U^{\gamma} (x,1|x)p^{\gamma} (1, x) + \int_{\theta_L^\gamma - \delta (\epsilon, \eta)}^{\theta_L^\gamma - 0} u (\theta, 1 - P (x|\theta)) p (x|\theta) (1 - \pi^\gamma (1|\theta)) dF(\theta) > 0,
\]
where the inequality follows from the fact that the integrand in the first integral is negative, whereas that in the second integral is positive. Because \( U^{\gamma} (x,1|x) \geq 0 \) for all \( x \), this implies that for any such \( x \), \( U^{\gamma,\eta} (x,1|x) \geq 0 \).

Next, consider \( x \in \{ x^*(\theta_L^\gamma - \delta (\epsilon, \eta)), x^*(\theta_L^\gamma) \} \). For any \( x \), and any \( \theta \), let

\[
q (\theta, x) \equiv |u (\theta, 1 - P (x|\theta))| / p (x|\theta).
\]

For any \( x \leq \bar{x} + \eta \), let \( \Delta U (x) \equiv U^{\gamma,\eta} (x,1|x)p^{\gamma,\eta} (1, x) - U^{\gamma} (x,1|x)p^{\gamma} (1, x) \). Note then that, for any \( x \in \{ x^*(\theta_L^\gamma - \delta (\epsilon, \eta)), x^*(\theta_L^\gamma) \} ,

\[
\Delta U (x) = \int_{\theta_L^\gamma - \delta (\epsilon, \eta)}^{\theta_L^\gamma} -u (\theta, 1 - P (x|\theta)) p (x|\theta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

+ \int_{\theta_L^\gamma}^{\theta_L^\gamma - \delta (\epsilon, \eta)} -u (\theta, 1 - P (x|\theta)) p (x|\theta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

+ \int_{\theta_L^\gamma}^{\theta_L^\gamma - \delta (\epsilon, \eta)} -u (\theta, 1 - P (x|\theta)) p (x|\theta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

\[
\Delta U (x) = \int_{\theta_L^\gamma}^{\theta_L^\gamma - \delta (\epsilon, \eta)} -u (\theta, 1 - P (x|\theta)) p (x|\theta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

+ \int_{\theta_L^\gamma}^{\theta_L^\gamma} q (\theta, x) / q (\theta, \bar{x} + \eta) q (\theta, \bar{x} + \eta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

+ \int_{\theta_L^\gamma}^{\theta_L^\gamma - \delta (\epsilon, \eta)} q (\theta, x) / q (\theta, \bar{x} + \eta) q (\theta, \bar{x} + \eta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

+ \int_{\theta_L^\gamma}^{\theta_L^\gamma - \delta (\epsilon, \eta)} q (\theta, x) / q (\theta, \bar{x} + \eta) q (\theta, \bar{x} + \eta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

\[
\Delta U (x) = \int_{\theta_L^\gamma}^{\theta_L^\gamma} q (\theta, x) / q (\theta, \bar{x} + \eta) q (\theta, \bar{x} + \eta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

= \int_{\theta_L^\gamma}^{\theta_L^\gamma} q (\theta, x) / q (\theta, \bar{x} + \eta) q (\theta, \bar{x} + \eta) f(\theta) (\pi^\gamma (1|\theta) - \pi^{\gamma,\eta} (1|\theta)) d\theta

= 0.
\]

\(^{58}\)This means that, for any \( z > 0 \), there exists \( \Delta > 0 \) such that, for any \( (\epsilon, \gamma) \) with \( 0 \leq \epsilon < \Delta \) and \( 0 \leq \gamma < \Delta \), and all \( x, |U^{\gamma,\eta} (x,1|x) - U^{\gamma,\eta} (x,1|x)| \leq z \), where, by definition, \( \Gamma^{\epsilon,\eta} = \Gamma^{\epsilon} \).
The first equality is by definition. The first inequality follows from the fact that (i) for any \( \theta \leq \theta_0(x) \), \( u(\theta, 1 - P(x|\theta)) < 0 \), whereas, for any \( \theta > \theta_0(x) \), \( u(\theta, 1 - P(x|\theta)) > 0 \), along with the fact that (ii) for \( \theta \in [\theta_0(x), \theta_H^\gamma] \), \( \pi'(1|\theta) \leq \pi^{\epsilon,\gamma,\eta}(1|\theta) \). Together, these two properties imply that

\[
\int_{\theta_0(x)}^{\theta_H^\gamma} -u(\theta, 1 - P(x|\theta)) p(x|\theta) f(\theta) \left( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \right) d\theta \\
\geq 0 \geq \frac{q(\theta_H^\gamma - \delta(\epsilon, \eta), x)}{q(\theta_H^\gamma - \delta(\epsilon, \eta), \bar{x} + \eta)} \int_{\theta_0(x)}^{\theta_H^\gamma} q(\theta, \bar{x} + \eta) f(\theta) \left( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \right) d\theta.
\]

The second inequality follows from the fact that \( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \) turns from positive to negative at \( \theta = \theta_H^\gamma - \delta(\epsilon, \eta) \leq \theta_0(x) \), along with the fact that, for any \( \theta \in [\theta_L^\gamma, \theta_0(x)] \), the function \( q(\theta, x)/q(\theta, \bar{x} + \eta) \) is non-increasing in \( \theta \) as implied by the log-supermodularity of \( |u(\theta, 1 - P(x|\theta))| p(x|\theta) \) over \( \{(\theta, x) \in [0, 1] \times \mathbb{R} : u(\theta, 1 - P(x|\theta)) \leq 0\} \), by virtue of part 2 of Condition M. Finally, the last two equalities follow from the fact that \( \theta_0(\bar{x} + \eta) > \theta_0(\bar{x}) > \theta_H \geq \theta_H^\gamma \), which implies that \( u(\theta, 1 - P(\bar{x} + \eta|\theta)) \leq 0 \) for all \( \theta \leq \theta_H^\gamma \), and hence that

\[
\int_{\theta_L^\gamma}^{\theta_H^\gamma} q(\theta, \bar{x} + \eta) f(\theta) \left( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \right) d\theta = \Delta U(\bar{x} + \eta)
\]

along with the fact that, by construction of the policy \( \Gamma^{\epsilon,\gamma,\eta} \), \( \Delta U(\bar{x} + \eta) = 0 \). Hence, for any \( x \in (x^*(\theta_H^\gamma - \delta(\epsilon, \eta)), x^*(\theta_H^\gamma)) \), \( \Delta U(x) \geq 0 \), which implies that \( U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) \geq 0 \).

Similar arguments imply that, for any \( x \in [x^*(\theta_H^\gamma), \bar{x} + \eta] \),

\[
\Delta U(x) = \int_{\theta_L^\gamma}^{\theta_H^\gamma} -u(\theta, 1 - P(x|\theta)) p(x|\theta) f(\theta) \left( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \right) d\theta \\
= \int_{\theta_L^\gamma}^{\theta_H^\gamma} \frac{q(\theta, x)}{q(\theta, \bar{x} + \eta)} q(\theta, \bar{x} + \eta) f(\theta) \left( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \right) d\theta \\
+ \int_{\theta_H^\gamma - \delta(\epsilon, \eta)}^{\theta_H^\gamma} \frac{q(\theta, x)}{q(\theta, \bar{x} + \eta)} q(\theta, \bar{x} + \eta) f(\theta) \left( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \right) d\theta \\
\geq \frac{q(\theta_H^\gamma - \delta(\epsilon, \eta), x)}{q(\theta_H^\gamma - \delta(\epsilon, \eta), \bar{x} + \eta)} \Delta U(\bar{x} + \eta) = 0,
\]

which implies that, for such \( x \) too, \( U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) \geq 0 \) \(^59\).

We conclude that, when \( \epsilon, \gamma, \eta \) are small, \( U^{\Gamma^{\epsilon,\gamma,\eta}}(x, 1|x) \geq 0 \) for all \( x \) and hence \( \Gamma^{\epsilon,\gamma,\eta} \in \mathcal{G} \).

We now show that, when property 3 in Condition M holds, the new policy yields the policy maker an expected payoff strictly higher than \( \Gamma' \). To see this, observe that, fixing \( (\gamma, \eta) \), for any \( \epsilon > 0 \), the

\(^59\) The first equality is by definition. The second equality follows from the fact that, for such \( x \), \( u(\theta, 1 - P(x|\theta)) \leq 0 \) for all \( \theta \leq \theta_H^\gamma \). The inequality follows from the fact that \( q(\theta, x)/q(\theta, \bar{x} + \eta) \) is non-increasing in \( \theta \) over \( [\theta_L^\gamma, \theta_H^\gamma] \) along with the fact that \( \pi'(1|\theta) - \pi^{\epsilon,\gamma,\eta}(1|\theta) \) changes sign only once, turning from non-negative to non-positive at \( \theta = \theta_H^\gamma - \delta(\epsilon, \eta) \).
policy maker’s payoff under the policy \( \Gamma^{\epsilon, \gamma, \eta} \) is equal to
\[
\mathcal{U}^P[\Gamma^{\epsilon, \gamma, \eta}] = \int_{-\infty}^{\theta_1^L + \epsilon} U^P(\theta, 0) dF(\theta) + \int_{\theta_1^L - \delta(\epsilon, \eta)}^{\theta_1^L} U^P(\theta, 1) dF(\theta)
+ \int_{(\theta_1^L + \epsilon, \theta_1^L - \delta(\epsilon, \eta)) \cup (\theta_1^L, +\infty)} \{ \pi'(1|\theta)U^P(\theta, 1) + (1 - \pi'(1|\theta))U^P(\theta, 0) \} dF(\theta).
\]
Differentiating \( \mathcal{U}^P[\Gamma^{\epsilon, \gamma, \eta}] \) with respect to \( \epsilon \), and taking the limit as \( \epsilon \to 0^+ \), we have that
\[
\lim_{\epsilon \to 0^+} \frac{d\mathcal{U}^P[\Gamma^{\epsilon, \gamma, \eta}]}{d\epsilon} = f(\theta_H^L)(1 - \pi'(1|\theta_H^L)) [U^P(\theta_H^L, 1) - U^P(\theta_H^L, 0)] \left( \lim_{\epsilon \to 0^+} \frac{\partial \delta(\epsilon, \eta)}{\partial \epsilon} \right)
- f(\theta_L^L)\pi'(1|\theta_L^L) [U^P(\theta_L^L, 1) - U^P(\theta_L^L, 0)]
= f(\theta_L^L)\pi'(1|\theta_L^L) \left( [U^P(\theta_H^L, 1) - U^P(\theta_H^L, 0)] \frac{p(\bar{x} + \eta|\theta_H^L) u(\theta_H^L, 1 - P(\bar{x} + \eta|\theta_H^L))}{p(\bar{x} + \eta|\theta_H^L) u(\theta_H^L, 1 - P(\bar{x} + \eta|\theta_H^L))} - [U^P(\theta_L^L, 1) - U^P(\theta_L^L, 0)] \right).
\]
Therefore, \( \lim_{\epsilon \to 0^+} \frac{d\mathcal{U}^P[\Gamma^{\epsilon, \gamma, \eta}]}{d\epsilon} > 0 \) if and only if
\[
\frac{U^P(\theta_H^L, 1) - U^P(\theta_H^L, 0)}{U^P(\theta_L^L, 1) - U^P(\theta_L^L, 0)} > \frac{p(\bar{x} + \eta|\theta_H^L) u(\theta_H^L, 1 - P(\bar{x} + \eta|\theta_H^L))}{p(\bar{x} + \eta|\theta_H^L) u(\theta_H^L, 1 - P(\bar{x} + \eta|\theta_H^L))}.
\]
Property 3 in Condition M, together with the fact that \( \bar{x} \leq \bar{x}_G \) (as proved above), guarantee this is the case. We conclude that, when \( \epsilon \) is small, the policy \( \Gamma^{\epsilon, \gamma, \eta} \in \mathcal{G} \) strictly improves upon \( \Gamma' \). Furthermore, the construction of \( \Gamma^{\epsilon, \gamma, \eta} \) above can be iterated to arrive to a monotone deterministic policy. Because any monotone deterministic policy \( \hat{\Gamma} \) with cut-off \( \hat{\theta} > \theta^* \) yields the policy maker a payoff strictly smaller than the monotone deterministic policy with cut-off \( \theta^* \) (and no monotone deterministic policy \( \hat{\Gamma} \) with cut-off \( \hat{\theta} < \theta^* \) is in \( \mathcal{G} \)), we conclude that any policy \( \Gamma' \in \arg \max_{\hat{\Gamma} \in \mathcal{G}} \{ \mathcal{U}^P[\hat{\Gamma}] \} \) is such that \( \pi'(1|\theta) = 0 \) for \( F \)-almost all \( \theta \leq \theta^* \) and \( \pi'(1|\theta) = 1 \) for \( F \)-almost all \( \theta > \theta^* \).

**Step 2.** Take any policy \( \Gamma' \in \arg \max_{\hat{\Gamma} \in \mathcal{G}} \{ \mathcal{U}^P[\hat{\Gamma}] \} \). The result in step 1 implies that \( \pi'(1|\theta) = 0 \) for \( F \)-almost all \( \theta \leq \theta^* \) and \( \pi'(1|\theta) = 1 \) for \( F \)-almost all \( \theta > \theta^* \). The result in the theorem then follows from observing that, given \( \Gamma' \), there exists a nearby deterministic monotone policy \( \Gamma^\hat{\theta} \in \mathcal{G} \) with cut-off \( \hat{\theta} = \theta^* + \varepsilon \), for \( \varepsilon > 0 \) but small, such that \( \Gamma^\hat{\theta} \) satisfies the perfect-coordination property (i.e., \( U^\Gamma^\hat{\theta}(x, 1|x) > 0 \) all \( x \)) and yields the policy maker a payoff arbitrarily close to that under \( \Gamma' \).

**References**

Alonso, R., Zachariadis, K., 2021. Persuading large investors. WP, LSE.
Alvarez, F., Barlevy, G., 2015. Mandatory disclosure and financial contagion. WP, NBER.


Gick, W., Pausch, T., 2012. Persuasion by stress testing: Optimal disclosure of supervisory information in the banking sector. Harvard University and Deutsche Bundesbank WP.


Li, F., Yangbo, S., Zhao, M., 2021. Global manipulation by local obfuscation. WP, UNC Chapel Hill.


Morris, S., Daisuke, O., Takahashi, S., 2020. Information design in binary-action supermodular games. WP, MIT.


Morris, S., Yang, M., 2019. Coordination under continuous stochastic choice. WP, MIT.


