Online Appendix
to *Persistent Private Information Revisited*

Appendices C–K present technical proofs and secondary results omitted from the main paper. Most of this material is presented in the same order that it is mentioned in the main text; Appendix K collects auxiliary mathematical facts.

C. Facts about AC Change-of-Measure

Following PPI (p. 1239) and Karatzas and Shreve (1998, p. 59), we view $\mathbf{P}^*$ and $\mathbf{P}^m$ as probability measures on the space $\mathbf{C}[0,\infty)$ of continuous paths. There exists a *density process* $\Gamma_t^m \equiv \mathrm{d}\,\mathbf{P}_t^m/\mathrm{d}\,\mathbf{P}_t^*$ iff $m \in \mathcal{M}^{\text{LAC}}$. By Karatzas and Shreve (1998, p. 191): (a) Γ^m is a continuous $\mathbf{P}^*$-local martingale and can be expressed as[1]

$$\Gamma_t^m \equiv \exp\left[\int_0^t \frac{(\Delta_\tau + \lambda m_\tau)}{\sigma}\,\mathrm{d}W_\tau^y - \frac{1}{2}\int_0^t \frac{(\Delta_\tau + \lambda m_\tau)^2}{\sigma^2}\,\mathrm{d}\tau\right],$$

where W^y is a standard Brownian motion under $\mathbf{P}^*$;[2] (b) $m \in \mathcal{M}^{\text{LAC}}$ iff Γ^m is a $\mathbf{P}^*$-martingale; and (c) $m \in \mathcal{M}^{\text{GAC}}$ iff Γ^m is a uniformly integrable (UI) $\mathbf{P}^*$-martingale, in which case there exists an infinite-horizon density $\Gamma_\infty^m := \mathrm{d}\,\mathbf{P}^m/\mathrm{d}\,\mathbf{P}^*$ and $\Gamma_t^m \to \Gamma_\infty^m$ $\mathbf{P}^*$-a.s. These facts imply that the strategies constructed in the proofs of Observations 1–2 are in the claimed feasible sets. Furthermore, as claimed in Sections 2.2 and 4.2:

Fact 3. The following hold:

(i) If $\lambda > 0$, every $m \in \mathcal{M}_\leqslant^{\text{GAC}}$ satisfies $m_t \equiv 0$.

(ii) $\mathcal{M}_\dagger^{\text{GAC}} \subseteq \mathcal{M}_\dagger^{\text{LAC}} \cap \mathcal{M}_\dagger^r$ for all $\dagger \in \{\leqslant, -\}$ and $r > 0$.

Proof. Part (i): Suppose there exists $m \in \mathcal{M}_\leqslant^{\text{GAC}} \backslash \{m^* \equiv 0\}$. Then $\Gamma_t^m \equiv \exp(X_t - \frac{1}{2}\langle X\rangle_t)$ is a UI $\mathbf{P}^*$-martingale, where $X_t := \int_0^t \frac{(\Delta_\tau+\lambda m_\tau)}{\sigma}\,\mathrm{d}W_\tau^y$ is a martingale and $\langle X\rangle_t = \int_0^t \frac{(\Delta_\tau+\lambda m_\tau)^2}{\sigma^2}\,\mathrm{d}\tau$ is its quadratic variation. On the event $\{m \not\equiv 0\}$, we have $\lim_{t\to\infty}\langle X\rangle_t = \infty$ because m is nonincreasing. Thus, there exists a time-changed Brownian motion B such that $X_t = B_{\langle X\rangle_t}$ (see Theorem 4.6 and Problem 4.7 in Karatzas and Shreve (1998, Ch. 3)). The SLLN for Brownian motion implies that $\lim_{t\to\infty} X_t/\langle X\rangle_t = 0$, so on the event $\{m \not\equiv 0\}$ we have $\Gamma_\infty^m = \lim_{t\to\infty}\exp\left[\langle X\rangle_t\left(X_t/\langle X\rangle_t - \frac{1}{2}\right)\right] = 0$. Hence, $\Gamma_\infty^m \lneqq 1$ on $\mathbf{C}([0,\infty))$ and therefore $\mathbf{P}^m(\mathbf{C}([0,\infty))) < 1$, a contradiction.

Part (ii): Trivially, $\mathcal{M}_\dagger^{\text{GAC}} \subseteq \mathcal{M}_\dagger^{\text{LAC}}$. The inclusions $\mathcal{M}_\dagger^{\text{GAC}} \subseteq \mathcal{M}_\dagger^r$ follow from y being a $\mathbf{P}^*$-OU process, Lemma K.3(iii) in Appendix K, and the definitions [**GAC**] and [**NP-*m***]. □

[1] This formula presumes that $\int_0^t \frac{(\Delta_\tau+\lambda m_\tau)}{\sigma}\,\mathrm{d}W_\tau^y$ is well-defined (e.g., $\Delta + \lambda m \in L_{\text{loc}}^2$).

[2] As in Section 3, $\mathrm{d}W_t^y := \frac{1}{\sigma}[\mathrm{d}y_t - (\mu - \lambda y_t)\mathrm{d}t]$, which is a $\mathbf{P}^*$-Brownian motion by definition of $\mathbf{P}^*$. When using the "weak formulation" of the agent's reporting problem to conduct change-of-measure (see Cvitanić and Zhang (2012)), PPI denotes W^y by W^* and the process W that drives b by $W_t^\Delta \equiv W_t^* - \int_0^t \left(\frac{\Delta_\tau+\lambda m_\tau}{\sigma}\right)\mathrm{d}\tau$. We maintain the (equivalent) notation W^y and W for simplicity.

D. PPI's Sufficient Conditions for IC

Theorem 4.1 in §4 of PPI presents sufficient conditions under which the first-order approach from §3 is valid in the finite-horizon setting from §2. That is, fixing a finite horizon $[0, T]$, PPI's Theorem 4.1 offers conditions under which we can conclude that a given contract is IC, assuming that (i) the agent's feasible set consists of all reporting strategies satisfying IML and finite-horizon AC change-of-measure ($\mathbf{P}_T^m \ll \mathbf{P}_T^*$), and (ii) the contract satisfies the equation [**A.1**] for q, the *downward FO-IC* condition $\gamma_t + p_t \geqslant 0$ (cf. Footnote 57), and the equation [**J.3**] for p (see also pp. 1244-45 in PPI). These conditions take the form of inequalities (displays (17) and (18) on p. 1247) to be satisfied by the y-adapted process $Q = (Q_t)_{t\in[0,T]}$ controlling the volatility of p in [**J.3**].

Issue 1: The proof of Theorem 4.1 requires IML. PPI presents the proof of Theorem 4.1 in §A.2 (pp. 1265-69). Three key steps of the proof rely on IML and would not go through as stated under NHB alone.

(a) On p. 1267, PPI writes: "when $Q_t \leqslant 0$, then we have both $Q_t m_t^2 \leqslant 0$ and $Q_t m_t \Delta_t \leqslant 0$." The final inequality requires that $m_t \Delta_t \geqslant 0$, which is implied by IML but can fail under NHB (which permits $m_t < 0 < \Delta_t$).

(b) Later on p. 1267, PPI writes: "Thus the optimality condition for truthtelling ($\Delta_t = 0$) is $Q_t m_t + \xi_t \geqslant 0$," where the inequality appears as display (A.9) and ξ is a co-state process in the agent's reporting problem. Direct inspection reveals that PPI's derivation of the inequality (A.9) from the preceding display (A.8) requires the IML constraint $\Delta_t \leqslant 0$. Without IML, to conclude that setting $\Delta_t = 0$ in (A.8) is optimal for the agent, one would need to strengthen (A.9) to the *equality* $Q_t m_t + \xi_t = 0$. Furthermore, truthful reporting corresponds to $m_t = 0$ rather than $\Delta_t = 0$; without IML, under many FO-IC contracts—including all DR-SICs—the agent will find it optimal to set $\Delta_t = +\infty$ in (A.8) when $m_t < 0$ (see Sections 6 and 7.3 and Appendix I).

(c) The final two displays on p. 1268 and the first display on p. 1269 invoke the inequality version of (A.9), and therefore also require IML. Without IML but maintaining the *equality* version of (A.9) and PPI's hypothesis that $m_\tau = m_t$ for all $\tau \geqslant t$ (middle of p. 1268), the final two displays on p. 1268, in particular (A.10), would need to hold as *equalities*. This is problematic because the inequality at the top of p. 1269, which yields (17) in the statement of Theorem 4.1, is not sufficient to ensure that the *equality* version of (A.10) holds. Furthermore, as noted in Point (b) above, without IML the equality version of (A.9) and the hypothesis that $m_\tau = m_t$ for all $\tau \geqslant t$ are both too demanding, so a different argument would be needed even if the equality version of (A.10) could be addressed.

In summary, the proof of Theorem 4.1 would not go through as stated if IML were weakened to NHB. We do not know whether the statement of Theorem 4.1 would remain valid, but conjecture that it would not.

Issue 2: The application of Theorem 4.1 to Contract PPI is incomplete. In §A.3.2 (p. 1271), PPI attempts to apply Theorem 4.1 to prove that Contract PPI is $\mathscr{M}^{\text{LAC}}_{\leqslant}$-IC when $\lambda > 0$. This conclusion is false (Observation 2). There are two issues with PPI's argument:

(a) *Theorem 4.1 pertains to the finite-horizon model, while Contract PPI arises in the infinite-horizon model.* PPI attempts to apply *an infinite-horizon variant of* Theorem 4.1 in which the key inequalities for the Q process (displays (17) and (18) on p. 1247) are modified by replacing the terminal time $T < \infty$ with $T = \infty$ wherever the former appears. PPI does not formally state or prove such a variant of Theorem 4.1.

(b) *The proof of Theorem 4.1, as stated, does not extend to the infinite-horizon model unless a sufficiently tight lower bound is imposed on the agent's misreports. In particular, it does not extend when the agent's infinite-horizon feasible set is* $\mathscr{M}^{LAC}_{\leqslant}$. To illustrate, we specialize to Contract PPI in the hidden endowment model. Following PPI's derivation of (A.4) on p. 1265, the agent's lifetime utility gain $V(m) - q_0$ from using strategy m instead of truthfully reporting is

[**D.1**]
$$\begin{aligned} V(m) - q_0 = \mathbf{E}_0^m \left[\int_0^T e^{-\rho t} \left[u(c_t - m_t) - u(c_t) \right] \mathrm{d}t + \int_0^T e^{-\rho t} \gamma_t \mathrm{d}W_t^y \right] \\ + e^{-\rho T} \mathbf{E}_0^m \left[\int_T^\infty e^{-\rho(t-T)} u(c_t - m_t) \mathrm{d}t - q_T \right], \end{aligned}$$

where [**D.1**] is an accounting identity that holds for *all* $T > 0$.[3] For a given $T > 0$, *if the strategy* m *satisfies* $m_t = m_T$ *for all* $t \geqslant T$, then by the same logic as PPI's (A.5) on p. 1266, the second line of [**D.1**] is bounded above by $e^{-\rho T} \mathbf{E}_0^m \left[p_T m_T \right]$.[4] This yields the overall bound: for all $T > 0$,

[**D.2**]
$$V(m) - q_0 \leqslant K(s, m, T) + e^{-\rho T} \mathbf{E}_0^m \left[p_T m_T \right],$$

where $K(s, m, T)$ denotes the first expectation in [**D.1**]. After using PPI's (A.4) to substitute out $e^{-\rho T} p_T m_T$, [**D.2**] reduces to PPI's (A.6), which is the basis for the rest of PPI's (finite-horizon) proof of Theorem 4.1. To adapt those later proof steps to derive the infinite-horizon variant of Theorem 4.1 that PPI applies to Contract PPI (viz., the infinite-horizon variant of (17) invoked on p. 1271), one must consider the

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[3][**D.1**] is the same as PPI's (A.4), specialized to the hidden endowment model and with time-T *continuation* payoffs in place of time-T *terminal* payoffs. As noted in Footnote 2, our W^y is PPI's W^*.

[4]PPI's (A.5) concerns $p_T m_T$ at the terminal time T. With an infinite horizon, let $U_T(m_T) := \mathbf{E}_T^{m_T} \left[\int_T^\infty e^{-\rho(t-T)} u(c_t - m_T) \mathrm{d}t \right]$ denote the agent's time-T continuation payoff under the continuation strategy $m_t \equiv m_T$. Then $U_T(\cdot)$ is smooth and concave, with $U'(m_T) = -\mathbf{E}_T^{m_T} \left[\int_T^\infty e^{-(\rho+\lambda)(t-T)} u'(c_t - m_T) \mathrm{d}t \right]$. Concavity yields $U_T(m_T) - U_T(0) \leqslant m_T U_T'(0)$ and [**3.1**]–[**3.2**] imply that $U_T(0) = q_T$ and $U_T'(0) = p_T$. Taking expectations yields the desired upper bound.

$T \to \infty$ limit of [**D.2**]. This adaptation is meaningful only if[5]

$$\textbf{[D.3]} \qquad \lim_{T\to\infty} e^{-\rho T}\, \mathbf{E}_0^m \left[p_T m_T\right] < \infty,$$

for otherwise subsequent calculations would involve expectations that are either infinite or ill-defined. *However, there exist strategies in $\mathscr{M}_{\leqslant}^{LAC}$ that violate* [**D.3**]. *In particular, any deterministic strategy $m \in \mathscr{M}_{\leqslant}^{LAC}$ that, for some fixed time $\hat{T} > 0$, satisfies $m_t = M < \underline{M} := -(\rho+\lambda)/(\theta\lambda)$ for all $t \geqslant \hat{T}$ violates* [**D.3**].[6] To see this, recall that $p_t \equiv k_0^* q_t$ under Contract PPI (where $k_0^* = \frac{\rho\theta}{\rho+\lambda}$), and use [**3.3**] and the identity $W_t^y \equiv W_t + \int_0^t \left(\frac{\lambda m_\tau + \Delta_\tau}{\sigma}\right) \mathrm{d}\tau$ to write

$$p_T m_T = k_0^* q_T^* m_T \exp\left[-k_0^*\left(m_T + \lambda \int_0^T m_t \mathrm{d}t\right)\right],$$

where $q_T^* := q_0 \exp\left(-\frac{1}{2}(k_0^*\sigma)^2 T - k_0^*\sigma W_T\right)$ is promised utility *under truthful reporting*. Thus, we have

$$\begin{aligned}
\lim_{T\to\infty} e^{-\rho T}\, \mathbf{E}_0^m \left[p_T m_T\right] &= p_0 M \cdot \lim_{T\to\infty} \exp\left[-\rho T - k_0^*\left(M + \lambda M(T-\hat{T}) + \lambda\int_0^{\hat{T}} m_t \mathrm{d}t\right)\right] \\
&= p_0 M e^{k_0^*\left(\lambda M(\hat{T}-1) - \lambda \int_0^{\hat{T}} m_t \mathrm{d}t\right)} \cdot \lim_{T\to\infty} \exp\left[-T\cdot(\rho + k_0^*\lambda M)\right] \\
&= \infty,
\end{aligned}$$

where the first equality uses the facts that (a) m is deterministic and satisfies $m_t = M$ for all $t \geqslant \hat{T}$ and (b) $\mathbf{E}_0^m \left[k_0^* q_T^*\right] = k_0^* q_0 = p_0$ because q^* is a martingale, and the last equality uses the fact that $M < \underline{M}$ implies $\rho + k_0^*\lambda M < 0$. We conclude that PPI's argument for the infinite-horizon variant of Theorem 4.1 requires the additional constraint that $m_t \geqslant \underline{M}$ for all $t \geqslant 0$.

E. Proof of Lemma 5.1

Note that $\hat{c}$ satisfies [**5.2**] and [**NP-A**] iff it satisfies the intertemporal constraint

$$\textbf{[E.1]} \qquad \int_0^\infty e^{-rt}\hat{c}_t \,\mathrm{d}t \leqslant \int_0^\infty e^{-rt} b_t \,\mathrm{d}t + A_0 \quad \mathbf{P}\text{-a.s.}$$

Given any $\xi \in \mathbb{R}$, [**E.1**] implies the following properties: (i) $\hat{c} \in \mathscr{A}(A_0, b_0)$ iff $\hat{c} + r\xi \in \mathscr{A}(A_0 + \xi, b_0)$, and (ii) $\mathscr{A}(A_0, b_0 + \xi) = \mathscr{A}(A_0 + \xi/(r+\lambda), b_0)$. Property (i) is immediate.

[5] IML implies $p_T m_T > 0$ for all $T \geqslant 0$.

[6] One such strategy is $m_t \equiv \max\{M, Mt/\hat{T}\}$; all such strategies, being eventually constant, satisfy the hypothesis used above to derive [**D.2**] from [**D.1**]. Also, note the parallel to Section 4.2: $\underline{M}$ is the same constant defined below [**4.1**] and the strategies considered here violate [**TVC**].

Property (ii) follows from inserting the closed-form solution for b_t (see Footnote 10) into [E.1]. Using these facts, we can characterize the agent's value function V^{SI} from [5.1] up to a parameter, $\gamma \in \mathbb{R}$, to be determined later.

Lemma E.1. Let $\xi \in \mathbb{R}$. The value function $V^{\mathrm{SI}} : \mathbb{R}^2 \to \mathbb{R}_{--}$ satisfies:

(i) $V^{\mathrm{SI}}(A_0 + \xi, b_0) = e^{-\theta r \xi} V^{\mathrm{SI}}(A_0, b_0)$.

(ii) $V^{\mathrm{SI}}(A_0, b_0 + \xi) = e^{-f(r;\lambda)\xi} V^{\mathrm{SI}}(A_0, b_0)$.

(iii) $V^{\mathrm{SI}}(A_0, b_0) = -\exp\big(-\theta r(A_0 + b_0/(r+\lambda) + \gamma)\big)$, where $-e^{-\theta r \gamma} := V^{\mathrm{SI}}(0,0)$.

Proof. Fix $(A_0, b_0) \in \mathbb{R}^2$. Note that $V^{\mathrm{SI}}(A_0, b_0) \in \mathbb{R}_{--}$ is well-defined: $V^{\mathrm{SI}}(A_0, b_0) < 0$ because $u(\cdot) < 0$ and [E.1] renders infeasible consumption processes approximating $\hat{c}_t \equiv +\infty$, and $V^{\mathrm{SI}}(A_0, b_0) > -\infty$ because there exist $\hat{c} \in \mathscr{A}(A_0, b_0)$ that deliver finite lifetime utility to the agent.[7] Point (i) follows from property (i) above, point (ii) follows from property (ii) above and point (i), and point (iii) follows from points (i) and (ii). □

Lemma E.1(iii) implies that $V^{\mathrm{SI}} \in \mathbf{C}^\infty(\mathbb{R}^2)$. Thus, letting (A_t, b_t) denote a generic state, standard arguments imply that V^{SI} is a classical solution to the HJB equation[8]

$$\begin{aligned} \rho V^{\mathrm{SI}}(A_t, b_t) = \sup_{c_t \in \mathbb{R}} \big[u(c_t) + (rA_t + b_t - c_t)V_A^{\mathrm{SI}}(A_t, b_t)\big] \\ + (\mu - \lambda b_t)V_b^{\mathrm{SI}}(A_t, b_t) + \tfrac{1}{2}\sigma^2 V_{bb}^{\mathrm{SI}}(A_t, b_t). \end{aligned} \tag{E.2}$$

This allows us to determine γ and the optimal (Markovian) control.

Lemma E.2. The following hold:

(i) The parameter $\gamma \in \mathbb{R}$ from Lemma E.1(iii) is

$$\gamma = \frac{\mu}{r(\lambda + r)} + \frac{\log(r)}{r\theta} - \left[\frac{r - \rho}{\theta r^2} + \frac{1}{2}\frac{(f(r;\lambda)\sigma)^2}{\theta r^2}\right]. \tag{E.3}$$

(ii) The supremum in [E.2] at state (A_t, b_t) is uniquely attained by

$$\hat{C}(A_t, b_t) := rA_t + \frac{r}{\lambda + r}b_t + \frac{\mu}{\lambda + r} - \left[\frac{r - \rho}{\theta r} + \frac{1}{2}\frac{(f(r;\lambda)\sigma)^2}{\theta r}\right]. \tag{E.4}$$

(iii) At every state (A_t, b_t), we have $u(\hat{C}(A_t, b_t)) = rV^{\mathrm{SI}}(A_t, b_t)$.

Proof. The FOC for [E.2] in state (A_t, b_t) is $u'(c_t) = V_A^{\mathrm{SI}}(A_t, b_t)$. Because u is exponential

[7] For one example, see the consumption process [5.3] constructed below.

[8] For instance, see Yong and Zhou (1999, Theorem 3.3) or Touzi (2018, Propositions 2.4-2.5).

and V^{SI} satisfies Lemma E.1(iii), it follows that $u(c_t) = rV^{\mathrm{SI}}(A_t, b_t)$. Equivalently,

$$c_t = rA_t + \frac{r}{\lambda + r} b_t + r\gamma - \frac{\log r}{\theta}. \tag{E.5}$$

Substituting [E.5] into [E.2] and solving for γ yields [E.3], hence point (i). Substituting [E.3] into [E.5] yields [E.4], hence point (ii). Point (iii) follows from the above observation that $u(c_t) = rV^{\mathrm{SI}}(A_t, b_t)$ and point (ii). □

Next, plugging [E.5] into the flow constraint [5.2] yields the following:

Lemma E.3. Define the asset process A^* by

$$\mathrm{d}A_t^* = \left(\frac{r - \rho}{\theta r} + \frac{1}{2} \frac{\sigma^2 f(r; \lambda)^2}{\theta r} + \frac{\lambda b_t - \mu}{\lambda + r} \right) \mathrm{d}t. \tag{E.6}$$

The consumption process $\hat{c}^*$ defined by $\hat{c}_t^* := \hat{C}(A_t^*, b_t)$ satisfies:

(i) Its induced asset process $A^{\hat{c}^*}$ satisfies $A^{\hat{c}^*} = A^*$.

(ii) It evolves as

$$\mathrm{d}\hat{c}_t^* = \left(\frac{r - \rho + \sigma^2 f(r; \lambda)^2/2}{\theta} \right) \mathrm{d}t + \frac{\sigma f(r; \lambda)}{\theta} \mathrm{d}W_t \tag{E.7}$$

(iii) It is feasible, i.e., is b-adapted and satisfies [5.2] and [NP-A].

Proof. Point (i) follows from plugging $\hat{c}_t^* = \hat{C}(A_t^*, b_t)$ into [5.2]. Point (ii) follows from noting that $\mathrm{d}\hat{c}_t^* = r\,\mathrm{d}A_t^* + \frac{r}{(r+\lambda)}\,\mathrm{d}b_t$ and plugging in [E.6] and [2.1]. For point (iii), only [NP-A] is nontrivial. Writing [E.6] in integrated form yields

$$A_t^* = A_0 + \left(\frac{r - \rho}{\theta r} + \frac{1}{2} \frac{\sigma^2 f(r; \lambda)^2}{\theta r} - \frac{\mu}{\lambda + r} \right) t + \frac{\lambda}{\lambda + r} \int_0^t b_\tau \,\mathrm{d}\tau. \tag{E.8}$$

When $\lambda = 0$, A^* is deterministic and affine in t, so $\lim_{t\to\infty} e^{-\alpha t} A_t^* = 0$ for every $\alpha > 0$. When $\lambda > 0$, the same conclusion holds because $\lim_{t\to\infty} e^{-\alpha t} \int_0^t b_\tau \,\mathrm{d}\tau = 0$ for all $\alpha > 0$ by Lemma K.3(iii) in Appendix K. □

Lemmas E.2 and E.3 immediately yield:

Corollary E.4. The strategy $\hat{c}^*$ from Lemma E.3 satisfies $u(\hat{c}_t^*) \equiv rV^{\mathrm{SI}}(A_t^{\hat{c}^*}, b_t)$.

The next two lemmas are useful technical facts:

Lemma E.5. Under $\hat{c}^*$ from Lemma E.3, $e^{(r-\rho)t} u'(\hat{c}_t^*)$, $e^{(r-\rho)t} u(\hat{c}_t^*)$, and $e^{(r-\rho)t} V^{\mathrm{SI}}(A_t^*, b_t)$ are martingales.

Proof. Note that $e^{(r-\rho)t}u(\hat{c}_t^*) = -e^{-\theta\hat{c}_0^*}\exp\left(-\frac{1}{2}\sigma^2(f(r;\lambda))^2 - \sigma f(r;\lambda)W_t\right)$ is a martingale, $u'(c) = -\theta u(c)$ by CARA, and $V^{\mathrm{SI}}(A_t^*, b_t) = \frac{1}{r}u(\hat{c}_t^*)$ by Corollary E.4. □

Lemma E.6. Under A^* from [E.6], $M_t := \int_0^t e^{-\rho\tau}V_b^{\mathrm{SI}}(A_\tau^*, b_\tau)\sigma\,\mathrm{d}W_\tau$ is a martingale.

Proof. It suffices to show $\mathbf{E}_0\left[\int_0^T \left(e^{-\rho t}V_b^{\mathrm{SI}}(A_t^*, b_t)\sigma\right)^2 \mathrm{d}t\right] < \infty$ for all $T > 0$. Lemma E.1(iii) implies that $V_b^{\mathrm{SI}}(A_t^*, b_t) \equiv \frac{(-r\theta)}{r+\lambda}V^{\mathrm{SI}}(A_t^*, b_t)$ and Lemma E.2(i) implies that $u(\hat{c}_t^*) = rV^{\mathrm{SI}}(A_t^*, b_t)$. Thus, by Fubini's Theorem, it suffices to show that $\int_0^T e^{-2\rho t}\,\mathbf{E}_0\left[u(\hat{c}_t^*)^2\right]\,\mathrm{d}t < \infty$. To that end, Lemma E.3(ii) implies that

$$u(\hat{c}_t^*)^2 = \exp(-2\theta\hat{c}_0^*)\exp(-2(r-\rho)t + \sigma^2 f^2(r;\lambda)t)\exp(-2\sigma^2 f^2(r;\lambda)t - 2\sigma f(r;\lambda)W_t).$$

Because $\exp(-2\sigma^2 f^2(r;\lambda)t - 2\sigma f(r;\lambda)W_t)$ is a martingale, we have $e^{-2\rho t}\,\mathbf{E}_0\left[u(\hat{c}_t^*)^2\right] = \exp(-2\theta\hat{c}_0^*)\exp(-2rt + \sigma^2 f^2(r;\lambda)t)$. Thus, $\int_0^T e^{-2\rho t}\,\mathbf{E}_0\left[u(\hat{c}_t^*)^2\right]\,\mathrm{d}t < \infty$, as desired. □

We now are in a position to prove Lemma 5.1 itself:

Proof of Lemma 5.1. We show that, given initial condition (A_0, b_0), the strategy $\hat{c}^*$ from Lemma E.3 attains lifetime utility $V^{\mathrm{SI}}(A_0, b_0)$. The Itô expansion of $e^{-\rho t}V^{\mathrm{SI}}(A_t^*, b_t)$ is

$$\text{[E.9]}\quad \begin{aligned} e^{-\rho T}V^{\mathrm{SI}}(A_t^*, b_t) = V^{\mathrm{SI}}(A_0, b_0) &+ \int_0^T e^{-\rho t}\left[\mathscr{L}^{\hat{c}^*}V^{\mathrm{SI}}(A_t^*, b_t) - \rho V^{\mathrm{SI}}(A_t^*, b_t)\right]\mathrm{d}t \\ &+ \int_0^T e^{-\rho t}V_b^{\mathrm{SI}}(A_t^*, b_t)\sigma\,\mathrm{d}W_t, \end{aligned}$$

where for any $v \in \mathbf{C}^2(\mathbb{R}^2)$, we let $\mathscr{L}^{\hat{c}^*}v(A,b) := (rA + b - \hat{c}^*)\partial_A v(A,b) + \lambda(\mu/\lambda - b)\partial_b v(A,b) + \frac{1}{2}\sigma^2\partial_{bb}v(A,b)$. Lemma E.2(ii) implies that $\mathscr{L}^{\hat{c}^*}V^{\mathrm{SI}}(A_t^*, b_t) - \rho V^{\mathrm{SI}}(A_t^*, b_t) = -u(\hat{c}_t^*)$. Substituting this into [E.9] and applying Lemma E.6 yields

$$\text{[E.10]}\qquad V^{\mathrm{SI}}(A_0, b_0) = \mathbf{E}_0\left[\int_0^T e^{-\rho t}u(\hat{c}_t^*)\,\mathrm{d}t\right] + \mathbf{E}_0\left[e^{-\rho T}V^{\mathrm{SI}}(A_t^*, b_t)\right].$$

Lemma E.5 shows that $e^{(r-\rho)t}V^{\mathrm{SI}}(A_t^*, b_t)$ is a martingale. Therefore,

$$\mathbf{E}_0\left[e^{-\rho T}V^{\mathrm{SI}}(A_t^*, b_t)\right] = e^{-rT}\,\mathbf{E}_0\left[e^{(r-\rho)T}V^{\mathrm{SI}}(A_t^*, b_t)\right] = e^{-rT}V^{\mathrm{SI}}(A_0, b_0),$$

so that letting $T \to \infty$ in [E.10] yields $V^{\mathrm{SI}}(A_0, b_0) = \mathbf{E}_0\left[\int_0^\infty e^{-\rho t}u(\hat{c}_t^*)\,\mathrm{d}t\right]$, as desired. □

F. Proof of Lemma 5.3

We must calculate $A_0 - \mathbf{E}_0[\int_0^\infty e^{-\rho t}(\rho - r)A_t^* \,\mathrm{d}t]$, where $A_0 := A_0(b_0, q_0, r)$ and the process A^* is from [E.6]. We claim that

$$\text{[F.1]} \qquad A_0 - (\rho - r)\int_0^\infty e^{-\rho t}A_t^* \,\mathrm{d}t = \hat{\Pi}(b_0, q_0, r) + \frac{(r-\rho)\lambda\sigma}{\rho(r+\lambda)(\rho+\lambda)}\int_0^\infty e^{-\rho t}\,\mathrm{d}W_t,$$

where $\hat{\Pi}(b_0, q_0, r)$ denotes the expression for $\Pi(b_0, q_0, r)$ on the RHS of the first line of [5.12]. To see this, note that $\int_0^\infty e^{-\rho t}A_t^* \,\mathrm{d}t = \frac{A_0}{\rho} + \frac{1}{\rho}\int_0^\infty e^{-\rho t}\,\mathrm{d}A_t^*$ because $\mathrm{d}(e^{-\rho t}A_t^*) = -\rho e^{-\rho t}A_t^* \,\mathrm{d}t + e^{-\rho t}\,\mathrm{d}A_t^*$ and, by (the proof of) Lemma E.3, $\lim_{T\to\infty} e^{-\rho T}A_T^* = 0$. Plugging in [E.6], the expression for $\int_0^\infty e^{-\rho t}b_t \,\mathrm{d}t$ in Lemma K.4, and simplifying yields [F.1]. A similar calculation yields

$$\text{[F.2]} \qquad A_0 - (\rho - r)\int_0^\infty e^{-\rho t}A_t^* \,\mathrm{d}t = \int_0^\infty e^{-\rho t}(\hat{c}_t^* - b_t)\,\mathrm{d}t.$$

Taking expectations in [F.1]–[F.2] and noting that $\mathbf{E}_0[\int_0^\infty e^{-\rho t}\,\mathrm{d}W_t] = 0$ yields [5.12].[9] □

G. Further Properties of SI Contracts

This appendix develops the properties of SI Contract described in Section 5.4.

Stationarity & State-Consistency. We say that a (direct-revelation) contract is *Stationary* if (i) it is FO-IC and (ii) $k_t := p_t/q_t \equiv k_0$ for some constant $k_0 > 0$. Strulovici (2022) defines the class of "state-consistent" renegotiation-proof contracts; Theorem 1 therein states, in our terminology, that a contract is state-consistent iff it is Stationary.

Lemma G.1. Suppose that $\lambda > 0$. A contract is Stationary with constant $k_0 > 0$ if and only if it is a DR-SIC with rate $r > 0$ such that $k_0 = f(r;\lambda)$.

Proof. For the"if" direction, note that under a DR-SIC with rate $r > 0$, the agent's promised utility satisfies $q_t \equiv V^{\text{SI}}(A_t^v, y_t)$ (where V^{SI} is defined in [5.10]) and hence

$$\text{[G.1]} \qquad \mathrm{d}q_t = (\rho - r)q_t\mathrm{d}t - f(r;\lambda)q_t\sigma\mathrm{d}W_t^y.$$

Furthermore, the Euler equation [5.7] and fact that $u'(c) = -\theta u(c)$ imply $p_t \equiv f(r;\lambda)q_t$. Thus, the contract satisfies [FO-IC] and $k_t \equiv f(r;\lambda)$. For the "only if" direction, fix a Stationary contract with $k_0 > 0$. By [A.1] and [FO-IC], promised utility satisfies

$$\text{[G.2]} \qquad \mathrm{d}q_t = (\rho q_t - u_t)\mathrm{d}t - k_0 q_t\sigma\mathrm{d}W_t^y.$$

[9] The process $M_t := \int_0^t e^{-r\tau}\,\mathrm{d}W_\tau$ is a uniformly integrable martingale because $\int_0^\infty (e^{-\alpha\tau})^2\,\mathrm{d}\tau < \infty$ (see Exercise 5.24 of Karatzas and Shreve (1998, p. 38)). Thus, $\mathbf{E}_0[\int_0^\infty e^{-\alpha t}\,\mathrm{d}W_t] = M_0 = 0$.

Comparing [G.1]–[G.2], it suffices to show that $u_t \equiv r(k_0;\lambda)q_t$, where $r(k_0;\lambda) := \frac{\lambda k_0}{\theta - k_0}$ is the inverse of $k_0(r) := f(r;\lambda)$. By the Martingale Representation Theorem, marginal promised utility [3.2] satisfies

$$\text{[G.3]} \qquad \mathrm{d}p_t = [(\rho+\lambda)p_t - \theta u_t]\,\mathrm{d}t + Q_t \sigma \mathrm{d}W_t^y$$

for some y-adapted process Q. By the Stationarity hypothesis (that $p_t \equiv k_0 q_t$) and the unique decomposition property for Itô processes, the drift (volatility) of p in [G.3] must a.e. equal k_0 times the drift (volatility) of q in [G.2]. Equating the drifts and using the identity $p_t \equiv k_0 q_t$ yields $u_t \equiv r(k_0;\lambda)q_t$, as desired. □

Long-Run Properties. Under an SI Contract, we say that the agent *converges to misery* if $u_t, V_t \to -\infty$ **P**-a.s. and say that he *converges to bliss* if $u_t, V_t \to 0$ **P**-a.s., where $u_t := u(\hat{c}_t^*)$ and $V_t := V^{\mathrm{SI}}(A_t^{\hat{c}^*}, b_t)$.

Theorem 3. *Under the optimal SI Contract, the following hold:*

(i) For each $\sigma > 0$, there exists a $\overline{\lambda}(\sigma) > 0$ such that the agent converges to misery if $\lambda > \overline{\lambda}(\sigma)$ and to bliss if $\lambda \in [0, \overline{\lambda}(\sigma))$.[10]

(ii) For each $\lambda > 0$, there exists a $\overline{\sigma}(\lambda) \geqslant 0$ such that the agent converges to misery if $\sigma > \overline{\sigma}(\lambda)$ and to bliss if $\sigma \in (0, \overline{\sigma}(\lambda))$. Furthermore, $\overline{\sigma}(\lambda) = 0$ if and only if $\lambda \geqslant \rho$.

By Lemma 5.3, the principal's optimization over SI Contracts can be written as

$$\text{[G.4]} \qquad \inf_{r>0} \left[-\frac{\log(r)}{\theta\rho} + \frac{r - \rho + \sigma^2 f(r;\lambda)^2/2}{\theta\rho^2} \right].$$

Let $(\lambda,\sigma) \mapsto r^*(\lambda,\sigma)$ denote an arbitrary selection from the argmin correspondence of [G.4], which is nonempty. Let $k^*(\lambda,\sigma) := f(r^*(\lambda,\sigma),\lambda)$. We require two lemmas.

Lemma G.2. Under the optimal SI Contract:

(i) For each $\sigma > 0$, $k^*(\cdot,\sigma)$ is strictly decreasing, with $\lim_{\lambda\to 0} k^*(\lambda,\sigma) = k^*(0,\sigma) = \theta$ and $\lim_{\lambda\to\infty} k^*(\lambda,\sigma) = 0$.

(ii) For each $\lambda > 0$, $k^*(\lambda,\cdot)$ is strictly decreasing, with $\lim_{\sigma\to 0} k^*(\lambda,\sigma) = f(\rho;\lambda)$ and $\lim_{\sigma\to\infty} k^*(\lambda,\sigma) = 0$.

Proof. Point (i): Let $\sigma > 0$ be given. For each $\lambda > 0$, let $r(k_0;\lambda) := \frac{\lambda k_0}{\theta - k_0}$ denote the inverse of $k_0(r) := f(r;\lambda)$. Changing variables in [G.4] from r to k and noting that $\frac{\partial^2}{\partial k \partial \lambda} r(k;\lambda) > 0$, Edlin and Shannon (1998, Theorem 1) (for minimization problems) implies that $k^*(\cdot,\sigma)$ is strictly decreasing. Next, recall from the proof of Theorem 1 that $r^*(\lambda,\sigma)$ satisfies the FOC [5.14] and $0 < r^*(\lambda,\sigma) \leqslant \rho$ for all $\lambda \geqslant 0$. (Furthermore,

[10] For $\lambda = \overline{\lambda}(\sigma)$, u_t and V_t are transient, with $\liminf_{t\to\infty} u_t, V_t = -\infty$ and $\limsup_{t\to\infty} u_t, V_t = 0$.

$r^*(0,\sigma) = \rho$, which yields $k^*(0,\sigma) = \theta$.) Multiplying [5.14] through by $\rho r^*(\lambda,\theta) > 0$ and rearranging yields

$$\rho = r^*(\lambda,\sigma) + \sigma^2 \cdot \frac{\lambda\theta^2 r^*(\lambda,\sigma)^2}{(r^*(\lambda,\sigma)+\lambda)^3}. \tag{G.5}$$

Because $r^*(\cdot,\sigma)$ is bounded, the second term in [G.5] goes to 0 as $\lambda \to \infty$. This implies $\lim_{\lambda\to\infty} r^*(\lambda,\sigma) = \rho$, and hence $\lim_{\lambda\to\infty} k^*(\lambda,\sigma) = \lim_{\lambda\to\infty} f(\rho;\lambda) = 0$. Finally, we show that $\lim_{\lambda\to 0} r^*(\lambda,\sigma) = \rho$, which implies $\lim_{\lambda\to 0} k^*(\lambda,\sigma) = \lim_{\lambda\to 0} f(\rho;\lambda) = \theta$. To this end, notice that $\underline{r} := \liminf_{\lambda\to 0} r^*(\lambda,\sigma) > 0$ (if not, [G.4] would explode as $\lambda \to 0$, contradicting the finite upper bound from setting $r = \rho$). Consequently, we have

$$0 \leqslant \liminf_{\lambda\to 0} \frac{\lambda\theta^2 r^*(\lambda,\sigma)^2}{(r^*(\lambda,\sigma)+\lambda)^3} \leqslant \limsup_{\lambda\to 0} \frac{\lambda\theta^2 r^*(\lambda,\sigma)^2}{(r^*(\lambda,\sigma)+\lambda)^3} \leqslant \limsup_{\lambda\to 0} \frac{\lambda\theta^2 r^*(\lambda,\sigma)^2}{\underline{r}^3} = 0,$$

where the equality uses $r^*(\cdot,\sigma) \leqslant \rho$. Display [G.5] then yields $\lim_{\lambda\to 0} r^*(\lambda,\sigma) = \rho$.
Point (ii): Let $\lambda > 0$ be given. Because $\frac{\partial^2}{\partial r \partial\sigma}\left[\sigma^2 f(r;\lambda)\right] > 0$, Edlin and Shannon (1998, Theorem 1) (for minimization problems) applied to [G.4] implies that $r^*(\lambda,\cdot)$ is strictly decreasing, which further implies that $k^*(\lambda,\cdot)$ is strictly decreasing. The limit properties follow from calculations similar to those in point (i) above, which are omitted. □

Lemma G.3. Let $D^* : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}$ be defined by

$$D^*(\lambda,\sigma) := r^*(\lambda,\sigma) - \rho + \frac{\sigma^2}{2} f\left(r^*(\lambda,\sigma);\lambda\right)^2. \tag{G.6}$$

Under the optimal SI Contract, the following hold:
(i) If $D^*(\lambda,\sigma) > 0$, then $\hat{c}^*_t \to \infty$ and $V_t, u_t \to 0$ **P**-a.s.
(ii) If $D^*(\lambda,\sigma) < 0$, then $\hat{c}^*_t \to -\infty$ and $V_t, u_t \to -\infty$ **P**-a.s.

Proof. By [5.3], $\hat{c}^*$ is a Brownian motion with drift $D^*(\lambda,\sigma)/\theta$. The long-run properties of $\hat{c}^*$ thus follow from the SLLN for Brownian motion (Lemma K.1 in Appendix K). The long-run properties of u_t and V_t then follow from CARA utility, [5.8], and the Continuous Mapping Theorem. □

Proof of Theorem 3. Recall from the proof of Theorem 1 that $r^*(\lambda,\sigma)$ satisfies the FOC [5.14], which as noted above is equivalent to [G.5]. Plugging [G.5] into [G.6] yields

$$D^*(\lambda,\sigma) = \underbrace{\sigma^2 f\left(r^*(\lambda,\sigma);\lambda\right)^2}_{>0} \cdot \left[\frac{1}{2} - \frac{\lambda}{r^*(\lambda,\sigma)+\lambda}\right]. \tag{G.7}$$

It follows from [**G.7**] that

$$D^*(\lambda,\sigma) > 0 \iff r^*(\lambda,\sigma) > \lambda \iff k^*(\lambda,\sigma) > \theta/2, \quad \textbf{[G.8]}$$

$$D^*(\lambda,\sigma) < 0 \iff r^*(\lambda,\sigma) < \lambda \iff k^*(\lambda,\sigma) < \theta/2. \quad \textbf{[G.9]}$$

Point (i): Let $\sigma > 0$ be given. By Lemma G.2(i), there exists a unique $\overline{\lambda}(\sigma) > 0$ such that $k^*(\lambda,\sigma) > \theta/2$ iff $\lambda \in \left[0, \overline{\lambda}(\sigma)\right)$ and $k^*(\lambda,\sigma) < \theta/2$ iff $\lambda > \overline{\lambda}(\sigma)$. The result then follows from [**G.8**]–[**G.9**] and Lemma G.3.

Point (ii): Let $\lambda > 0$ be given. By Lemma G.2(ii), there exists a unique $\overline{\sigma}(\lambda) \geqslant 0$ such that $k^*(\lambda,\sigma) > \theta/2$ iff $\sigma \in (0, \overline{\sigma}(\lambda))$ and $k^*(\lambda,\sigma) < \theta/2$ iff $\sigma > \overline{\sigma}(\lambda)$. Furthermore, $\lim_{\sigma\to 0} k^*(\lambda,\sigma) = f(\rho;\lambda)$, and $f(\rho;\lambda) > \theta/2$ iff $\rho > \lambda$. Thus, $\overline{\sigma}(\lambda) > 0$ iff $\rho > \lambda$. The result then follows from [**G.8**]–[**G.9**] and Lemma G.3. ☐

H. Hidden Savings

In this appendix, we consider the *hidden savings* variant of PPI's hidden endowment model in which (a) the agent can directly self-insure via the market at rate ρ, and (b) both the agent's endowment and trading activity are his private information (as in Allen 1985 and Cole and Kocherlakota 2001).

Given a contract s (as in Section 2.1) and a feasible set of (extended) misreporting strategies $F \subseteq \mathscr{M}_{\text{ext}}$ (as in Section 6.1), the agent chooses an $m \in F$ and a b-adapted consumption strategy $\hat{c}$ subject to the constraint that the induced b-adapted *asset process* $A^{m,\hat{c}}$ solves the equation

$$A_t^{m,\hat{c}} = \left(\rho A_t^{m,\hat{c}} + b_t + s_t - \hat{c}_t\right)\mathrm{d}t$$

and satisfies the no Ponzi condition [**NP-A**] at the market rate $r = \rho$. The agent's joint strategy $(m, \hat{c})$ is *optimal given contract* s if it maximizes his lifetime utility from the consumption process $\hat{c}$ among all strategies satisfying the above constraints.

A contract is F*-HS-IC* if it makes truthful reporting—i.e., some joint strategy $(m^* \equiv 0, \hat{c})$—optimal for the agent.[11] It is F*-NS-IC* it makes truthful reporting and no savings—i.e., the joint strategy $(m^* \equiv 0, \hat{c} = s + b)$—optimal for the agent. It is *NS-FO-IC* if (i) it is FO-IC and (ii) conditional on truthful reporting, the consumption process $\hat{c} = s + y$ satisfies the agent's Euler equation [**5.7**] at rate $r = \rho$. Intuitively, properties (i)–(ii) defining NS-FO-IC contracts are the infinitesimal optimality conditions implied by F-NS-IC.[12] Finally, *Contract PPI is implementable as an F-HS-IC contract* if there

[11] In an analogous discrete-time setting, Doepke and Townsend (2006) show that it is without loss of generality (in terms of implementable consumption processes and payoffs) to focus on F-HS-IC contracts.

[12] Cf. Footnotes 56 and 57 for possible technical caveats to this intuition.

is an F-HS-IC contract under which the agent's optimal joint strategy $(m, \hat{c})$ satisfies $m = m^* \equiv 0$ (truthful reporting) and $\hat{c} = c$ from [**3.4**] with $y = y^* = b$ (consumption is the same as that under Contract PPI and truthful reporting). We define implementability as an F-NS-IC contract in the obvious analogous manner.

Theorem 4. *Given any $\lambda \geqslant 0$, Contract PPI satisfies the following properties:*

(i) *It is the unique (hence, optimal) NS-FO-IC contract.*

(ii) *It is implementable as an F-HS-IC contract for any $F \subseteq \{m : m$ is b-adapted$\}$.*

Proof. Point (i): Under any NS-FO-IC contract, the agent's flow utility $u_t \equiv u(s_t + y_t)$ and marginal flow utility $u'(s_t + y_t) \equiv -\theta u_t$ are P^*-martingales. Plugging this into [**3.1**]–[**3.2**] and using Tonelli's Theorem to interchange the order of integration yields $q_t \equiv u_t/\rho$ and $p_t \equiv \theta u_t/(\rho + \lambda)$. The only FO-IC contract with these properties is Contract PPI. *Point (ii):* As noted in Section 5.3, Contract PPI can be implemented as an SI Contract with zero taxes ($r = \rho$); moreover, this can be done with deterministic flow transfers rather than a lump-sum transfer at $t = 0$, per Footnote 36. This implies the present result because (a) those transfers are independent of the agents' reports by construction and (b) from the agent's perspective, saving via the ambient market is a perfect substitute for saving via the principal at rate ρ. □

The intuition for Theorem 4(i) is familiar from Allen's (1985) two-period model. With hidden savings, the agent only cares about the net present value (NPV) of the contract's transfers, so HS-IC requires that the agent receive the same NPV along every path of reports. Thus, it is *as if* all transfers were made in lump-sum at time $t = 0$, as in the SI Contract with no taxes ($r = \rho$), which implements Contract PPI. Two aspects of Theorem 4(ii) warrant elaboration:

- Theorem 4(ii) states that Contract PPI is implementable even when the agent's misreporting strategies are permitted to violate the no Ponzi constraint [**NP-*m***]. This might seem to contradict Observation 2 and Theorem 2, but it does not. The above proof shows that, with hidden savings, Contract PPI can be implemented *without communication* via determinstic transfers by *having the agent save and consume outside of the contract*. In such implementations, the no Ponzi constraint on assets [**NP-A**]—which is necessary for the agent's self-insurance problem to be well-posed—effectively imposes the same restrictions on the agent's consumption that [**NP-*m***] does in PPI's model without hidden savings. This suggests that [**NP-*m***] is needed for PPI's model to be well-behaved: without it, we would reach the unreasonable conclusion that Contract PPI is not IC in the original model (without hidden savings) but is HS-IC in the hidden savings model, wherein the agent has access to more deviations.
- Theorem 4(ii) does *not* state that Contract PPI is always implementable as an F-NS-IC contract. In fact, the obvious adaptation of Observation 2 implies that it is

not $\left[\mathscr{M}^{\mathrm{LAC}}_{\leqslant} \cap \mathscr{M}^{r}\right]$-NS-IC for any $r > \rho$. Again, this suggests that [**NP-*m***] (with an appropriately chosen rate r) is needed for PPI's model to be well-behaved: without it, restricting attention to NS-IC contracts would be *with* loss of generality, undercutting a key simplification on which much of the hidden savings literature is based (e.g., Cole and Kocherlakota 2001; DeMarzo and Sannikov 2006; He et al. 2017).

I. Incentive Compatibility of DR-SICs without Jump Reports

In this appendix, we consider the following corollary of Theorem 2:

Theorem 5. *For any given $r > 0$, every DR-SIC (b_0, q_0, r) is $\mathscr{M}^r$-IC. Per Fact 1, such contracts are also F-IC for any smaller strategy space $F \subseteq \mathscr{M}^r$.*

Our goal is to prove Theorem 5 from first principles, *without reference to the "extended" reporting problem from Section 6*, and to highlight some subtleties that arise in such an analysis. We first introduce the requisite definitions, then discuss the relevant subtleties, and finally present the proof (sketch) of Theorem 5.

Preliminaries. Fix a DR-SIC with rate $r > 0$. When the agent's strategy space is $\mathscr{M}^r$, she is constrained to misreports with absolutely continuous sample paths, viz., $m_t \equiv \int_0^t \Delta_\tau \mathrm{d}\tau$. Thus, as in PPI, her reporting problem can be viewed as one of stochastic control with states $(A_t^v, y_t, m_t) \in \mathbb{R}^3$ and control $\Delta_t \in \mathbb{R}$. As in Appendix B, $V^{\mathrm{NJ}}(A_t^v, y_t, m_t)$ denotes the agent's value function in this problem.

For a smooth function $\psi \in \mathbf{C}^2(\mathbb{R}^3)$ and $\Delta_t \in \mathbb{R}$, define the *infinitesimal generator*[13]

$$\begin{aligned}\mathscr{L}^{\Delta_t}\psi(A_t^v, y_t, m_t) := {}& [\mu - \lambda \cdot (y_t - m_t) + \Delta_t]\, \psi_y(A_t^v, y_t, m_t) + \Delta_t \cdot \psi_m(A_t^v, y_t, m_t) \\ &+ \left[\bar{A}(r;\lambda) + \frac{\lambda}{r+\lambda} y_t\right] \psi_A(A_t^v, y_t, m_t) + \frac{\sigma^2}{2} \cdot \psi_{yy}(A_t^v, y_t, m_t),\end{aligned}$$

where $\bar{A}(r;\lambda)$ is the constant from [**5.5**]; define the *Hamiltonian*

$$\mathscr{H}(A_t^v, y_t, m_t \mid \psi) := u(\hat{C}(A_t^v, y_t) - m_t) + \sup_{\Delta_t \in \mathbb{R}} \left[\mathscr{L}^{\Delta_t}\psi(A_t^v, y_t, m_t)\right], \qquad [\mathscr{H}]$$

where $\hat{C}(A_t^v, y_t)$ is from [**5.4**]; and define the *("no jump") HJB equation*

$$\rho\psi(A_t^v, y_t, m_t) = \mathscr{H}(A_t^v, y_t, m_t \mid \psi). \qquad [\textbf{HJB-NJ}]$$

The relevant notion of a "solution" to [**HJB-NJ**] will be that of a *supersolution*.

Definition I.1. A (continuous) locally bounded function $F : \mathbb{R}^3 \to \mathbb{R}$ is:

[13] We have $\mathrm{d}m_t = \Delta_t \mathrm{d}t$, $\mathrm{d}y_t = \mathrm{d}b_t + \mathrm{d}m_t$ (where $\mathrm{d}b_t$ satisfies [**2.1**] and $b_t \equiv y_t - m_t$), and $\mathrm{d}A_t^v = [\bar{A}(r;\lambda) + \frac{\lambda}{r+\lambda} y_t]\mathrm{d}t$ by [**6.1**].

(i) A *viscosity supersolution* of [**HJB-NJ**] if

$$\rho\psi(A_t^v, y_t, m_t) \geqslant \mathscr{H}(A_t^v, y_t, m_t \mid \psi) \tag{I.1}$$

for all $(A_t^v, y_t, m_t) \in \mathbb{R}^3$ and $\psi \in \mathbf{C}^2(\mathbb{R}^3)$ such that (A_t^v, y_t, m_t) is a minimizer of $F - \psi$.

(ii) A *classical supersolution* of [**HJB-NJ**] if it is a viscosity supersolution and in $\mathbf{C}^2(\mathbb{R}^3)$.

(iii) A *classical solution* to [**HJB-NJ**] if it is a classical supersolution and satisfies [**I.1**] (for $\psi = F$) with equality everywhere.

Definition I.1 is standard in the stochastic control literature (e.g., Pham 2009, Ch. 3–4; Touzi 2018, Ch. 2 and 6). For essentially any (maximization) control problem, if the value function is locally bounded (but possibly non-smooth), it is necessarily a viscosity supersolution of the relevant HJB equation (Touzi 2018, Proposition 6.2). If the value function is also smooth, it is necessarily a classical supersolution of the HJB equation (Touzi 2018, Proposition 2.4).

Discussion. Generally, even if a value function is smooth, additional regularity conditions on the Hamiltonian are needed to conclude that it is a classical solution, rather than just a supersolution (Touzi 2018, Proposition 2.5).[14] It is known that *the requisite regularity conditions typically fail in settings where the control variable is unbounded and enters linearly in the infinitesimal generator, as in the present formulation of the agent's reporting problem* (e.g., Pham 2009, Sec. 3.4.2 and 4.5). In such cases, the value function only satisfies the HJB equation in the weaker sense of being a supersolution.

To illustrate, recall from Lemma B.1 that, given the agent's value function $V^{\mathrm{DR}} \in \mathbf{C}(\mathbb{R}^2)$ from Section 6.4 in the extended reporting problem (with feasible set $\mathscr{M}^r_{\mathrm{ext}}$), we can deduce that $V^{\mathrm{NJ}}(A_t^v, y_t, m_t) = V^{\mathrm{DR}}(A_t^v, y_t - m_t)$, and hence $V^{\mathrm{NJ}} \in \mathbf{C}(\mathbb{R}^3)$. This fact motivates the following observation:

Lemma I.2. The function $F(A_t^v, y_t, m_t) := V^{\mathrm{DR}}(A_t^v, y_t - m_t)$ is a classical supersolution of [**HJB-NJ**]. However, it is *not* a classical solution: [**I.1**] (with $\psi = F$) holds with equality at (A_t^v, y_t, m_t) iff $m_t = 0$.

Proof. By [**6.9**], we have $V^{\mathrm{DR}}(A_t^v, y_t - m_t) = \hat{V}^{\mathrm{DR}} \exp\left[-\theta r\left(A_t^v + \frac{b_t}{r+\lambda}\right)\right]$ for $\hat{V}^{\mathrm{DR}} = -\frac{1}{r}\exp\left[\theta \bar{A}(r;\lambda)\right]$ (where $\bar{A}(r;\lambda)$ is from [**5.5**]). Thus, for F defined as above, we have $F_A = -\theta r F$, $F_y = F_A/(r+\lambda)$, $F_m + F_y = 0$, and $F_{yy} = \theta^2 r^2 F/(r+\lambda)^2$. Furthermore, for each $(A_t^v, y_t, m_t) \in \mathbb{R}^3$, $u(\hat{C}(A_t^v, y_t) - m_t) = \exp\left[\frac{\theta\lambda}{r+\lambda} m_t\right] \cdot rF(A_t^v, y_t - m_t)$. These

[14] These conditions concern the Hamiltonian's continuity, when viewed as a function of the partial derivatives of ψ. For all other HJB equations stated in this paper (viz., [**6.9**] in Section 6.4, [**E.2**] in Appendix E, and [**J.6**] and [**J.9**] in Appendix J.3.1) it can be shown that the regularity conditions in Touzi (2018, Proposition 2.5) are satisfied because the control variables enter the strictly concave/convex "flow return" functions, yielding interior optima and allowing one to solve for the relevant Hamiltonians in closed-form. This justifies our focus on classical solutions elsewhere in the paper.

properties imply that

$$\rho F(A_t^v, y_t, m_t) - \mathscr{H}(A_t^v, y_t, m_t \mid F) = \underbrace{-rF(A_t^v, y_t, m_t)}_{>0} \cdot \underbrace{\left(\exp\left[\frac{\theta\lambda}{r+\lambda} m_t\right] - 1 - \frac{\theta\lambda}{r+\lambda} m_t\right)}_{\geqslant 0,\text{ with equality iff } m_t = 0}$$

where (strict) positivity of the last term follows from strict convexity of $x \mapsto e^x$. □

One way to "restore equality" in the agent's HJB equation is to reformulate the agent's problem by expanding his strategy space and treating $m_t \in \mathbb{R}$ as a control (rather than state) variable, as in Section 6. Another way is to keep $\Delta_t \in \mathbb{R}$ as the control, but reformulate [**HJB-NJ**]. For instance, noting that $\mathscr{H}(A_t^v, y_t, m_t \mid \psi) < \infty$ iff $\psi_y(A_t^v, y_t, m_t) + \psi_m(A_t^v, y_t, m_t) = 0$, [**HJB-NJ**] is equivalent to the variational inequality

$$\min\left\{\rho\psi(\cdot) - \mathscr{H}(\cdot \mid \psi), \left|\psi_y(\cdot) + \psi_m(\cdot)\right|\right\} = 0$$

described in Pham's (2009) treatment of *singular control* problems. Alternatively, one can reformulate [**HJB-NJ**] as the variational inequality in the associated *impulse control* problem (Oksendal and Sulem 2019) where setting $\Delta_t = \pm\infty$ is viewed as inducing a jump in m_t, as in Strulovici (2022).

Proof of Theorem 5. We work directly with [**I.1**], the inequality version of [**HJB-NJ**]. We describe the main steps, only sketching some details for brevity.

Step 1: Shape of Value Function. It is not *a priori* clear that the value function V^{NJ} coincides with V^{DR} (cf. Lemma B.1). But by applying the same controls at different states, it can be shown that

$$V^{\text{NJ}}(A_t^v, y_t, m_t) = \hat{V}^{\text{NJ}} \cdot h(m_t) \cdot \exp\left[-\theta r\left(A_t^v + \frac{y_t}{r+\lambda}\right)\right] \qquad \textbf{[I.2]}$$

for some constant $\hat{V}^{\text{NJ}} < 0$ and convex function $h : \mathbb{R} \to \mathbb{R}_{++}$. Without loss of generality, we can normalize $h(0) := 1$.

Step 2: Determining the h ***Function.*** It is not *a priori* clear that h is smooth. But Step 1 implies that V^{NJ} is locally bounded, so Touzi (2018, Proposition 6.2) implies that V^{NJ} is a viscosity supersolution of [**HJB-NJ**]. By standard smooth approximation results,[15] there exists a dense subset $D \subseteq \mathbb{R}$ such that, at every point $m_t \in D$, there exists a function $\phi_{(m_t)} \in \mathbf{C}^2(\mathbb{R})$ satisfying $\phi_{(m_t)}(m_t) = h(m_t)$ and $\phi_{(m_t)}(\cdot) \geqslant h(\cdot)$; hence, h is differentiable at m_t and $h'(m_t) = \phi'_{(m_t)}(m_t)$. Thus, for any (A_t^v, y_t, m_t) with $m_t \in D$, the

[15] See, for instance, Lemma 8(g), Theorem 9, and associated discussion in Katzourakis (2015, Ch. 2).

function

$$\text{[I.3]} \qquad \psi_{(m_t)}(m_t)(A_t^v, y_t, m_t) := \hat{V}^{\mathrm{NJ}} \cdot \phi_{(m_t)}(m_t) \cdot \exp\left[-\theta r\left(A_t^v + \frac{y_t}{r+\lambda}\right)\right]$$

satisfies the conditions of Definition I.1(i). Plugging [I.3] into [$\mathscr{H}$], we see that $\mathscr{H}(A_t^v, y_t, m_t \mid \psi_{(m_t)}) < \infty$—a necessary condition for [I.1]—iff $\phi'_{(m_t)}(m_t) = f(r;\lambda)\phi_{(m_t)}(m_t)$, which is equivalent to $h'(m_t) = f(r;\lambda)h(m_t)$. As $h'(\cdot) = f(r;\lambda)h(\cdot)$ on the dense set $D \subseteq \mathbb{R}$ and h is convex (hence continuous and directionally differentiable on $\mathbb{R}$), it can be shown that $h'(\cdot) = f(r;\lambda)h(\cdot)$ on $\mathbb{R}$. The unique solution to this ODE satisfying $h(0) = 1$ is

$$\text{[I.4]} \qquad h(m_t) = \exp\left[f(r;\lambda)m_t\right].$$

Step 3: Determining the $\hat{V}^{NJ}$ Constant. Combining [I.1], [I.2], and [I.4] yields $\hat{V}^{\mathrm{NJ}} \geqslant -\frac{1}{r}\exp\left[\theta\bar{A}(r;\lambda)\right]$. We conclude that this inequality holds with equality because $V^{\mathrm{DR}}(A_t^v, y_t - m_t) \geqslant V^{\mathrm{NJ}}(A_t^v, y_t, m_t)$ by construction (cf. Section 6.4).[16]

Step 4: Optimal Strategy. Together, [I.2] and [I.4] imply that, for all states (A_t^v, y_t, m_t), the supremum in [$\mathscr{H}$] is attained at $\Delta_t = 0$ (in fact, at any $\Delta_t \in \mathbb{R}$). Step 3 and the same calculations underlying Lemma I.2 imply that V^{NJ} satisfies [I.1] with equality when $m_t = 0$. Thus, integrating [I.1] forward from an truthful initial state $(A_0^v, y_0, m_0 = 0)$ implies that truthful reporting attains $V^{\mathrm{NJ}}(A_0^v, y_0, m_0 = 0)$, i.e., [IC] holds. □

J. Optimal FO-IC Contracts

This appendix presents supporting details for the discussion of fully optimal contracts in Section 7.1. In Appendix J.3.1 (see Remark J.10), we also independently verify Steps 1–3 and 5 of PPI's derivation of Contract PPI, as described in Appendix A.

J.1. Domain of Implementable (Marginal) Promised Utilities

Lemma J.1. For each $q_0 < 0$, the following hold:

(i) If $\lambda > 0$, then under any FO-IC contract $k_t \in (0,\theta)$ for all $t \geqslant 0$. Furthermore, for each $k_0 \in (0,\theta)$ there exists an FO-IC contract with $k_t \equiv k_0$.

(ii) If $\lambda = 0$, then under any FO-IC contract $k_t = \theta$ for all $t \geqslant 0$. Furthermore, an FO-IC contract exists.

[16] This is the only place in the proof that we reference the extended reporting problem from Section 6; alternatively, we could reach the same conclusion by appealing to the (equivalent) self-insurance problem from Section 5.1. We do not know if this step can be avoided: [I.1] only places a lower bound on $\hat{V}^{\mathrm{NJ}}$ and *in principle* could hold as a strict inequality everywhere, so it seems necessary to appeal elsewhere for an upper bound on $\hat{V}^{\mathrm{NJ}}$. (One could instead conjecture that $\hat{V}^{\mathrm{NJ}} = -\frac{1}{r}\exp\left[\theta\bar{A}(r;\lambda)\right]$ and then appeal to a "verification theorem" as desribed in Section 4.2, but this would entail additional technical restrictions on the agent's strategy space beyond those embodied in $\mathscr{M}^r$. See Pham (2009, Theorem 3.5.3) for details.)

Proof. Point (i): Let $\lambda > 0$ and fix an FO-IC contract. Since $\lambda > 0$ and $u(\cdot) < 0$, we have $u_t < e^{-\lambda t} u_t < 0$ for all $t > 0$, so [3.1]–[3.2] imply that $p_t > \theta q_t$. Dividing through by $q_t < 0$ yields $k_t = p_t/q_t < \theta$, while $q_t, p_t < 0$ implies $k_t > 0$. DR-SICs demonstrate the existence claim (see [5.15] and recall that $f(\cdot;\lambda)$ has range $(0,\theta)$).

Point (ii): Let $\lambda = 0$ and fix an FO-IC contract. Since $\lambda = 0$, we have $u_t \equiv e^{-\lambda t} u_t$. Thus, [3.1] and [3.2] imply that $p_t = \theta q_t$, and dividing through by $q_t < 0$ yields $k_t = p_t/q_t = \theta$. Contract PPI demonstrates the existence claim. $\square$

Lemma J.1 allows us to define the *domain* D of implementable (q,p) pairs by

[J.1]
$$D := \begin{cases} \{(q,p) \in \mathbb{R}^2_{--} : p/q \in (0,\theta)\} & \text{if } \lambda > 0 \\ \{(q,p) \in \mathbb{R}^2_{--} : p = \theta q\} & \text{if } \lambda = 0. \end{cases}$$

J.2. Permanent Shocks

Theorem 6. *If $\lambda = 0$, then Contract PPI satisfies the following properties:*

(i) *It is the unique optimal FO-IC contract.*

(ii) *It is F-IC for any feasible set $F \subseteq \{m : m$ is b-adapted$\}$.*

Proof. Point (i). For any FO-IC contract, Lemma J.1 implies that $p_t \equiv \theta q_t$. Display [A.1] then implies that, under truthful reporting, promised utility satisfies

[J.2]
$$q_t = q_0 \exp\left[\int_0^t \left(\rho - \beta_\tau - \frac{\sigma^2\theta^2}{2}\right) \mathrm{d}\tau - \sigma\theta W_t\right],$$

where $\beta_t \equiv u_t/q_t$. The agent's recommended consumption process is then $c_t \equiv \bar{c}(q_t,\beta_t) := -\log(-q_t\beta_t)/\theta$. Thus, substituting the transfer process $s_t \equiv \bar{c}(q_t,\beta_t) - y_t$ into [2.2] and ignoring terms that do not involve β, the principal minimizes

$$\mathsf{E}_0^*\left[\int_0^\infty e^{-\rho t}\left(-\log(\beta_t) + \int_0^t \beta_\tau \mathrm{d}\tau\right)\mathrm{d}t\right]$$

over all b-adapted, strictly positive β processes. This objective is strictly convex, so the optimal β process is unique, deterministic, and satisfies the pointwise FOC[17]

$$\frac{\mathrm{d}}{\mathrm{d}\beta_t}\left[-\log(\beta_t) + \frac{\beta_t}{\rho}\right] = 0 \quad \text{for all } t \geqslant 0.$$

[17]Focusing on deterministic β and applying integration by parts, the objective in [J.3] becomes $\int_0^\infty e^{-\rho t}(-\log(\beta_t) + \beta_t/\rho)\,\mathrm{d}t$. Note that the flow cost in this transformed objective satisfies an Inada condition at $\beta_t = 0$, implying that the optimal process must be strictly positive. Thus, the pointwise first-order condition from this transformed objective is [J.3].

Thus, the optimal β is $\beta_t \equiv \rho$, which upon substitution into [J.2] yields Contract PPI.
Point (ii). Under Contract PPI, $c_t = \bar{c}(q_0;\rho) + \bar{A}(\rho;0)t + y_t$ when $\lambda = 0$. Thus, the transfer process $s_t \equiv c_t - y_t$ is deterministic, i.e., report-independent. $\square$

Remark J.2. Theorem 6(i) can be strengthened: *Contract PPI is the unique optimal contract satisfying $\gamma_t + p_t \geqslant 0$ for all $t \geqslant 0$*, the one-sided variant of [FO-IC] that is appropriate under NHB or IML (cf. Footnote 57). We adapt the above proof as follows. Defining $\hat{k}_t := -\gamma_t/q_t$, the constraint becomes $\hat{k}_t \geqslant \theta$. Promised utility satisfies $q_t = q_0 \exp\left[\int_0^t \left(\rho - \beta_\tau - \frac{\sigma^2\hat{k}_\tau^2}{2}\right)\mathrm{d}\tau - \int_0^t \sigma\hat{k}_\tau \mathrm{d}W_\tau\right]$. The principal's problem is then additively separable in β and $\hat{k}$. The optimization over β is unchanged. The portion of the principal's objective involving $\hat{k}$ is $\mathsf{E}_0^*\left[\int_0^\infty e^{-\rho t}\left(\frac{\sigma^2}{2}\int_0^t \hat{k}_\tau^2 \mathrm{d}\tau + \sigma\int_0^t \hat{k}_\tau \mathrm{d}W_\tau\right)\right]$. The stochastic integral vanishes in expectation, and (constrained) minimization of the first term yields $\hat{k}_t \equiv \theta$.

J.3. Transient Shocks

Let $J(y_0, q_0, p_0)$ denote the principal's value function over FO-IC contracts given the initial condition (y_0, q_0, p_0) (see [J.4] below for details). We call $J : \mathbb{R} \times D \to \mathbb{R}$ the principal's *FO value function*.

Definition J.3. The environment is *regular* if:

(i) An optimal FO-IC contract (as defined in Appendix A) exists.

(ii) The principal's FO value function J is twice continuously differentiable.

Regularity is a technical assumption, which is implicitly adopted in PPI. It is needed to analyze the principal's FO problem with standard stochastic control techniques.

Theorem 7. *Suppose that $\lambda > 0$. If the environment is regular, then the optimal DR-SIC is not an optimal FO-IC contract.*

We prove Theorem 7 in Appendix J.3.1 below. Recall that DR-SICs are equivalent to Stationary contracts, i.e., those with constant $k_t = p_t/q_t$ processes (Lemma G.1). Thus, Theorem 7 equivalently states that the optimal FO-IC contract is not Stationary, consistent with Implication 2 at the end of Appendix A. Of course, if the first-order approach is valid—viz., every IC contract is FO-IC, and the optimal FO-IC contract is IC—then Theorem 7 also implies that the fully optimal contract outperforms the optimal DR-SIC/Stationary contract. It is an open question whether the first-order approach is valid in PPI's model with $\lambda > 0$.

J.3.1. *Proof of Theorem 7*

By the Martingale Representation Theorem, the agent's promised utility process q (defined in [3.1]) under any FO-IC contract satisfies [A.1] and FO-IC, and his marginal promised utility process p (defined in [3.2]) satisfies

$$\text{[J.3]} \qquad \mathrm{d}p_t = \left[(\rho+\lambda)p_t - \theta u_t\right]\mathrm{d}t + Q_t \sigma \mathrm{d}W_t^y,$$

where $\sigma \mathrm{d}W_t^y \equiv \mathrm{d}y_t - (\mu - \lambda y_t)\mathrm{d}t$ and Q is a y-adapted process.

Definition J.4. The principal's *auxiliary first-order (FO) problem* is

$$\text{[J.4]} \qquad J(y_0, q_0, p_0) := \inf_{(c,Q)\in \mathscr{A}_P(y_0,q_0,p_0)} \mathsf{E}_0^* \left[\int_0^\infty e^{-\rho t}\left(c_t - b_t\right)\,\mathrm{d}t\right]$$

where $\mathscr{A}_P(y_0, q_0, p_0)$ consists of the y-adapted processes (c, Q) such that [A.1] and [J.3] have unique solutions satisfying [FO-IC] and the transversality conditions: for all $t \geqslant 0$, $\lim_{T\to\infty} \mathsf{E}_t^*\left[e^{-\rho(T-t)}q_T\right] = \lim_{T\to\infty} \mathsf{E}_t^*\left[e^{-\rho(T-t)}p_T\right] = 0$.[18] The *first-stage FO problem* is

$$\text{[J.5]} \qquad \inf_{p_0<0 \text{ s.t. } (q_0,p_0)\,\in\, D} J(y_0, q_0, p_0),$$

and the *FO problem* is the joint optimization [J.4]–[J.5].

Definition J.3(ii) requires that $J \in \mathbf{C}^2(\mathbb{R}\times D)$. Standard arguments (Yong and Zhou 1999, Theorem 3.3; Touzi 2018, Propositions 2.4–2.5) then imply that J is a classical solution to the HJB equation

$$\text{[J.6]} \qquad \begin{aligned} \rho J(y_t, q_t, p_t) = \min_{(c_t,Q_t)\in\mathbb{R}^2} \Big[& c_t - y_t + J_y(y_t, q_t)\cdot(\mu - \lambda y_t) + J_q(y_t, q_t, p_t)\cdot(\rho q_t - u(c_t)) \\ & + J_p(y_t, q_t, p_t)\cdot((\rho+\lambda)p_t + \theta u(c_t)) \\ & + \frac{\sigma^2}{2} J_{yy}(y_t, q_t, p_t) + \frac{\sigma^2 p_t^2}{2} J_{qq}(y_t, q_t, p_t) + \frac{\sigma^2 Q_t^2}{2} J_{pp}(y_t, q_t, p_t) \\ & - \sigma^2 p_t J_{yq}(y_t, q_t, p_t) + \sigma^2 Q_t J_{yp}(y_t, q_t, p_t) - \sigma^2 p_t Q_t J_{qp}(y_t, q_t, p_t)\Big]. \end{aligned}$$

(Cf. display (19) on p. 1249 of PPI.) We wish to rewrite [J.6] in terms of (y_t, q_t, k_t). This requires two lemmas, the latter of which appears in PPI as a conjecture.

Lemma J.5. Under any FO-IC contract and truthful reporting, the $k_t \equiv p_t/q_t$ process satisfies

$$\text{[J.7]} \qquad \mathrm{d}k_t = \left[(\beta_t + \lambda)\,k_t - \theta\beta_t + \sigma^2 k_t\left(k_t^2 - \hat{Q}_t\right)\right]\,\mathrm{d}t + \sigma\left[k_t^2 - \hat{Q}_t\right]\,\mathrm{d}W_t,$$

where the process $\hat{Q} = (\hat{Q}_t)_{t\geqslant 0}$ is defined as $\hat{Q}_t := -Q_t/q_t$.

Proof. Apply Ito's lemma to [A.1] and [J.3] under truthtelling ($W_t^y \equiv W_t$). □

[18] As in Section 2, we also implicitly restrict attention to (c, Q) processes such that the double-integral defining $J(y_0, q_0, p_0)$ is well-defined.

Lemma J.6. Let $\lambda > 0$. If the environment is regular, then J satisfies $J(y_t, q_t, p_t) \equiv \hat{J}(y_0, q_0, p_0/q_0)$, where

[**J.8**]
$$\hat{J}(y_0, q_0, k_0) := -\frac{y_0}{\rho + \lambda} - \frac{\log(-q_0)}{\rho\theta} + h(k_0)$$

for some function $h \in \mathbf{C}^2\left((0, \theta)\right)$.

Proof. Regularity implies that J is well-defined and finite-valued and, given [**J.8**], also that $h \in \mathbf{C}^2\left((0, \theta)\right)$. Let $(q_0, p_0) \in D$, $y_0 \in \mathbb{R}$, and $\alpha > 0$ be given for Steps 1-2 below.
Step 1: We assert that $J(y_0, q_0, p_0) = J(0, q_0, p_0) - y_0/(\rho + \lambda)$. Let $(c, Q) \in \mathscr{A}_P(y_0, q_0, p_0)$ be given. Define $g_t := y_0 e^{-\lambda t}$, $\tilde{c}_t(y) := c_t(y + g)$, and $\tilde{Q}_t(y) := Q_t(y + g)$.[19] We have $(\tilde{c}, \tilde{Q}) \in \mathscr{A}_P(0, q_0, p_0)$. Let $\mathbf{P}^{*,(y_0)}$ denote the distribution over report paths starting from y_0 and $\mathbf{P}^{*,(0)}$ denote the distribution starting from $\tilde{y}_0 = 0$, assuming truthful reporting. The law of (c, Q) under $\mathbf{P}^{*,(y_0)}$ equals the law of $(\tilde{c}, \tilde{Q})$ under $\mathbf{P}^{*,(0)}$. Thus,

$$\mathbf{E}_0^{*,(y_0)}\left[\int_0^\infty e^{-\rho t}\left(c_t - y_t\right)\,\mathrm{d}t\right] = \mathbf{E}_0^{*,(0)}\left[\int_0^\infty e^{-\rho t}\left(\tilde{c}_t - y_t\right)\,\mathrm{d}t\right] - \underbrace{\int_0^\infty e^{-\rho t} g_t\,\mathrm{d}t}_{= y_0/(\rho+\lambda)}.$$

Step 2: We assert that $J(y_0, \alpha q_0, \alpha p_0) = J(y_0, q_0, p_0) - \log(\alpha)/(\rho\theta)$. Let $(c, Q) \in \mathscr{A}_P(y_0, q_0, p_0)$ be given. Define $\tilde{c}_t := c_t - \log(\alpha)/\theta$ and $\tilde{Q}_t := \alpha Q_t$. Note that $u(\tilde{c}_t) \equiv \alpha u(c_t)$. Display [**A.1**], [**J.3**], and [**FO-IC**] then imply that $(\tilde{c}, \tilde{Q}) \in \mathscr{A}_P(y_0, \alpha q_0, \alpha p_0)$. The distribution over endowment paths is the same at both initial states, so the principal's cost of $\tilde{c}$ at $(y_0, \alpha q_0, \alpha p_0)$ equals her cost of c at (y_0, q_0, p_0) plus $\int_0^t e^{-\rho t}\left(-\log(\alpha)/\theta\right)\,\mathrm{d}t = -\log(\alpha)/(\theta\rho)$.
Step 3: Fix $(q_0, p_0) \in D$ and $y_0 \in \mathbb{R}$. Combining Steps 1 and 2 with $\alpha = -p_0/q_0$ yields

$$J(y_0, q_0, p_0) = -\frac{y_0}{\rho + \lambda} - \frac{\log(-q_0)}{\rho\theta} + J\left(0, -1, -p_0/q_0\right).$$

Defining $h(k_0) := J\left(0, -1, -k_0\right)$ and $\hat{J}$ as in [**J.8**] completes the proof. □

Using Lemmas J.5 and J.6, we can write the HJB equation [**J.6**] in terms of (y_t, q_t, k_t) as

[**J.9**]
$$\begin{aligned}\rho\hat{J}(y_t, q_t, k_t) = \min_{\beta_t > 0, \hat{Q}_t \in \mathbb{R}} &\Big\{\overline{c}(q_t, \beta_t) - y_t - \frac{1}{\rho+\lambda}\left[\mu - \lambda y_t\right] - \frac{1}{\rho\theta}\left[\rho - \beta_t\right] \\ &+ h'(k_t)\left[\left(\beta_t + \lambda\right)k_t - \theta\beta_t + \sigma^2 k_t\left(k_t^2 - \hat{Q}_t\right)\right] \\ &+ \frac{\sigma^2 k_t^2}{2\rho\theta} + \frac{\sigma^2(k_t^2 - \hat{Q}_t)^2}{2} h''(k_t)\Big\},\end{aligned}$$

[19] That is, the path of $(\tilde{c}, \tilde{Q})$ when the agent reports the endowment path $\hat{y} \in \mathbf{C}\left([0, \infty)\right)$ is the same as the path of (c, Q) when he reports the endowment path $\hat{y} + g$.

where $\beta_t = u(c_t)/q_t$ and $\bar{c}(q_t, \beta_t) = -\log(-\beta_t q_t)/\theta$.

Lemma J.7. Let $\lambda > 0$. If the environment is regular, then any optimal $\beta^\dagger$ satisfies $\beta_t^\dagger \equiv \hat{\beta}(k_t)$, where $\hat{\beta} : (0, \theta) \to \mathbb{R}$ is defined by

[**J.10**]
$$\hat{\beta}(k_t) := \frac{1}{1/\rho + \theta(k_t - \theta)h'(k_t)}.$$

Proof. Eliminating terms on the RHS of [**J.9**] yields the following minimization over β_t, which any optimal $\beta^\dagger$ process must a.e. satisfy

[**J.11**]
$$\min_{\beta_t > 0} \left[-\log(\beta_t) + \frac{\beta_t}{\rho} + \beta_t \cdot h'(k_t)(k_t - \theta) \right].$$

The unique solution to [**J.11**] is interior and characterized by the FOC [**J.10**]. □

Lemma J.8. Let $\lambda > 0$. If the environment is regular, then any $k_0^\dagger \in \arg\min_{k_0 \in (0,\theta)} h(k_0)$ satisfies $\hat{\beta}(k_0^\dagger) = \rho$.

Proof. Because $h \in \mathbf{C}^2((0, \theta))$ by Definition J.3(i) and Lemma J.6, any such $k_0^\dagger$ must satisfy the FOC $h'(k_0^\dagger) = 0$. Plugging this into [**J.10**] completes the proof. □

Lemma J.9. Let $\lambda > 0$. If the environment is regular and an optimal FO-IC contract is Stationary (i.e., induces a constant k process), then that contract is Contract PPI.

Proof. Consider any optimal FO-IC contract that is Stationary. Lemma J.5 and the unique decomposition property for Ito processes imply that $k_t^2 \equiv \hat{Q}_t^\dagger$ (so k has zero volatility) and thus that $\beta_t^\dagger(\lambda + k_t) - \theta\beta_t^\dagger \equiv 0$ (so k has zero drift). Furthermore, [**J.5**] and [**J.8**] imply that $k_t \equiv k_0^\dagger \in \arg\min_{k_0 \in (0,\theta)} h(k_0)$. Then Lemma J.8 implies that $\hat{\beta}_t^\dagger \equiv \rho$ and Lemma G.1 implies that $k_t \equiv f(\rho; \lambda) = k_0^*$. This yields Contract PPI. □

Proof of Theorem 7. Suppose the environment is regular. Theorem 1(i) implies that the optimal SI Contract / DR-SIC strictly dominates Contract PPI. Every DR-SIC is FO-IC (by construction) and Stationary (by Lemma G.1). If an optimal FO-IC were Stationary, then Lemma J.9 would imply that it is Contract PPI, a contradiction. □

Remark J.10. The above work confirms Steps 1–3 and 5 of PPI's derivation of Contract PPI, as described in Appendix A. The conjecture in Step 1 is established by Lemma J.6. Step 2 follows from Lemma J.6 and [**J.9**]. Step 3 follows from plugging the optimal $\hat{Q}_t$ from [**J.9**] into [**J.7**]. Step 5 is Lemma J.9.

K. Properties of Brownian Motion and OU Process

This appendix collects auxiliary facts about Brownian motion (BM) and OU processes.

Lemma K.1 (SLLN for BM)**.** Let $W = (W_t)_{t\geqslant 0}$ be a standard Brownian motion. Then $\lim_{t\to\infty} W_t/g(t) = 0$ for any function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} t/g(t) < \infty$.

Proof. The $g(t) = t$ case is standard (e.g., Problem 9.4 in Karatzas and Shreve (1998, p. 104)). Thus, more generally, $\lim_{t\to\infty} W_t/g(t) = \lim_{t\to\infty} (W_t/t) \cdot \lim_{t\to\infty} (t/g(t)) = 0$. $\square$

Let b be an OU process as defined by the equation [2.1], the solution to which is

[**K.1**]
$$b_t = b_0 e^{-\lambda t} + \mu \underbrace{\left(\frac{1-e^{-\lambda t}}{\lambda}\right)}_{=t \text{ when } \lambda = 0} + \underbrace{e^{-\lambda t}\int_0^t \sigma e^{\lambda\tau}\, \mathrm{d}W_\tau}_{=: X_t}.$$

We invoke $X = (X_t)_{t\geqslant 0}$ from [K.1] below. Note that $X_t \equiv b_t$ when $b_0 = \mu = 0$.

Lemma K.2. Given X_t from [K.1], the following holds for each $t \geqslant 0$:

(i) When $\lambda = 0$,

[**K.2**]
$$\int_0^t b_\tau\, \mathrm{d}\tau = b_0 t + \frac{1}{2}\mu t^2 + \sigma\left(tW_t - \int_0^t \tau\, \mathrm{d}W_\tau\right)$$

(ii) When $\lambda > 0$,

[**K.3**]
$$\int_0^t b_\tau\, \mathrm{d}\tau = b_0\left(\frac{1-e^{-\lambda t}}{\lambda}\right) + \frac{\mu}{\lambda}\left(t - \frac{1-e^{-\lambda t}}{\lambda}\right) + \frac{\sigma W_t - X_t}{\lambda}$$

Proof. In both points (i) and (ii), the deterministic terms follow from straightforward integration of the first two terms in [K.1]. The stochastic terms follow from stochastic integration by parts calculations:

Point (i): When $\lambda = 0$, we have $X_t = \sigma W_t$. Itô's lemma yields applied to tW_t yields $tW_t = \int_0^t W_\tau\, \mathrm{d}\tau + \int_0^t \tau\, \mathrm{d}W_\tau$. Thus, $\int_0^t X_\tau\, \mathrm{d}\tau = \sigma\left(tW_t - \int_0^t \tau\, \mathrm{d}W_\tau\right)$, as desired.

Point (ii): When $\lambda > 0$, we have $X_t = e^{-\lambda t}\int_0^t \sigma e^{\lambda\tau}\, \mathrm{d}W_\tau =: e^{-\lambda t} Y_t$. Itô's lemma applied to X_t yields $\mathrm{d}X_t = -\lambda X_t\, \mathrm{d}t + e^{-\lambda t}\, \mathrm{d}Y_t = -\lambda X_t\, \mathrm{d}t + \sigma\, \mathrm{d}W_t$. Putting this in integral form and rearrangning yields $\int_0^t X_\tau\, \mathrm{d}\tau = (\sigma W_t)/\lambda - X_t/\lambda$, as desired. $\square$

Lemma K.3. The following hold (almost surely):

(i) If $\lambda > 0$, then $\lim_{t\to\infty} b_t/t = 0$. If $\lambda = 0$, then $\lim_{t\to\infty} b_t/t = \mu$,

(ii) For all $\lambda \geqslant 0$ and $\alpha > 0$, $\lim_{t\to\infty} e^{-\alpha t} b_t = 0$, and

(iii) For all $\lambda \geqslant 0$ and $\alpha > 0$, $\lim_{t\to\infty} e^{-\alpha t}\int_0^t b_\tau\, \mathrm{d}\tau = 0$.

Proof. We consider each point of the lemma in turn.

Point (i): When $\lambda = 0$, the result is immediate from [K.1] and Lemma K.1. When $\lambda > 0$, we have

$$\lim_{t\to\infty} \frac{1}{t}\left[b_0 e^{-\lambda t} + \mu\left(\frac{1-e^{-\lambda t}}{\lambda}\right)\right] = 0.$$

Thus, it suffices to show that $\lim_{t\to\infty} X_t/t = 0$ for X_t in [K.1]. Defining the *time-change* $v(t) := e^{2\lambda t} - 1$ (with inverse $t(v) := \log(v+1)/(2\lambda)$), the process $B_v := \frac{\sqrt{2\lambda}}{\sigma} e^{\lambda t} X_{t(v)}$ is a standard BM (Karatzas and Shreve 1998, p. 174). By construction,

$$\frac{X_t}{t} = \frac{\sigma\sqrt{2\lambda}B_{v(t)}}{\sqrt{v(t)}\log(v(t))}. \tag{K.4}$$

It suffices to show that the RHS of [K.4] goes to zero as $v \to \infty$. To this end, the Law of the Iterated Logarithm (Mörters and Peres 2010, p. 119) implies that

$$\limsup_{v\to\infty} \frac{|B_v|}{\sqrt{2v\log(\log(v))}} = 1 \tag{K.5}$$

and L'Hôpital's rule implies that

$$\lim_{v\to\infty} \frac{\sqrt{2v\log(\log(v))}}{\sqrt{v}\log(v)} = \lim_{v\to\infty} \frac{1}{\sqrt{2}\log(v)\sqrt{\log(\log(v))}} = 0. \tag{K.6}$$

Combining [K.5] with [K.6] yields the desired conclusion that

$$\limsup_{v\to\infty} \frac{|B_v|}{\sqrt{v}\log(v)} = \limsup_{v\to\infty} \frac{|B_v|}{\sqrt{2v\log(\log(v))}} \frac{\sqrt{2v\log(\log(v))}}{\sqrt{v}\log(v)} = 0.$$

Point (ii): Let $\alpha > 0$ be given. Then

$$\limsup_{t\to\infty} e^{-\alpha t}|b_t| = \limsup_{t\to\infty} \frac{|b_t|}{t} \cdot \frac{t}{e^{\alpha t}} = \mathbf{1}(\lambda = 0)|\mu| \cdot \lim_{t\to\infty} \frac{t}{e^{\alpha t}} = 0$$

where the second equality follows from point (i).
Point (iii): First, suppose that $\lambda > 0$. Lemma K.2(ii) yields

$$e^{-\alpha t}\int_0^t b_\tau \,\mathrm{d}\tau = e^{-\alpha t}\left[b_0\left(\frac{1-e^{-\lambda t}}{\lambda}\right) + \frac{\mu}{\lambda}\left(t - \frac{1-e^{-\lambda t}}{\lambda}\right)\right] + \frac{\sigma}{\lambda}e^{-\alpha t}W_t - \frac{e^{-\alpha t}}{\lambda}X_t$$

The first term clearly goes to zero as $t \to \infty$. The second term also goes to zero by Lemma K.1. The third term goes to zero by point (ii) of the present lemma. Next, suppose that $\lambda = 0$. Lemma K.2(i) yields

$$e^{-\alpha t}\int_0^t b_\tau \,\mathrm{d}\tau = e^{-\alpha t}\left[b_0 t + \frac{1}{2}\mu t^2\right] + \sigma e^{-\alpha t} t W_t - \sigma e^{-\alpha t}\int_0^t \tau \,\mathrm{d}W_\tau \tag{K.7}$$

The first term clearly goes to zero as $t \to \infty$, as does the second term because

$$\lim_{t\to\infty} e^{-\alpha t} t W_t = \lim_{t\to\infty} \frac{W_t}{e^{\alpha t/2}} \cdot \underbrace{\lim_{t\to\infty} \frac{t}{e^{\alpha t/2}}}_{=0}$$

and $\lim_{t\to\infty} W_t/e^{\alpha t/2} = 0$ by Lemma K.1. For the final term in [K.7], define $Z_t := \int_0^t \tau \, \mathrm{d}W_\tau$. Using the time-change $\phi(t) := t^3/3$ (with inverse $t(\phi) := (3\phi)^{1/3}$), the process $\hat{B}_{\phi(t)} := Z_t$ is a standard BM. Therefore,

$$\lim_{t\to\infty} e^{-\alpha t} Z_t = \lim_{\phi\to\infty} e^{-\alpha t(\phi)} \hat{B}_\phi = \lim_{\phi\to\infty} \frac{\hat{B}_\phi}{\phi} \cdot \frac{\phi}{\exp(\alpha(3\phi)^{1/3})} = 0,$$

where the last equality is by Lemma K.1 and L'Hôpital's rule. □

Lemma K.4. For any $\alpha > 0$, the following holds:

[**K.8**] $$\int_0^\infty e^{-\alpha t} b_t \, \mathrm{d}t = \frac{b_0}{\alpha+\lambda} + \frac{\mu}{\alpha(\alpha+\lambda)} + \frac{\sigma}{\alpha+\lambda} \int_0^\infty e^{-\alpha t} \, \mathrm{d}W_t$$

Proof. We integrate [K.1], discounted by $e^{-\alpha t}$. The first two terms yield

[**K.9**] $$\int_0^\infty e^{-\alpha t} \left[b_0 e^{-\lambda t} + \mu \underbrace{\left(\frac{1-e^{-\lambda t}}{\lambda} \right)}_{= t \text{ when } \lambda = 0} \right] \mathrm{d}t = \frac{b_0}{\alpha+\lambda} + \frac{\mu}{\alpha(\alpha+\lambda)}$$

For the final $\int_0^\infty e^{-\alpha t} X_t \, \mathrm{d}t$ term, applying Itô's lemma twice yields

$$e^{-\alpha T} X_T = -(\alpha+\lambda) \int_0^T e^{-\alpha t} X_t \, \mathrm{d}t + \int_0^T e^{-\alpha t} \sigma \, \mathrm{d}W_t.$$

Rearranging and letting $T \to \infty$, we obtain

[**K.10**] $$\int_0^\infty e^{-\alpha t} X_t \, \mathrm{d}t = \frac{\sigma}{\alpha+\lambda} \int_0^\infty e^{-\alpha t} \, \mathrm{d}W_t - \underbrace{\frac{1}{\alpha+\lambda} \lim_{T\to\infty} e^{-\alpha T} X_T}_{= 0 \text{ by Lemma K.3(ii)}}.$$

Combining [K.1], [K.9], and [K.10] completes the proof. □