COMPARATIVE STATICS WITH THE INTERVAL DOMINANCE ORDER: SOME EXTENSIONS[1]

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Extended Abstract: In Quah and Strulovici (2007) (henceforth to be referred to as QS07), we identified a natural way of ordering functions, which we called the *interval dominance order*, and showed that this concept is useful in the theory of monotone comparative statics and also in statistical decision theory. This ordering on functions is weaker than the standard one based on the single crossing property (Milgrom and Shannon, 1994) and so monotone comparative statics results based on this property apply in some settings where the single crossing property does not hold. We also showed that certain basic results in statistical decision theory which are important in economics - specifically, the complete class theorem of Karlin and Rubin (1956) and the results connected with Lehmann's (1988) concept of informativeness - generalize to payoff functions that obey the interval dominance order.

These incomplete notes gather together some new results not found in QS07. The reader is advised to read these notes in conjunction with that earlier paper. The notes are divided into three sections:

Section 1 defines the interval dominance order for functions defined on lattices and develops a theory of monotone comparative statics around this property. The results here generalize the ones in QS07, which only considered the interval dominance order on functions defined on the real line.

Section 2 uses the interval dominance order to study the impact of the discount rate in a problem of optimal stopping; it generalizes to a stochastic setting Example 2 in QS07.

Section 3 has more applications of the theory developed in Section 1.

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1. The interval dominance order

Let X be a set and $\geq$ a relation defined on X. We say that $(X, \geq)$ is a *partially ordered set* (or *poset* for short) if $\geq$ obeys the following properties: (i) if $x \geq y$ and $y \geq z$ then $x \geq z$ (transitivity); (ii) for all x in X, $x \geq x$ (reflexivity); and (iii) if $x \geq y$ and $y \geq x$ then $x = y$ (anti-symmetry). For any two elements x' and x'', we denote the set $\{x \in X : x' \leq x \leq x''\}$ by $[x', x'']$. A subset J of X is an *interval of* X if, whenever x' and x'' are in J, the set $[x', x'']$ is also in J. It is clear that sets of the form $[x', x'']$ are also intervals.

The partially ordered set $(X, \geq)$ is a *lattice* if any two points x and y in X have a supremum and an infimum. The supremum of x and y refers to that element in X which is greater than x and y (with respect to $\geq$) and smaller any other point that is also greater than x and y. We denote the supremum by $x \vee y$. (Note that it follows from anti-symmetry that the supremum must be unique.) Similarly, the infimum of x and y refers to the unique element in X which is smaller than x and y and greater than any other element that is also smaller than both x and y. We denote the infimum by $x \wedge y$. A subset S of X is a *sublattice of* $(X, \geq)$ if for any x and y in S, the points $x \vee y$ and $x \wedge y$ are also in S. Note that a sublattice of $(X, \geq)$ is a lattice in its own right (with the inherited order).[3]

The most familiar example of a lattice is R^l, endowed with the *product order*, i.e., for y and x in R^l, $y \geq x$ if $y_i \geq x_i$ for $i = 1, 2, ..., l$. With this order, it is not hard to see that ith-entry of $x \vee y$ is just $\max\{x_i, y_i\}$. Similarly, the ith-entry of $x \wedge y$ is $\min\{x_i, y_i\}$. From this point onwards, whenever a subset X of R^l is considered, we shall assume that the order $\geq$ on X is the product order.

Definition and properties of IDO

Let f and g be two functions mapping the partially ordered set $(X, \geq)$ to R. Following Milgrom and Shannon (1994), we say that g dominates f by the *single*

[3] For more information on lattices that are relevant to comparative statics, see Topkis (1998).

crossing property (which we denote by $g \succeq_{SC} f$) if for all x'' and x' such that $x'' > x'$, the following holds:

$$f(x'') - f(x') \geq (>)\, 0 \implies g(x'') - g(x') \geq (>)\, 0. \tag{1}$$

For two real-valued functions f and g defined on X, we say that g dominates f by the *interval dominance order* (or, for short, g I-dominates f, with the notation $g \succeq_I f$) if (1) holds for x'' and x' such that $x'' > x'$, whenever $f(x'') \geq f(x)$ for all x in the interval $[x', x'']$.

It is clear that the interval dominance order (IDO) is weaker than ordering by SCP and our previous paper has given an example of functions that are comparable by IDO but not by SCP. We now turn to some properties of this new way of ordering functions. It is obvious that the order is transitive, i.e., if $h \succeq_I g$ and $g \succeq_I f$ then $h \succeq_I f$. Our first result gives some other, quite natural, properties of this order. It is not hard to check that the properties listed in Proposition 1 below are also true of ordering by SCP; establishing them for IDO is slightly more involved. The proposition requires a mild regularity condition: the function $f : X \to R$ is said to be *regular* if $\text{argmax}_{x \in [x', x'']} f(x)$ is nonempty for any points x' and x'' with $x'' > x'$.[4]

PROPOSITION 1: *Let f, g, and h be three regular real-valued functions defined on the poset $(X, \geq)$.*
(i) If $h \succeq_I g$ and $h \succeq_I f$, then $h \succeq_I F(g, f)$, provided $F : R^2 \to R$ is increasing in the following sense: $F(\bar{x}, \bar{y}) \geq (>)\, F(x, y)$ whenever $(\bar{x}, \bar{y}) \geq (\gg)\, (x, y)$.[5]
(ii) If $h \succeq_I f$ and $g \succeq_I f$, then $F(g, h) \succeq_I f$.
(iii) If $g \succeq_I f$, then $g \succeq_I F(g, f) \succeq_I f$.
(iv) If $g \succeq_I f$, then $g \succeq_I \max\{g, f\} \succeq_I f$ and $g \succeq_I \min\{g, f\} \succeq_I f$

[4]Suppose the set X is a subset of R^l such that $X \cap [x', x'']$ is always closed, and thus compact, in R^l (with the respect to the Euclidean topology). Then f is regular if it is upper semi-continuous with respect to the relative topology on X.

[5]By $(\bar{x}, \bar{y}) \gg (x, y)$, we mean that $\bar{x} > x$ and $\bar{y} > y$. Note that our restriction on F allows for the possibility that $F(\bar{x}, y) = F(x, y)$ when $\bar{x} > x$.

The proof of this proposition requires Lemma 1 below. Lemma 1 says that the interval dominance order can be equivalently defined in terms which are (loosely speaking) the mirror image of the one we chose.

LEMMA 1: *Let f and g two regular real-valued functions defined on the poset $(X, \geq)$. Then $g \succeq_I f$ if and only if the following property holds:*

(M) if $g(x') \geq g(x)$ for x in $[x', x'']$ then

$$g(x') - g(x'') \geq (>)\, 0 \implies f(x') - f(x'') \geq (>)\, 0.$$

Proof: Suppose $x' < x''$ and $g(x') \geq g(x)$ for x in $[x', x'']$. There are two possible ways for property (M) to be violated. One possibility is that $f(x'') > f(x')$. By regularity, we know that $\text{argmax}_{x \in [x', x'']} f(x)$ is nonempty; choosing x^* in this set, we have $f(x^*) \geq f(x)$ for all x in $[x', x^*]$, with $f(x^*) \geq f(x'') > f(x')$. Since $g \succeq_I f$, we must have $g(x^*) > g(x')$, which is a contradiction.

The other possible violation of (M) occurs if $g(x') > g(x'')$ but $f(x') = f(x'')$. By regularity, we know that $\text{argmax}_{x \in [x', x'']} f(x)$ is nonempty, and if f is maximized at x^* with $f(x^*) > f(x')$, then we are back to the case considered above. So assume that x' and x'' are both in $\text{argmax}_{x \in [x', x'']} f(x)$. Since $f \succeq_I g$, we must have $g(x'') \geq g(x')$, contradicting our initial assumption.

So we have shown that (M) holds if $g \succeq_I f$. The proof that (M) implies $g \succeq_I f$ is similar. QED

Proof of Proposition 1: Notice that the maps $(x, y) \mapsto \max\{x, y\}$ and $(x, y) \mapsto \min\{x, y\}$ are both increasing in the sense defined in the proposition, so (iv) follows from (iii). It is also clear that (iii) follows from (i) and (ii). Choosing $h = g$ in (i), we obtain $g \succeq_I F(g, f)$. Choosing $h = f$ in (ii), we obtain $F(g, f) \succeq_I f$. We now turn to the proofs of (i) and (ii).

To prove (ii), assume that $f(x'') \geq f(x)$ for x in $[x', x'']$. Since both h and g

I-dominates f, we have $h(x'') \geq (>)\, h(x')$ and $g(x'') \geq (>)\, g(x')$ if $f(x'') \geq (>)\, f(x')$. Therefore $F(g(x''), h(x'')) \geq (>)\, F(g(x'), h(x'))$ if $f(x'') \geq (>)\, f(x')$, as required.

To prove (i), we rely on Lemma 1 and establish the 'mirror' property (M) instead. Assume that $h(x') \geq h(x)$ for x in $[x', x'']$. Since both f and g are I-dominated by h, we have $f(x') \geq (>)\, f(x'')$ and $g(x') \geq (>)\, g(x'')$ if $h(x') \geq (>)\, h(x'')$. Therefore $F(g(x'), f(x')) \geq (>)\, F(g(x''), f(x''))$ if $h(x') \geq (>)\, h(x'')$. QED

An immediate consequence of Proposition 1(iii) is that if $g \succeq_I f$ then

$$g \succeq_I \frac{1}{2}f + \frac{1}{2}g \succeq_I f,$$

which in turn implies that

$$\frac{3}{4}g + \frac{1}{4}f = \frac{1}{2}g + \frac{1}{2}\left[\frac{1}{2}f + \frac{1}{2}g\right] \succeq_I \frac{1}{2}f + \frac{1}{2}g.$$

Our next result is an extension of this observation.

Let $\{f(\cdot, s)\}_{s \in S}$ be a family of functions defined on the poset $(X, \geq)$ and parameterized by s in S, an interval in R. We call this an *interval dominance ordered family* (or IDO family, for short) if $f(\cdot, s'')$ I-dominates $f(\cdot, s')$ whenever $s'' > s'$. Given a density function λ on S, we define

$$F(x; \lambda) = \int_{s \in S} f(x, s)\lambda(s)ds.$$

The density function γ on S is a *monotone likelihood ratio (MLR) shift* of λ if $\gamma(s)/\lambda(s)$ is increasing in s.

PROPOSITION 2: *Let $\{f(\cdot, s)\}_{s \in S}$ be a family of regular functions parameterized by s in S, an interval in R, with each function $f(\cdot, s)$ mapping $(X, \geq)$ to R. Suppose that this is an IDO family; then $F(\cdot; \gamma) \succeq_I F(\cdot; \lambda)$ if the density function γ is an MLR shift of the density function λ.*

This result appears as Theorem 2 in QS07. Even though throughout that paper we maintained the assumption that X is a subset of R, nothing in the proof of Theorem 2 requires that assumption. That proof remains valid for any poset $(X, \geq)$. Proposition

2 is useful in problems involving uncertainty where the agent has to choose an action x before the state of the world is realized; $F(x;\lambda)$ and $F(x;\gamma)$ represent the expected payoff functions before and after a shift in probability towards the higher states. The result is intuitive: it says that a shift in probability weight towards higher states - those associated with more dominant payoff functions - will lead to a more dominant expected payoff function.

The next result gives a simple sufficient condition for checking I-dominance.

Proposition 3: *Suppose X is an open and convex subset of $(R^l, \geq)$ and let f, $g : (X, \geq) \to R$ be two differentiable functions. Then $g \succeq_I f$ if there is an increasing and positive function $\alpha : X \to R$ such that*

$$\frac{\partial g}{\partial x_i}(x) \geq \alpha(x) \frac{\partial f}{\partial x_i}(x) \text{ for } i = 1, 2, ..., l. \tag{2}$$

Note that QS07 has the one-dimensional version of Proposition 3, where X is an interval of R. QS07 also gives an example to show that condition (2) does not imply that g SCP-dominates f. The proof of Proposition 3 relies on the following lemma, whose proof can be found in QS07.

Lemma 2: *Suppose $[x', x'']$ is a compact interval of R and $\hat{\alpha}$ and h are real-valued functions defined on $[x', x'']$, with h integrable and $\hat{\alpha}$ increasing (and thus integrable as well). If $\int_x^{x''} h(s)ds \geq 0$ for all x in $[x', x'']$, then*

$$\int_{x'}^{x''} \hat{\alpha}(s)h(s)ds \geq \hat{\alpha}(x') \int_{x'}^{x''} h(s)ds. \tag{3}$$

Proof of Proposition 3: Let $x'' \geq x'$ be two elements in X and suppose that $f(x'') \geq f(x)$ for all x in $[x', x'']$. Define the function $H(\cdot; f) : [0, 1] \to R$ by $H(t; f) = f(x' + t(x'' - x'))$ and the function $H(\cdot; g)$ analogously. Then

$$H(1; f) \geq H(t; f) \text{ for } t \text{ in } [0, 1]. \tag{4}$$

Condition (2) guarantees that

$$H'(t;g) \geq \alpha(x'' + t(x'' - x'))H'(t;f) = \hat{\alpha}(t)H'(t;f),$$

where $\hat{\alpha} : [0,1] \to R$ is given by $\hat{\alpha}(t) = \alpha(x'' + t(x'' - x'))$. It follows from (4) that $\int_t^1 H'(s;f)ds \geq 0$ for all t in $[0,1]$. Therefore, by Lemma 2,

$$\begin{aligned} g(x'') - g(x') &= H(1;g) - H(0;g) \\ &= \int_0^1 H'(s;g)ds \\ &\geq \hat{\alpha}(0)\int_0^1 H'(s;f)ds \\ &= \hat{\alpha}(0)\left[H(1;f) - H(0;f)\right] \\ &= \hat{\alpha}(0)\left[f(x'') - f(x')\right]. \end{aligned}$$

Thus $g(x'') \geq (>) g(x')$ if $f(x'') \geq (>) f(x')$, so g I-dominates f. QED

We turn now to another way of checking for multidimensional I-dominance in the case where X is an open and convex sublattice of $(R^l, \geq)$ and $f : (X, \geq) \to R$ is a quasiconcave function, i.e., $f^{-1}(r)$ is a convex set in X for any r in R. To build intuition for our result, consider firstly the case of a differentiable and quasiconcave function f defined on an open interval J of R with the following additional property: $f'(x^*) = 0$ implies that x^* is in $\text{argmax}_{x \in J} f(x)$. This is a mild regularity assumption on a quasiconcave function and is always satisfied if f is concave. With this condition on f, it easy to see that $g \succeq_I f$ if the following holds: $f'(x)\,(\geq) > 0 \Rightarrow g'(x)\,(\geq) > 0$.

It turns out that this result can be extended to a multidimensional setting. We denote the set of variables $\{1, 2, ..., l\}$ by L. For any x^* in X and $K \subset L$, we may define the set $X_K(x^*) = \{x_K \in R^{|K|} : (x_K, x^*_{-K}) \in X\}$. The quasiconcave function $f : X \to R$ is said to be *well-behaved* if it is differentiable and obeys the following condition: if

$$\frac{\partial f}{\partial x_i}(x) = 0$$

for all i in $K \subset L$, then x^*_K is in $\text{argmax}_{x_K \in X_K(x^*)} f(x_K, x^*_{-K})$. In other words, at any point x^*, if the first order conditions are satisfied for some subset K of the variables, then f when considered as a function of those variables alone, with the other variables fixed at x^*_{-K}, is maximized at x^*_K. Note that this condition holds if f is concave, and holds also for any quasiconcave function f which is a differentiably strictly increasing transformation of a concave function.[6]

Proposition 4: *Suppose that X is a convex and open sublattice of $(R^l, \geq)$ and that the function $f : (X, \geq) \to R$ is a well-behaved quasiconcave function. Then the differentiable function $g : (X, \geq) \to R$ I-dominates f if at any point x,*

$$\frac{\partial f}{\partial x_i}(x)\ (\geq) > 0 \implies \frac{\partial g}{\partial x_i}(x)\ (\geq) > 0. \tag{5}$$

Proof: Assume that $x'' \geq x'$ and $f(x'') \geq f(x)$ for all x in $[x', x'']$. Suppose that I-dominance is violated. There are two possible violations: either (a) $g(x'') > g(x')$ or (b) $f(x'') > f(x')$ but $g(x'') = g(x')$. Consider case (a). Since g is differentiable, the set $\text{argmax}_{x \in [x', x'']} g(x)$ is nonempty, so choose x^* in this set. We denote by K the set $\{i \in L : x''_i > x^*_i\}$; hence for i not in K, $x''_i = x^*_i$. We claim that $\partial f(x^*)/\partial x_i = 0$ for i in K. Firstly, notice that $\partial f(x^*)/\partial x_i \leq 0$ for i in K, because if for some j in K we have $\partial f(x^*)/\partial x_j > 0$ then f is strictly locally increasing in direction j at the point x^*. By (5), g must also be strictly increasing in direction j at this point. Thus there is x^{**} in $[x^*, x'']$ with $g(x^{**}) > g(x^*)$, which contradicts the assumption that x^* is in $\text{argmax}_{x \in [x', x'']} g(x)$. Suppose now, that for some j in K, $\partial f(x^*)/\partial x_j < 0$. In this case, there is some point $\tilde{x}$ in the chord joining x^* and x'' such that $f(x^*) > f(\tilde{x}')$; on the other hand, $f(x'') \geq f(x^*)$ by assumption. Thus f is not quasiconcave, which is a contradiction.

Since $\partial f(x^*)/\partial x_i = 0$ for i in K, we know that x^*_K is in $A = \text{argmax}_{x_K \in X_K(x^*)} f(x_K, x^*_{-K})$. The set $X_K(x^*)$ contains x''_K and thus x''_K is also in A. Indeed, by the quasiconcavity of f, any point x in the chord is also in A. Thus for any point x on this chord, we

[6]Formally, if $\tilde{f}$ is concave and $h : R \to R$ satisfies $h' > 0$ then $f = h \circ \tilde{f}$ is a well-behaved quasiconcave function.

have $\partial f(x)/\partial x_i = 0$ for i in K. This implies that $\partial g(x)/\partial x_i \geq 0$ for i in K. Thus $g(x'') \geq g(x^*)$, contradicting our initial assumption that $g(x^*) > g(x'')$.

We turn now to case (b). Suppose that x' is not in $A = \text{argmax}_{x \in [x', x'']} g(x)$. Then there is an element x^* in A such that $g(x^*) > g(x'')$. The proof then proceeds as in case (a). On the other hand, suppose that x' is in A. Then proceeding as we do in case (a), we conclude that $\partial f(x')/\partial x_i = 0$ for i in K, where K is the set $\{i \in L : x''_i > x'_i\}$. But this implies that x'_K is in $\text{argmax}_{x_K \in X_K(x')} f(x_K, x''_{-K})$. Note that x''_K is also in $X_K(x')$; this contradicts the assumption that $f(x'') > f(x')$. QED

Comparative Statics using IDO

Proposition 5: *Suppose that f and g are real-valued functions defined on the poset $(X, \geq)$ and $g \succeq_I f$. If x' is in $\text{argmax}_{x \in X} f(x)$ and x'' is in $\text{argmax}_{x \in X} g(x)$ with $x'' \leq x'$, then x' in $\text{argmax}_{x \in X} g(x)$ and x'' in $\text{argmax}_{x \in X} f(x)$.*

In the case when the optimal choice is unique, Proposition 5 says that as the objective function changes from f to g, the optimal choice *cannot fall*. When the poset is a subset of R^l with the product order, Proposition 5 guarantees that it is impossible for the optimal value of *every* variable to fall. However, it does not exclude the possibility that the optimal value of some variable will fall. To exclude the latter case will require additional assumptions on the objective functions.

Proof of Proposition 5: Since x' is in $\text{argmax}_{x \in X} f(x)$, $f(x') \geq f(x)$ for all x in $[x'', x']$. Since $g \succeq_I f$, we have $g(x') \geq g(x)$ for all x in $[x'', x']$. In particular, $g(x') \geq g(x'')$, so x' is also in $\text{argmax}_{x \in X} g(x)$. To show that x'' is in $\text{argmax}_{x \in X} f(x)$, we assume to the contrary that $f(x') > f(x'')$. Then $g \succeq_I f$ implies that $g(x') > g(x'')$, contradicting our assumption that x'' is in $\text{argmax}_{x \in X} g(x)$. QED

If we wish to have a conclusion that is stronger than Proposition 5, we will need to make additional assumptions on the objective function. Let $(X, \geq)$ be a lattice. The function $f : (X, \geq) \to R$ is said to be *supermodular* if

$$f(x \vee y) - f(y) \geq f(x) - f(x \wedge y) \text{ for all } x \text{ and } y \text{ in } X. \tag{6}$$

If X is an open set and f is C^2, then supermodularity (with respect to the product order) is guaranteed if $\partial^2 f/\partial x_i \partial x_j \geq 0$ for $i \neq j$. Supermodularity is not an ordinal property; its ordinal analog (due to Milgrom and Shannon (1994)) is *quasisupermodularity*, which requires the following property on f: for any x and y in X,

$$f(x) \ \geq (>) \ f(x \wedge y) \implies f(x \vee y) \ \geq (>) \ f(y). \tag{7}$$

Given what we have done so far, it is natural to generalize the notion of quasisupermodularity in the following way: the function f is *I-quasisupermodular* if for any x and y such that $f(x) \geq f(x')$ for all x' in $[x \wedge y, x]$, we have (7).

Quasisupermodularity can be thought of as a type of single crossing property which holds 'within' the function f. The same is true of I-quasisupermodularity, except that I-dominance replaces the single crossing property. To be precise, let X be an open and convex sublattice of R^l and consider a real-valued function f defined on X. Then f is quasisupermodular (I-quasisupermodular) if and only if the following property holds: for any subset K of $L = \{1, 2, ..., l\}$, the restricted function $f(\cdot, \bar{x}_{-K})$ SCP-dominates (I-dominates) $f(\cdot, \hat{x}_{-K})$ whenever $\bar{x}_{-K} \geq \hat{x}_{-K}$ (with the domain of these functions being any interval in $R^{|K|}$ over which both functions are defined).

This means that the results we have developed to check for I-dominance, Propositions 3 and 4, can also be used to check for I-supermodularity. Our next result gives a sufficient condition for I-supermodularity and follows immediately from Proposition 4.

PROPOSITION 6: *Let X be an open and convex sublattice of $(R^l, \geq)$ and $f : (X, \geq) \to R$ a well-behaved quasiconcave function. Then f is I-quasisupermodular if for any k in L,*

$$\frac{\partial f}{\partial x_k}(\bar{x}) \geq (>) \ 0 \implies \frac{\partial f}{\partial x_k}(\tilde{x}) \geq (>) \ 0,$$

where $\tilde{x}_k = \bar{x}_k$ and $\tilde{x}_i \geq \bar{x}_i$ for $i \neq k$.

The final result of this section is a comparative statics theorem that strengthens the conclusion of Proposition 5, by imposing stronger assumptions on the objective function. The result gives conditions under which $\text{argmax}_{x\in X} g(x)$ is 'higher' than $\text{argmax}_{x\in X} f(x)$. The notion of 'higher' when comparing sets is formalized through the *strong set order*; the set S'' dominates S' in the strong set order (denoted by $S'' \geq S'$) if for any x' in S' and x'' in S'', the supremum $x' \vee x''$ is in S'' and the infimum $x' \wedge x''$ is in S'. Note that $x' \vee x'' \geq x'$ and $x' \wedge x'' \leq x''$; thus the following is true: if S'' and S' are nonempty and $S'' \geq S'$, then for any x^* in S' there is x^{**} in S'' such that $x^{**} \geq x^*$ and for any x^{**} in S'' there is x^* in S' such that $x^{**} \geq x^*$. So it does make intuitive sense for us to say that S'' is 'higher' than (or has 'increased' from S' if $S'' \geq S'$.

THEOREM 1: *Suppose that f and g are real-valued functions defined on the lattice $(X, \geq)$ and that $g \succeq_I f$. Then if either f or g are I-quasisupermodular, we obtain*

$$\text{argmax}_{x\in J} g(x) \geq \text{argmax}_{x\in J} f(x), \tag{8}$$

where J is an interval and sublattice of X. In particular (8) holds if $J = X$.

It is instructive to compare this theorem with Theorem 4 in Milgrom and Shannon (1994), in which they obtain (8) under a different set of assumptions. Compared to Milgrom and Shannon's result, Theorem 1 imposes weaker assumptions on the objective functions; instead of quasisupermodularity of the objective function, we assume I-quasisupermodularity and instead of requiring that g dominates f by the single crossing property, we assume I-dominance. On the other hand, the stronger assumptions made in Milgrom and Shannon's theorem guarantee (8) whenever J is a sublattice while, in our theorem, J must also be an interval.

Proof of Theorem 1: We shall prove the case where g is I-quasisupermodular; the other case is similar and we shall omit it. Suppose x' is in $\text{argmax}_{x\in J} f(x)$ and y is in $\text{argmax}_{x\in J} g(x)$. We first show that $x' \vee y$ is in $\text{argmax}_{x\in J} g(x)$. If not, we have $g(x' \vee y) < g(y)$. Given that g is I-quasisupermodular, there must be z in $[x' \wedge y, x']$ such that $g(z) > g(x')$. However, this is impossible: since x' maximizes f in J, we have $f(x') \geq f(x)$ for all x in $[z, x'] \subset J$, and with g I-dominating f, we obtain $g(x') \geq g(z)$.

We also need to show that $x' \wedge y$ is in $\text{argmax}_{x \in J} f(x)$. If not, $f(x') > f(x' \wedge y)$. Since x' is in $\text{argmax}_{x \in J} f(x)$, we also $f(x') \geq f(z)$ for all z in $[x', x' \wedge y] \subset J$. By I-dominance, we have $g(x') > g(x' \wedge y)$ and $g(x') \geq g(z')$ for all z in $[x' \wedge y, x']$. Note that g is I-quasisupermodular, thus $g(x' \vee y) > g(y)$, contradicting the assumption that y maximizes g. QED

2. The Stochastic Optimal Stopping Problem

Our objective in this section is to develop a comparative statics result for stochastic optimal stopping problems, analogous to the one we have already established for the deterministic optimal stopping problem (Example 2 in QS07).

Let $\{x_t\}$ be a (possibly time-heterogeneous) diffusion process adapted to some filtration $\mathcal{F} = \{F_t\}$ on the probability space (Ω, P). An agent with the instantaneous payoff function $u : (x, t) \to u(x, t) \in R$ solves the following optimal stopping problem:

$$\text{maximize } E\left[\int_0^{\hat{\tau}} \alpha(s) u(x(s), s) ds\right] \text{ subject to } \hat{\tau} \in \mathcal{T}, \tag{9}$$

where $\mathcal{T}$ denotes the set of stopping times adapted to $\mathcal{F}$ and taking values in the (possibly infinite) time interval $T = [0, \bar{t}]$. E denotes the expectation operator and we assume that the *discount function* $\alpha : T \to R$ is deterministic and strictly positive. We assume throughout that $E[(\int_0^{\bar{t}} \alpha(s) u(x(s), s) ds)^2]$ is finite, which guarantees that all expectations (including conditional expectations) are well-defined, and that Fubini's Theorem can be used.

Before proceeding with the analysis, we state the following lemma.

Lemma 3: *Let τ and $\hat{\tau}$ denote two stopping times, and consider the events*

$$\begin{aligned} A &= \left\{\omega \in \Omega : \hat{\tau} \leq \tau \textit{ and } E\left[\int_{\hat{\tau}}^{\tau} \alpha(s) u(x(s), s) ds | F_{\hat{\tau}}\right] < 0\right\} \quad \textit{and} \\ B &= \left\{\omega \in \Omega : \hat{\tau} \geq \tau \textit{ and } E\left[\int_{\hat{\tau}}^{\tau} \alpha(s) u(x(s), s) ds | F_{\hat{\tau}}\right] > 0\right\}. \end{aligned}$$

If τ solves (9), then $P(A) = P(B) = 0$.

Proof: Let $A_\varepsilon = \{\omega \in \Omega : \hat{\tau} \leq \tau \text{ and } E\left[\int_{\hat{\tau}}^{\tau} \alpha(s)u(x(s),s)ds|F_{\hat{\tau}}\right] \leq -\varepsilon\}$ for $\varepsilon > 0$. If we prove that $P(A_\varepsilon) = 0$ for all ε, continuity of P and monotonicity of A_ε will imply that $P(A) = P(\cap A_\varepsilon) = 0$. Suppose on the contrary that $P(A_\varepsilon) > 0$ for some $\varepsilon > 0$, and let $\tau^* = \hat{\tau}1_{\omega\in A_\varepsilon} + \tau 1_{\omega\notin A_\varepsilon}$. Since A_ε is $F_{\hat{\tau}}$ measurable, τ^* is a stopping time. Moreover, letting $g(s) = \alpha(s)u(x(s),s)$ and $A_\varepsilon^c = \Omega \setminus A_\varepsilon$,

$$\begin{aligned}
E\left[\int_0^{\tau^*} \alpha(s)u(x(s),s)ds\right] &= P(A_\varepsilon^c)E\left[\int_0^{\tau^*} g(s)ds|A_\varepsilon^c\right] + P(A_\varepsilon)E\left[\int_0^{\tau^*} g(s)ds|A_\varepsilon\right] \\
&= P(A_\varepsilon^c)E\left[\int_0^{\tau} g(s)ds|A_\varepsilon^c\right] + P(A_\varepsilon)E\left[\int_0^{\tau^*} g(s)ds|A_\varepsilon\right] \\
&\geq P(A_\varepsilon^c)E\left[\int_0^{\tau} g(s)ds|A_\varepsilon^c\right] + P(A_\varepsilon)(E\left[\int_0^{\tau} g(s)ds|A_\varepsilon\right] + \varepsilon) \\
&= E\left[\int_0^{\tau} g(s)ds\right] + P(A_\varepsilon)\varepsilon,
\end{aligned}$$

which contradicts optimality of τ. The second inequality is proved similarly. QED

We define the function $v : R_+ \to R$ by

$$v(s) = E[u(x(s),s)1_{s<\tau}]\,; \tag{10}$$

$v(s)$ represents the expected payoff rate at time s, where the payoff is zero in the event that $s \geq \tau$. Denoting $W(\alpha) = E\left[\int_0^{\tau} \alpha(s)u(x(s),s)ds\right]$, Fubini's theorem implies that

$$W(\alpha) = \int_0^{\bar{t}} \alpha(s)v(s)ds. \tag{11}$$

Our first result is a simple consequence of the fact that, at every point in time, the expected payoff of an optimizing agent looking forward must be non-negative.

Lemma 4: *For all t in $[0,\bar{t})$, $\int_t^{\bar{t}} \alpha(s)v(s)ds \geq 0$.*

Proof: By definition of v,

$$\begin{aligned}
\int_t^{\bar{t}} \alpha(s)v(s)ds &= E\left[\int_t^{\bar{t}} \alpha(s)u(x(s),s)1_{s<\tau}ds\right] \\
&= E\left[E\left[\int_t^{\bar{t}} \alpha(s)u(x(s),s)1_{s<\tau}1_{t<\tau}ds|F_t\right]\right] \\
&= E\left[E\left[\int_t^{\tau} \alpha(s)u(x(s),s)ds|F_t\right]1_{t<\tau}\right].
\end{aligned}$$

Optimality of τ and Lemma 3 imply that the inner expectation is almost surely nonnegative if $t < \tau$. Therefore, the random variable $E[\int_t^{\bar{t}} \alpha(s)u(x(s),s)1_{s<\tau}ds|F_t]1_{t<\tau}$ is always nonnegative, and so is its expectation. QED

Lemma 4 leads to the following result.

THEOREM 2: *Let τ and τ' be solutions to the optimal stopping problem (9) when the discount function is α and β respectively. If $\beta(s)/\alpha(s)$ is increasing in s, then $\tau \vee \tau'$ is also an optimal stopping time for the discount function β.*

Proof: Consider any diffusion path ω, and $t < \tau(\omega)$. It is enough to show that

$$E\left[\int_t^{\tau} \beta(s)u(x(s),s)|F_t\right] \geq 0$$

as it implies that waiting until τ is at least weakly better than stopping immediately. We can set without loss of generality $t = 0$, since the problem could otherwise be restated with the origin of time at t. We wish to show that $\int_0^{\bar{t}} \beta(s)v(s)ds \geq 0$ with v as defined by (10). Lemma 4 guarantees that the hypothesis of Lemma 2 is satisfied, so we obtain

$$\int_0^{\bar{t}} \beta(s)v(s)ds = \int_0^{\bar{t}} \frac{\beta(s)}{\alpha(s)}\alpha(s)v(s)ds \geq \frac{\beta(0)}{\alpha(0)}\int_0^{\bar{t}} \alpha(s)v(s)ds = W(\alpha) \geq 0. \qquad (12)$$

. QED

As in the deterministic case, the proof of this theorem also shows that the value $W(\cdot)$ of the stopping problem decreases with the interest rate or, equivalently, increases with patience.

COROLLARY 1 (Value Monotonicity): *Suppose that the discount functions α and β satisfy $\alpha(0) = \beta(0) = 1$. Then $W(\beta) \geq W(\alpha)$.*

Proof: By definition, $W(\beta)$ is the utility achieved at an optimal stopping time τ' for the discount function β. So $W(\beta) \geq E_x[\int_0^{\tau} \beta(s)u(x(s),s)ds] = \int_0^{\bar{t}} \beta(s)v(s)ds$. The result then follows (12). QED

Theorem 2 and Corollary 1 tell us that as the future gets discounted less, the optimal horizon gets longer and the optimal value gets larger, independently of the particular diffusion process and payoff function under consideration. Suppose that the discount functions are exponential, i.e., $\alpha(s) = \exp(-\bar{\alpha}s)$ and $\beta(s) = \exp(-\bar{\beta}s)$, where α and β are positive scalars. Then the ratio $\beta(s)/\alpha(s)$ is increasing in s if $\bar{\beta} < \bar{\alpha}$. More generally, if we allow for a nonconstant discount rate, we have $\alpha(s) = \exp(-\int_0^s r_\alpha(z)dz)$ where the function r_α is positive and deterministic. Writing a similar expression for $\beta(s)$, it is easy to check that $\beta(s)/\alpha(s)$ is increasing in s if $r_\alpha(z) > r_\beta(z)$ for all z in $T = [0, \bar{t}]$.

As an application of Theorem 2, consider the model of endogenous-default setting introduced by Leland (1994) and generalized by Manso et al. (2004). Equity holders of a firm must pay a coupon rate $c(x)$ to debtholders, where c is nonincreasing in some performance measure x, and receive a payout rate $\delta(x)$, with δ nondecreasing in x.[7] The performance measure $\{x_t\}$ is a time-homogeneous diffusion (for example, geometric Brownian motion, or a mean-reverting) process. The shareholder problem is thus to solve

$$W(x, r) = \sup_{\hat{\tau} \in \mathcal{T}} E_x \left[\int_0^{\hat{\tau}} e^{-rt}(\delta(x_t) - c(x_t))dt \right].$$

Given the time-homogeneous, Markov structure of the problem, and since $\delta - c$ is nondecreasing it is easy to show that optimal default takes the form of a hitting time $\tau_{A_B(r)} = \inf\{t : x_t \leq A_B(r)\}$; $A_B(r)$ is called the *default-triggering level* of the firm, and is independent of the initial asset level x. Theorem 2 says that $A_B(r)$ is nondecreasing in r, and Corollary 1 says that $W(x, r)$ is nonincreasing in r. We can check this result directly when $\delta(x) = \delta x$, $c(x) = c$, and x is the geometric Brownian motion with drift μ and volatility σ. In this case, standard computations (see, for

[7] For standard debt, c is a constant. However, in many contracts such as performance-pricing loans or step-up bonds, c increases as some performance measure of the firm deteriorates. This measure maybe the credit rating, or directly related to the earnings (EBITDA, price-earning ratio, etc.) of the issuing firm. See Manso, et al. (2004) for examples. The model can easily be modified to account for tax and bankruptcy costs.

example, Manso et al. (2004)) show that

$$A_B(r) = \frac{\gamma(r)}{\gamma(r)+1}\left(1-\frac{\mu}{r}\right)\frac{c}{\delta},$$

where $\gamma(r) = (m + \sqrt{m^2 + 2r\sigma^2})/\sigma^2$ and $m = \mu - \sigma^2/2$. Since A_B increases in γ and r, and $\gamma(r)$ increases in r, necessarily A_B increases in r. In general, A_B cannot be computed explicitly. However, Corollary 1 ensures that monotonicity with respect to the interest rate holds for very general asset processes and coupon and payout profiles.

3. Applications of IDO in higher dimensions

Example 3.1. Consider a criminal who has to decide on the scale (x) of his illegal activity, which earns him a revenue of $u(x) > 0$. The probability that he is caught increases with the scale of his activity, in the form of an exponential distribution with mean $1/\theta$. So the probability of him not being caught is $e^{-x/\theta}$, and if he is caught, his revenue is zero, which gives him an expected revenue of $u(x)e^{-x/\theta}$. We also assume that an operation of scale x incurs a cost of $C(x)$; furthermore, he can choose θ by paying a bribe of $B(\theta)$ to the authorities. Assume that both B and C are increasing functions. The criminal's overall utility function $U : R^2_{++} \to R$ has the form

$$U(x,\theta) = u(x)e^{-x/\theta} - C(x) - B(\theta). \tag{13}$$

Suppose the revenue function changes from u to v: what conditions relating u and v will guarantee that $V \succeq_I U$ (where V is the overall utility function, as defined in (13), associated with v)?

Re-writing $V(x,\theta) = [v(x)/u(x)]\,[u(x)e^{-x/\theta}] - C(x) - B(\theta)$, and differentiating this expression, we see that

$$V_x(x,\theta) \geq \frac{v(x)}{u(x)} U_x(x,\theta) \tag{14}$$

if $v(x) \geq u(x) > 0$. Subject to this condition, we can also check easily that

$$V_\theta(x,\theta) \geq \frac{v(x)}{u(x)} U_\theta(x,\theta). \tag{15}$$

Using Proposition 3, we see that $V \succeq_I U$ provided $\alpha(\theta, x) = v(x)/u(x)$ is increasing. This last condition is equivalent to $xv'(x)/v(x) \geq xu'(x)/u(x)$, i.e., the elasticity of revenue with respect to criminal activity (x) has increased. Proposition 5 then tells us that the optimal values of x and θ cannot both fall with the change from u to v.

We may wish to go beyond this, to find conditions under which $\text{argmax}_{R^2_{++}} V(x, \theta) \geq \text{argmax}_{R^2_{++}} U(x, \theta)$. This can be obtained using Theorem 1, but it will require additional conditions on the objective functions. The next result sets out these conditions.

Proposition 7: *Suppose that U and V are defined by (13). Assume also that u, v, C and B are all differentiable, with $B'(x) > 0$ and $C'(x) > 0$. Then*

$$\text{argmax}_{R^2_{++}} V(x, \theta) \geq \text{argmax}_{R^2_{++}} U(x, \theta)$$

if the following conditions hold:
(i) for all $x > 0$, we have $u(x) > 0$, $v(x) > 0$, and $v(x)/u(x)$ increasing with x;
(ii) the elasticity coefficient $v'(x)/v(x)$ is strictly decreasing and continuous in x, with $\lim_{x \to 0} v'(x)/v(x) = \infty$ and $\lim_{x \to \infty} v'(x)/v(x) = 0$.

Proof: We define $R(x, \theta) = v(x)e^{-x/\theta}$. Then

$$R_x(\theta, x) = \frac{e^{-x/\theta}}{v(x)} \left[\frac{v'(x)}{v(x)} - \frac{1}{\theta} \right].$$

By assumption (ii), there exists a unique $x^*(\theta)$ at which $R(\cdot, \theta)$ is maximized and $x^*(\theta)$ increases with θ. We define $X = \{(\theta, x) \in R^2_{++} : x \leq x^*(\theta)\}$. Since $v'(x)/v(x)$ is decreasing in x, we have

$$\frac{v'(x)}{v(x)} \geq \frac{1}{\theta} \tag{16}$$

for x in X. Since c is a strictly increasing function, the value of x that maximizes $V(\cdot, \theta)$ must lie in $\{x \in R_+ : x \leq x^*(\theta)\}$. With (14) and condition (i), Proposition 3 tells us that $V(\cdot, \theta) \succeq_I U(\cdot, \theta)$, so the value of x that maximizes $U(\cdot, \theta)$ must also lie in $\{x \in R_+ : x \leq x^*(\theta)\}$. We conclude that $\text{argmax}_{x \in X} V(x, \theta) = \text{argmax}_{x \in R^2_{++}} V(x, \theta)$ and $\text{argmax}_{x \in X} U(x, \theta) = \text{argmax}_{x \in R^2_{++}} U(x, \theta)$.

We now show that for any (x, θ) in X, $V_{x,\theta}(x, \theta) \geq 0$, i.e., V is supermodular in X. The result then follows from Theorem 1, given that we have already shown that $V \succeq_I U$ is guaranteed by condition (i). Straightforward calculation gives us

$$V_{x\theta}(x, \theta) = \frac{e^{-x/\theta}}{\theta^2 v(x)} \left[x \left(\frac{v'(x)}{v(x)} - \frac{1}{\theta} \right) + 1 \right].$$

By (16), we have $V_{x\theta}(x, \theta) > 0$. QED

More examples will follow in later versions of this paper .

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